AN EXAMPLE OF STOCHASTIC INSTABILITY OF NONLINEAR OSCILLATIONS

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The aim of this communication is to once more focus attention on a specific form of nonlinear system instability -- the so-called stochastic instability. This type of instability must be taken into consideration especially during the development of the so-called nonlinear accelerators which are recently attracting considerable attention.

The paper presents certain results of numerical calculations for the undamped nonlinear oscillations caused by an external periodic perturbation. The motion of the oscillator is described by an exact system of difference equations:

$$\omega_{n+1} = \omega_n - \varepsilon\omega_n \,\mathrm{sign}\,(\psi_n - 1/2), \qquad (1)$$

$$\psi_{n+1} = \left\{\psi_n + \omega_{n+1} + \frac{\varepsilon}{4} - \varepsilon\,|\,\psi_n - 1/2\,|\right\}.$$

The equations are in the difference form because the external perturbation is chosen in the form of very short pulses whose relative magnitudes are characterized by a parameter ε. The reduction of the differential equations to difference equations allows a reduction of the calculational error to a minimum determined by the rounding off errors. This is essential for the study of the behavior of the oscillator over extended intervals of time.

In the system of equations (1), ω (w) is the frequency which, because of nonlinearity, is characterized by the energy of the oscillator; ψ is the phase of the oscillations, having a period 1; index n is the number of the pulse; { } denote the fractional part of the argument.

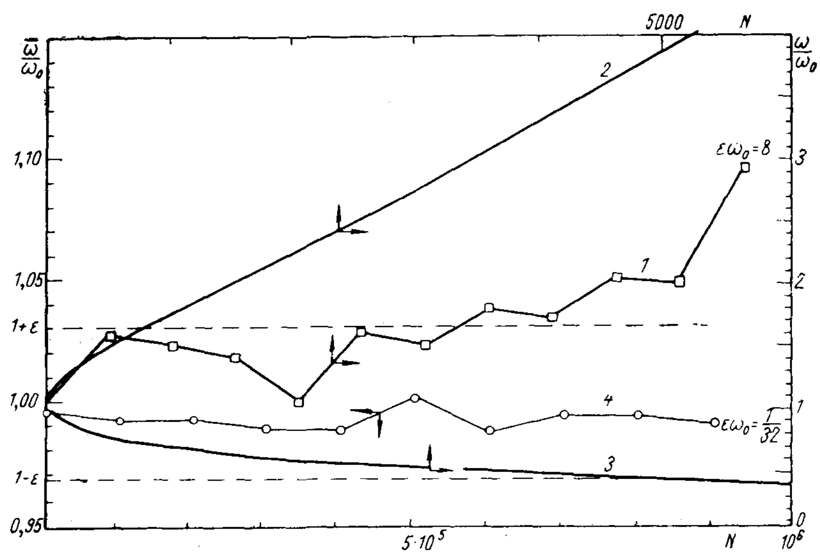
The results of the integration of the system (1) are shown in the figure for two values of the characteristic parameter $\varepsilon\omega$. For $\varepsilon\omega \ll 1$ and $\omega_0 - k \ll 1$ ($k \neq 0$, integer) the difference equations may be approximately substituted by differential equations [1]:

$$\left.\begin{aligned}\omega' &\approx -\varepsilon\omega v(\psi),\\ \psi' &\approx \omega - k,\end{aligned}\right\} \tag{2}$$

where $v(\psi)$ is a stepwise function with unit amplitude, while the differentiation is carried over n. The system (2) may be transformed into a phase equation

$$\psi'' + \varepsilon\omega v(\psi) \approx 0, \tag{3}$$

from which it can be seen that, within the approximation under consideration, the frequency (and energy) of the oscillator describes limited so-called phase oscillations with a frequency and amplitude $\sim\sqrt{\varepsilon\omega}$. The rigorously bounded solution of the exact system (1) (for $t \longrightarrow \infty$) with $\varepsilon\omega \ll 1$ is extremely complicated. A



The example of stochastic instability of nonlinear oscillations:

$\omega(w)$ -- The frequency of the nonlinear oscillator depending on the energy; N -- the number of oscillations; 1 -- the motion of the oscillator for $\omega_0 = 64$; $\varepsilon = 1/8$; $\psi_0 = 1/4$; 2, 3 -- average changes in frequency during stochastic motion; 4 -- the motion of the oscillator with $\omega_0 = 1$; $\varepsilon = 1/32$; $\psi_0 = 1/4$. Dashed line -- the magnitude of a single pulse for the last mentioned case (the lowest scale N refers to curves 3 and 4, the upper to curves 1, 2).

solution can be obtained from similar works [2, 3] although this was never rigorously proved. The results of the present calculations are likewise in agreement with such a solution. The figure contains the value of the oscillator frequency averaged over phase oscillations (curve 4). Even over the extent of a million oscillations, we did not observe any tendency towards a systematic change, and the oscillations did not exceed 1 percent.

The motion for $\varepsilon\omega \gg 1$ (curve 1) is of a completely different character. Even during only 500 oscillations, the frequency changes by almost a factor of three although the relative pulse intensity (recalculated per single oscillation) is 16 times smaller than in the previous case. In agreement with deliberations mentioned earlier [4, 5], this time the motion must have a stochastic character or, as it is customary to say in the theory of dynamic systems, it must be a motion with statistical fluctuations.

Alternative number	ω_0	ε	$\varepsilon\omega_0$	N	η	$\overline{\eta^2}_T^{1/2}$	ξ	$(\overline{\xi^2})^{1/2}$	$\overline{(\xi^2)}_T^{1/2}$	$\bar{\psi}-\frac{1}{2}$	$[\overline{(\Delta\psi)^2}]_T^{1/2}$	Number of cases
1	64	1/8	8	500	+0,17	0,11	+0,41	5,7	0,63	—	—	11
2	64	1/8	8	1000	+0,17	0,23	—0,49	0,67	1,20	—	—	5
3	7000	0,01	70	$3{,}5\cdot10^6$	—0,08	0,04	—0,22	3,4	0,74	0,01	0,05	40
4	7000	0,01	70	$7\cdot10^6$								

Note. $\eta=\frac{\bar{\omega}}{\bar{\omega}_T}-1;\ \eta_T=0;\ \xi=\frac{\overline{(\Delta\omega)^2}}{\overline{(\Delta\omega)^2_T}}-1;\ \bar{\xi}_T=0;\ \Delta\omega=\omega-\bar{\omega}_T;\ \Delta\psi=\psi-{}^1/_2.$ Index "T" denotes the theoretical value. The quantities without a "T" index are from numerical calculations.

The change in frequency (and energy) of the oscillator in this case can be described naturally by the distribution function $f(\omega, t)$ which for $\varepsilon \ll 1$ obeys the Fokker-Planck-Kolmogorov kinetic equation:

$$\frac{\partial f}{\partial t}=\frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\omega^2}(\omega^2 f). \tag{4}$$

For inital and boundary conditions

$$\left.\begin{aligned} &f(\omega, 0)=\delta(\omega-\omega_0),\\ &f(\infty, t)=f(0, t)=0 \end{aligned}\right\} \tag{5}$$

the solution of (4) is in the form of

$$f(\omega, t)=(4\pi\tau)^{-1/2}\left(\frac{\omega_0}{\omega}\right)^2\exp\times$$
$$\times\left\{-\frac{\left(\ln\frac{\omega}{\omega_0}-\tau\right)^2}{4\tau}\right\}. \tag{6}$$

A quantitative estimate of the stochastic character of the motion was carried out by comparison of the mean value of the frequency $\bar{\omega}$ and the root mean square spread $\overline{(\Delta\omega)^2}=\overline{(\omega-\omega_0)^2}$ from the results of calculations utilizing the values from the distribution (6). The results of the comparison are given below. In addition, we showed the characteristics of the phase, ψ_n, distribution which should be uniform for a stochastic motion.

It is obvious from the results of the present work that the stability region corresponds to $\varepsilon\omega \sim 1$. In this manner, we confirm the general criterion for stochastic behavior obtained earlier [4, 5]. Apparently, the first criterion of such a type of instability was already established in 1953 [6] by numerical calculations; however, as far as we know, the study of the observed instability was not carried out and its connection with the stochastic property was not taken into consideration. In our notation, the instability criterion [6] has the form $\varepsilon\omega > \pi$. Note that in the general case there is no sharp limit on the $\varepsilon\omega$ parameter. For $\varepsilon\omega \sim 1$ depending on the initial conditions, one can generate motions of various types including also the purely resonant one [5].

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