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COLLECTIVE EXCITATIONS IN FERMI SYSTEMS

by V. M. GALITSKY

The present paper is devoted to collective excitations in Fermi systems at zero temperature. As applied to various specific systems, this problem was considered in the papers by Pines and Schrieffer. The purpose of this paper is to give a discussion of the problem from a general point of view and a study of the general method of considering collective excitations.

1. In many cases, weakly excited states of a system of interacting particles may be described approximately as an assemblage of elementary excitations, quasiparticles. The elementary excitations do not represent stationary states of the system, which fact results in damping of the quasi-particles. Damping of a quasiparticle with momentum P is proportional to

$$|P - P_F|^2, \tag{1}$$

so that a description of excited states in terms of quasiparticles is all the more exact, the closer the quasiparticle momenta are to P_F.

In a Fermi system, the quasiparticles have spin $\frac{1}{2}$ and, consequently, are Fermi quasiparticles. This means that no single quasiparticles can be created or destroyed. For this reason, the simplest excited state is an assemblage of two quasiparticles or of a quasiparticle and a quasihole. An essential feature of such states is the interaction between quasiparticles. If this interaction leads to scattering of quasiparticles, the excitation energy is equal to the energy of the quasiparticles at infinity:

$$E = \varepsilon(p_1) - \varepsilon(p_2), \quad \boldsymbol{P} = \boldsymbol{p}_1 - \boldsymbol{p}_2. \tag{2}$$

However, in a number of cases the interaction leads to the appearance of states which can be interpreted as bound states of quasiparticles. Such excited states are what we call collective excitations.

Collective excitations are conveniently investigated by means of the two-particle Green's function:

$$iK(12; 34) = \langle T\{\psi(1)\psi^+(2)\psi(3)\psi^+(4)\}\rangle. \tag{3}$$

In the case $t_1 = t_2 > t_3 = t_4$ the function can be written in the form

$$iK(12; 34) = \textstyle\sum_s \chi_s(12)\bar{\chi}_s(34), \tag{4}$$

where

$$\chi_s(12) = \langle 0|\psi(1)\psi^+(2)|s\rangle \tag{5}$$

can be interpreted as a wave function describing the behaviour of a particle and a hole in the S-state. After a Fourier transformation, expression (4) assumes the form of a Lehmann expansion for the function K, which shows that the two-particle Green's function has poles corresponding to the energy of particle and hole. To the bound states — collective excitations — there corresponds an isolated pole of the function K and the wave function

$$\chi_{\text{c.e.}}(1, 2) = \langle 0|\psi(1)\psi^+(2)|c.e.\rangle. \tag{4'}$$

This wave function is a matrix element between the ground state of the system and the state of collective excitation of the density matrix, and in the classical limit it goes into a Fourier component of the distribution function. Thus, the problem of investigating collective excitations reduces to a consideration of the two-particle Green's function and to finding the isolated poles of this function. The solution of this problem turns out to be different in two possible cases: repulsion and attraction between particles.

2. Let us first consider the case of repulsion [1]). In this case the ground state of free particles passes into the ground state of the system when the interaction is switched on adiabatically. Therefore, use may be made of the ordinary procedure of the S-matrix and Green's functions.

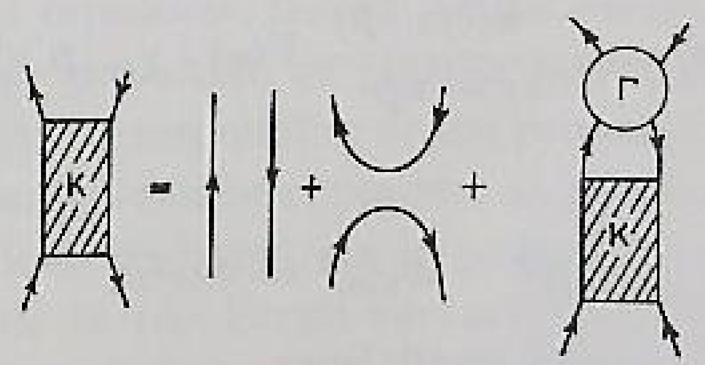

Fig. 1

As has been shown in the works of Schwinger [2]) and Gell-Mann and Low [3]), the equation for function K is of the form shown in fig. 1 or, analytically,

$$K(12; 34) = -\,iG(1-4)G(2-3) + iG(1-2)G(3-4) + \\ + i\int G(1-5)G(6-2)\Gamma(56; 78)K(78; 34)\mathrm{d}5\mathrm{d}6\mathrm{d}7\mathrm{d}8, \tag{6}$$

where Γ is a quadrupole irreducible with respect to a particle and a hole. The inhomogeneous term of this equation describes the propagation of non-interacting particle and hole and does not contain frequencies corresponding to bound states. For this reason, extraction of the function χ, which describes collective excitations, leads to the following homogeneous equation for $\chi_{\text{c.e.}}$:

$$\chi_{\text{c.e.}}(1, 2) = i\int G(1-5)G(6-2)\Gamma(56; 78)\chi_{\text{c.e.}}(78)\mathrm{d}5\mathrm{d}6\mathrm{d}7\mathrm{d}8. \tag{7}$$

We have thus obtained an eigenvalue problem, the solution of which yields the energy of the collective excitations and the wave function.

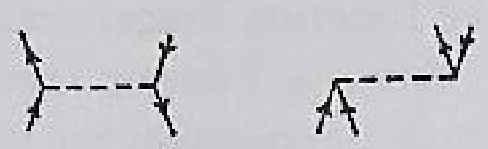

Fig. 2

As a first example let us consider the case of weak interaction. For excitations with total spin zero, the irreducible quadrupole is determined, to a first approximation, by two graphs in fig. 2, and the equation for the function χ can be written in the following form:

$$\chi_{\kappa\omega}(\mathbf{p}) = \frac{n_0(\mathbf{p}+\kappa/2) - n_0(\mathbf{p}-\kappa/2)}{\omega - \kappa\mathbf{p}} \cdot \cdot \{\int V(\mathbf{q})\chi_{\kappa\omega}(\mathbf{p}+\mathbf{q})\mathrm{d}\mathbf{q} - 2V(\kappa)\int \chi_{\kappa\omega}(\mathbf{p}')\mathrm{d}\mathbf{p}'\}. \tag{8}$$

Here κ and ω are the total momentum and excitation frequency, $\mathbf{p}$ is the relative momentum of the particle and hole, $\hbar = m = 1$.

In the case of Coulomb interaction, $V(\kappa)$ has a pole at $\kappa \to 0$, and so the second term is much greater than the first. Confining ourselves to the case of small κ we obtain the kinetic equation in the self-consistent field approximation:

$$\chi_{\kappa\omega}(\mathbf{p}) = \frac{-\kappa(\partial f_0/\partial \mathbf{p})}{\omega - \kappa\mathbf{p}} V(\kappa)\int \chi_{\kappa\omega}(\mathbf{p}')\mathrm{d}\mathbf{p}' \tag{9}$$

with the solution

$$\omega^2 = \omega_{\mathrm{p.o.}}{}^2 = 4\pi e^2 n,$$

which corresponds to plasma oscillations.

In the case of short-range forces, $V(\kappa)$ and $V(q)$ may be replaced by the zeroth Fourier component of the potential V_0, and the equation for χ takes on the form:

$$\chi_{\kappa\omega}(\mathbf{p}) = \tfrac{1}{2}\frac{-\kappa(\partial f_0/\partial \mathbf{p})}{\omega - \kappa\mathbf{p}} V_0\int \chi_{\kappa\omega}(\mathbf{p}')\mathrm{d}\mathbf{p}'. \tag{10}$$

In this case we obtain what is known as zero sound:

$$\omega = S\kappa, \quad S = P_F(1 + 2\mathrm{e}^{-1/\xi}) \tag{11}$$

$$\xi = \frac{P_F V_0}{8\pi^2}.$$

For excitations with total spin unity, Γ is determined only by the first graph of fig. 2; for this reason, for these excitations an equation of type (8) does not contain the second term. The resulting equations have no solutions.

As a second illustration, we consider a dilute gas with a large short-range interaction potential. In this case, in place of the potential we must use the group of "ladder" diagrams for two particles shown in fig. 3, that is, the effective potential [4]). The irreducible quadrupole Γ is determined by the

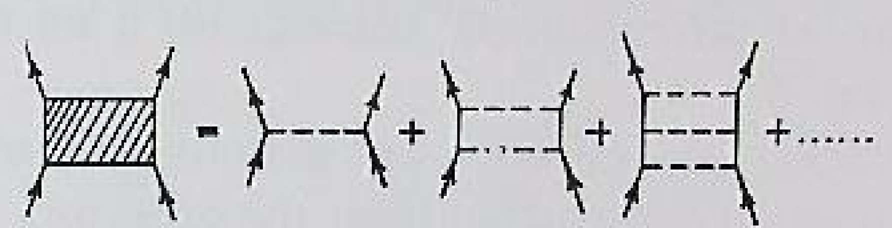

Fig. 3

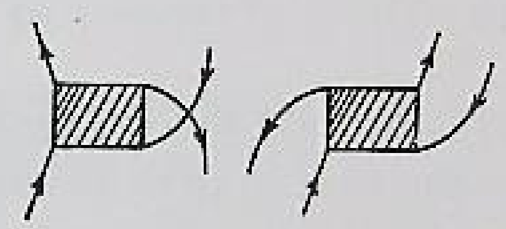

Fig. 4

two graphs of fig. 4. To the first approximation (as regards powers of density), the effective potential is equal to the real part of the scattering amplitude taken with inverse sign and divided by 4π. Introducing the notation $a = -\mathrm{Re}\, f/4\pi$ (for the case of hard spheres, a is equal to their radius), we obtain equation (10), in which V_0 is substituted by a. Accordingly, the expression for frequency of zero sound is obtained from (11) by a similar substitution.

Thus, in the case of repulsion, the collective excitations represent either zero sound (short-range potential) or plasma waves (Coulomb interaction). The absence of solutions corresponding to the propagation of ordinary sound obtained in hydrodynamics can be explained as follows. As can be seen from equation (8), the particles participating in collective excitation are situated in a narrow layer close to the Fermi surface of width of the order of K. Therefore, the damping of these particles is proportional to K^2, and the mean free path, to λ^2. Consequently, for long waves the mean free path is always greater than the wavelength of sound, and the hydrodynamical approximation cannot be applied.

3. Let us consider the case of attraction [5]). In this case, there is a Bose condensate of coupled pairs in the ground state, and the ordinary graphical method is incorrect [6]). To account for the rearrangement caused by pairing, the original Hamiltonian with direct interaction between particles must be transformed to the operators of quasiparticles α and α^+ applying for instance the Bogolyubov method [7]). After transformation, the Hamiltonian assumes the form:

$$H = E_0 + H_0 + H' \tag{12}$$

where H_0 is the Hamiltonian of free quasiparticles.

$$H_0 = \textstyle\sum_p \varepsilon(p)(\alpha_{p0}{}^+\alpha_{p0} + \alpha_{p1}{}^+\alpha_{p1})$$
$$\varepsilon(p) = \sqrt{(\tfrac{1}{2}p^2 - \mu)^2 + \Delta^2} \tag{13}$$

and H' is the Hamiltonian of interaction between quasi-particles. This interaction is obtained from the original interaction by means of a canonical transformation and transition to a N-product. As a result of the canonical transformation, there appears (in place of operator a) either of the operators α or α^+, so that to the interaction H' corresponds a set of different vertex parts depicted in fig. 5.

For a consideration of collective excitations we introduce the two-particle Green's function constructed from the operators of quasi-particles:

$$iK(12;34) = \langle T\{\alpha_1\alpha_2\alpha_3{}^+\alpha_4{}^+\}\rangle. \tag{14}$$

Let us now consider the graphs for function K on the assumption of the smallness of interaction. In the first approximation, the interaction Hamiltonian H' contains only one graph (fig. 5, graph a). In the second approximation there appear three graphs shown in fig. 6. It is not difficult to see that graphs a and b are of the same order of magnitude as the graph of the first approximation. Indeed, the total momentum of the quasi-particles in the intermediate state of the graphs is fixed and is equal to the excitation momentum κ. Integration with respect to the relative momentum in the case of interest of small κ yields $\ln \bar{\omega}/\Delta$; this compensates the smallness of the interaction potential. The appearance of this logarithm is natural and follows from the existence of bound states of particles with total momentum different from zero but small [8]). In contrast, in graph c, in which the total momentum of the quasi-particles is not given in the intermediate state, the total momenta are essentially large and the compensating logarithm is absent.


a b c

Fig. 5

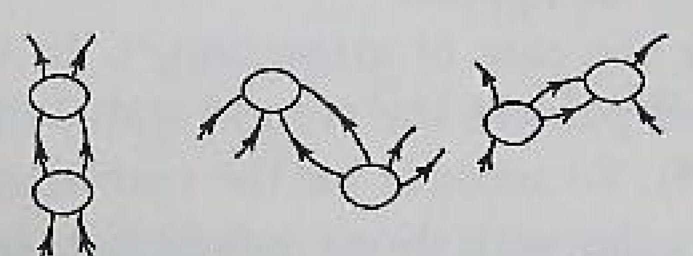
a b c

Fig. 6

Thus, the two-quasi-particle Green's function, K, is determined by an infinite sequence of graphs a and b, fig. 6. This sequence is similar to the system of graphs for a single-particle Green's function in a Bose gas [9]). For this reason, the subsequent consideration is best conducted in similar manner, by introducing (in addition to the K function) a second two-particle

Green's function,

$$i\bar{K}(12;34) = \langle T\{\alpha_1{}^+\alpha_2{}^+\alpha_3{}^+\alpha_4{}^+\}\rangle. \tag{15}$$

For functions K and $\bar{K}$ it is easy to construct a set of equations similar to (5–2) of reference [9]). This system is depicted graphically in fig. 7. Discarding

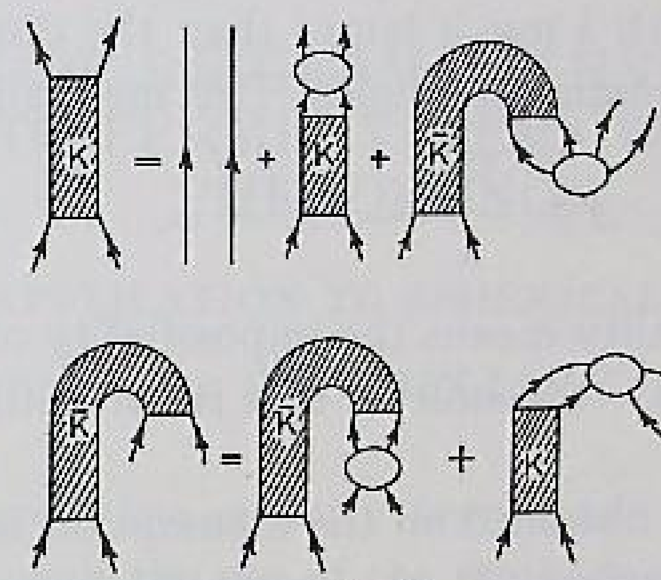

Fig. 7

the inhomogeneity, we obtain the equations for the functions:

$$\begin{aligned}\chi_{\kappa\omega}(p) &= \langle 0|\alpha_{p+(\kappa/2),\,0}\,\alpha_{p-(\kappa/2),\,1}|\text{c.e.}\rangle \\ \varphi_{\kappa\omega}(p) &= \langle 0|\alpha^+_{p-(\kappa/2),\,0}\,\alpha^+_{p+(\kappa/2),\,1}|\text{c.e.}\rangle,\end{aligned} \tag{16}$$

which are of the following form,

$$\chi_{\kappa\omega}(p) = \frac{1}{\varepsilon(p+\kappa/2)+\varepsilon(p-\kappa/2)-\omega}\cdot \{\int \mathrm{d}p'\gamma_{11}(pp')\chi_{\kappa\omega}(p') - \int \mathrm{d}p'\gamma_{12}(pp')\varphi_{\kappa\omega}(p')\}$$

$$\varphi_{\kappa\omega}(p) = -\frac{1}{\varepsilon(p+\kappa/2)+\varepsilon(p-\kappa/2)+\omega}\cdot \{\int \mathrm{d}p'\gamma_{21}(pp')\chi_{\kappa\omega}(p') - \int \mathrm{d}p'\gamma_{22}(pp')\varphi_{\kappa\omega}(p')\}, \tag{17}$$

where κ and ω are the momentum and energy of excitation.

$$\begin{aligned}\gamma_{11} = \gamma_{22} &= V\cdot(u_{p+(\kappa/2)}u_{p'-(\kappa/2)} + v_{p+(\kappa/2)}v_{p'-(\kappa/2)})(u_{p-(\kappa/2)}u_{p'+(\kappa/2)} + \\ &\qquad + v_{p-(\kappa/2)}v_{p'+(\kappa/2)}) \\ \gamma_{12} = \gamma_{21} &= V\cdot(u_{p+(\kappa/2)}v_{p'-(\kappa/2)} - v_{p+(\kappa/2)}u_{p'-(\kappa/2)})(u_{p-(\kappa/2)}v_{p'+(\kappa/2)} - \\ &\qquad - v_{p-(\kappa/2)}u_{p'+(\kappa/2)})\end{aligned} \tag{18}$$

Due to the degeneracy of the kernels (18), the system of integral equations reduces to an algebraic set, the condition of solubility of which yields an equation for determining the dependence $\omega(\kappa)$. For small momenta we obtain

$$\omega = S\kappa, \quad S^2 = \frac{P_F{}^2}{3} \tag{19}$$

that is, ordinary sound. The appearance, in a system with attraction, of excitations of the type of ordinary sound may be explained as follows. The formation of coupled pairs signifies that each particle is enclosed in a space of the order of the volume of the pair. In other words, the mean free path of the particles is of the order of the dimensions of the pair, P_F/Δ. This is why for waves with wavelength λ much larger than the dimensions of the pair, the hydrodynamical approximation holds. This inequality can be written as

$$\Delta \gg \frac{P_F}{\lambda} \sim SK = \omega. \tag{20}$$

In such form, this inequality means the impossibility of decay of the sound quantum into excitations — a condition that is not fulfilled in the absence of pairing.

The solution of (19) is obtained on the assumption of constancy of interaction on the Fermi surface. If the potential has spherical harmonics different from zero, the system (17) also has solutions that correspond to collective excitations with angular momentum different from zero. These excitations have been considered by N. N. Bogolyubov, V. V. Tolmachev and D. V. Shirkov [10]), and Schrieffer [11]).

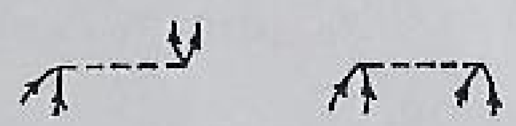

Fig. 8

When Coulomb interaction is taken into account, expression (18) does not hold. In this case, it is necessary to take into consideration that the vertex parts *a* and *b* (fig. 5) contain in particular graphs of fig. 8. These graphs include the nonshielded Coulomb potential, since the polarization graphs are already taken into account in the set of equations in fig. 7 *). Solution of the equations with Coulomb interaction leads to plasma waves (see also [11]).

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*) In other words, the screened potential is a reduced graph in our system.

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EXTENSION OF DIAGRAM TECHNIQUE OF FIELD THEORY TO FERMI SYSTEMS WITH PAIRING PHENOMENON

(WITH APPLICATION TO SPHERICAL NUCLEI)

by S. T. BELIAEV

Considerable progress has recently been made in the theory of many-particle systems through the use of the methods of quantum field theory. However, it has not always been possible to utilize directly the methods of field theory. Two important problems may serve to illustrate when ordinary graphical methods are not applicable, namely, a non-ideal Bose gas and a superconductor, that is, a Fermi system with attraction.

The existence of a large number of particles in the $p = 0$ state (condensate) does not permit applying ordinary methods in the case of a Bose gas. This difficulty may be circumvented by considering (in place of the initial system) only particles outside the condensate [1]) [2]). However, the new system is no longer closed, and transitions become possible of pairs of particles $(\boldsymbol{p}, -\boldsymbol{p})$ into the condensate and reverse processes, for a description of which it is necessary to introduce, in addition to the ordinary Green's functions

$$G = -i\langle T\{\Psi\Psi^+\}\rangle, \tag{1}$$

also two other functions:

$$F = -i\langle T\{\Psi\Psi\}\rangle; \quad F^+ = -i\langle T\{\Psi^+\Psi^+\}\rangle \tag{2}$$

and to construct all graphs with the aid of functions (1) and (2).

The case of a superconductor has much in common with a Bose gas. Here, likewise, in the ground state is a condensate (consisting of Cooper pairs) which is an obstacle to the applications of ordinary graphical methods. By analogy with the Bose gas, it is also possible in this case to introduce Green's functions (2) [3]) and then consider the totality of graphs constructed from (1) and (2). It should be stressed that, whereas for a Bose system this method is rigorously substantiated [1]), for a superconductor we are still lacking such proof, and the substantiation is rather intuitive.

For definiteness we consider below the case of a finite Fermi system, a spherical nucleus, although the general formulas can very easily be re-written for an infinite system.