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RADIATIVE EFFECTS IN ELECTRON-ELECTRON COLLISIONS

V. M. GALITSKY and S. A. KHEIFETS

Institute of Physics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR

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Abstract: An experiment in electron-electron scattering in the c.m. system is considered. Scattering angle and radiative energy distributions are obtained.

1. Formulation of the Problem

The infrared divergence in quantum electrodynamics is known to lead to the impossibility of purely elastic processes. Consider, e.g., the scattering in the c.m. system of two electrons with initial energies and momenta $p_{1\mu}(E, \boldsymbol{p})$ and $p_{2\mu}(E, -\boldsymbol{p})$ and final energies and momenta $q_{1\mu}(q_{10}, \boldsymbol{q}_1)$ and $q_{2\mu}(q_{20}, \boldsymbol{q}_2)$. Let us introduce for the sake of convenience the vector Q_μ with the components $\boldsymbol{Q} = \boldsymbol{q}_1 + \boldsymbol{q}_2$ and $Q_0 = q_{10} + q_{20}$. The probability for this process is usually represented as an elastic scattering probability

$$dW = (2\pi)^4 W_0 d\Omega_0 dQ_0 d\boldsymbol{Q} \tfrac{1}{2} E^2 \delta(\boldsymbol{Q}) \delta(2E - Q_0) \tag{1}$$

plus the probability of inelastic processes with the radiation of one or more quanta. Consideration of the radiation corrections cancels the first term of this sum (1). This means that in quantum electrodynamics the distribution in the angle $(\pi - \theta)$ between final electrons cannot in principle have the shape of a δ function but must be replaced by a certain sufficiently narrow distribution which transforms into a δ-function only when $\alpha \to 0$. The same applies to the energy distribution in the quantity $2E - Q_0$. It can be expected that the deviation from δ-shaped distributions for small angles θ and small energy differences $2E - Q_0$ is caused by the radiation of a large number of soft quanta, whereas the large angle and energy distribution regions can be determined by the radiation of single hard quanta. Though these effects can be expected to take place in all electrodynamic processes we shall discuss for the sake of definiteness the above experiment in electron scattering through scattering angles $\theta_{\text{scat}} \gg m/E$ (experiments with colliding beams). The aim of the investigation is the determination of the initial sections of the distributions since these determine the choice of the registering equipment aperture in the planned experiments. It is shown below, however, that the formulae obtained describe with fairly good accuracy the behaviour of the distribution functions in the large angle and energy regions as well.

3. General Formulae

Our aim is to determine the probability for an electron-electron collision which causes the radiation of an arbitrary number of soft quanta. The total probability, i.e. the elastic scattering probability plus the probability for scattering with the radiation of an arbitrary number of quanta has the form

$$\mathrm{d}W = W_0\delta(Q)\delta(2E-Q_0)\mathrm{d}q_1\mathrm{d}q_2+\mathrm{d}q_1\mathrm{d}q_2\int W_1\delta(Q+k_1)\delta(2E-Q_0-\omega)\mathrm{d}k_1$$
$$+\ldots+\mathrm{d}q_1\mathrm{d}q_2\int W_n\delta(Q+k_1+\ldots+k_n)\delta(2E-Q_0-\omega_1-\ldots-\omega_n)\mathrm{d}k_1\ldots\mathrm{d}k_n$$
$$+\ldots, \tag{2}$$

where W_n is the probability of scattering with emission of n quanta. Let us introduce a parameter Δ denoting the energy limit of classically radiated quanta. Then for quanta with energies $\omega < \Delta$ we have

$$W_n = W_0\frac{1}{n!}(W_k)^n, \tag{3}$$

where the classical quantum radiation probability W_k is

$$W_k = \frac{\alpha}{4\pi^2}\frac{1}{\omega}\left(\frac{p_{1\mu}}{p_1k}+\frac{p_{2\mu}}{p_2k}-\frac{q_{1\mu}}{q_1k}-\frac{q_{2\mu}}{q_2k}\right)^2. \tag{4}$$

The probability for the radiation of many quanta with energies $\omega > \Delta$ is small. Using the representation of δ functions through a Fourier integral in the parameters ξ, η we can rewrite eq. (2) in the form

$$\mathrm{d}W = W_0\mathrm{d}q_1\mathrm{d}q_2\int\mathrm{d}\xi\mathrm{d}\eta\, e^{iQ\cdot\xi+i(2E-Q_0)\eta}\exp\left\{\int_0^\Delta W_k e^{ik\cdot\xi-i\omega\eta}\mathrm{d}k\right\}$$
$$+\mathrm{d}q_1\mathrm{d}q_2\int_\Delta^E\mathrm{d}k_1 W_1(q_1q_2k_1)e^{i(Q+k_1)\cdot\xi+i(2E-Q_0-\omega_1)\eta}\exp\left\{\int_0^\Delta W_k e^{ik\cdot\xi-i\omega\eta}\mathrm{d}k\right\}\mathrm{d}\xi\mathrm{d}\eta$$
$$+\ldots, \tag{5}$$

where the terms corresponding to the radiation of two and more non-classical quanta have not been written out. With the same accuracy, representing $\exp\{\int_0^\Delta W_k\}$ in the form $\exp\{\int_0^E W_k-\int_\Delta^E W_k\}$ and expanding in powers of $\{\int_\Delta^E W_k\}$ we can write eq. (5) as

$$\mathrm{d}W = W_0\mathrm{d}q_1\mathrm{d}q_2\int\mathrm{d}\xi\mathrm{d}\eta\, e^{iQ\cdot\xi+i(2E-Q_0)\eta}\exp\left\{\int_0^E W_k e^{ik\cdot\xi-i\omega\eta}\mathrm{d}k\right\}$$
$$+\mathrm{d}q_1\mathrm{d}q_2\int_0^E\mathrm{d}k_1[W_1-W_0W_{k_1}]e^{i(Q+k_1)\cdot\xi+i(2E-Q_0-\omega_1)\eta}$$
$$\times\left\{\int_0^E W_k e^{ik\cdot\xi-i\omega\eta}\mathrm{d}k\right\}\mathrm{d}\xi\mathrm{d}\eta+\ldots. \tag{6}$$

In the integral over k_1 the lower limit can be assumed to be zero since the expression in square brackets vanishes when $\omega_1 < \Delta$ (see eq. (3)). The second term (and the other terms which have not been written out) correspond to the radiation of non-classical quanta. The distribution for small angles between scattered electrons must be determined by the irradiation of soft quanta, i.e. the first term in eq. (6). If the radiation corrections from the virtual quanta are taken into account the index of the exponent can be replaced by the expression

$$\int_0^E W_k(e^{ik\cdot\xi-i\omega\eta}-1)\mathrm{d}k. \tag{7}$$

Let us also pass from $\mathrm{d}q_1\mathrm{d}q_2$ to $\mathrm{d}n_0\mathrm{d}Q_0\mathrm{d}Q$ directing the unit vector n_0 along the momentum of one of the scattered electrons q_1:

$$\mathrm{d}q_1\mathrm{d}q_2 = \mathrm{d}n_0\mathrm{d}Q_0\mathrm{d}Q\frac{(Q_0^2-Q^2)^2(Q_\perp^2+(Q_0-Q_n)^2)}{8(Q_0-Q_n)^4}, \tag{8}$$
$$Q_n = Qn_0,\quad [Q_\perp n_0]=0,\quad 0<Q_0<\infty,\quad Q^2\leqq Q_0^2.$$

The distribution in Q_0 gives an energy spread, i.e. the difference of $q_{10}+q_{20}$ from $2E$; the distribution in $\mathrm{d}Q_\perp$ is an angular distribution. Since the expected distribution must be close to the distributions in the form of δ functions, i.e. must be sufficiently narrow, we can put in the Jacobian of the transition approximately $Q_0 \approx 2E$; $Q = 0$ so that

$$\mathrm{d}q_1\mathrm{d}q_2 = \tfrac{1}{2}\mathrm{d}n_0\mathrm{d}Q_0\mathrm{d}QE^2. \tag{9}$$

Comparison of eqs. (6) and (9), taking into account eq. (7), with elastic scattering probability (1) shows that the distribution function has the form

$$\mathrm{d}f = \mathrm{d}Q_0\mathrm{d}Q_\perp\frac{1}{(2\pi)^3}\int\mathrm{d}\xi_\perp\mathrm{d}\eta\, e^{iQ_\perp\cdot\xi_\perp+i(2E-Q_0)\eta}$$
$$\times\exp\left\{\int_0^E\mathrm{d}k\,W_k(e^{ik\cdot\xi-i\omega\eta}-1)\right\}. \tag{10}$$

This distribution when $\alpha\to 0$ ($W_k\to 0$) has in fact the form of a product of δ functions. The contribution from non-classical quanta (the second term in eq. (6)) is estimated below.

3. Energy Distribution

The energy distribution can be obtained from eq. (10) by integration over $\mathrm{d}Q_\perp$, $Q^2 < Q_0^2$. Then we obtain a non-vanishing function only in the region $\xi \lesssim 1/Q_0$ as an integrand depending on ξ. This makes it possible to omit $e^{ik\cdot\xi}$ in the exponent index in eq. (10) since essential η are of the order $[2E-Q_0]^{-1} \gg Q_0^{-1}$. Thus the energy distribution is given by the integral

$$df_E = \frac{dQ_0}{2\pi}\int_{-\infty}^{\infty} d\eta\, e^{i(2E-Q_0)\eta} \exp\left\{\int_0^E dk(e^{-i\omega\eta}-1)W_k\right\}.$$

When calculating the integral in the exponent we can assume that $\boldsymbol{q}_1+\boldsymbol{q}_2 = 0$. For the sake of simplicity we consider scattering through $\frac{1}{2}\pi$ so that $\boldsymbol{p}\cdot\boldsymbol{q} = 0$. If we keep only the terms giving the products of logarithms, the terms in eq. (4) proportional to m^2 can be omitted and the crossing terms (pq) can be written as

$$[\omega^2E^2-(\boldsymbol{p}\cdot\boldsymbol{k})^2]^{-1}[\omega^2E^2-(\boldsymbol{q}\cdot\boldsymbol{k})^2]^{-1} = \frac{1}{\omega^2E^2}\left[\frac{1}{E^2\omega^2-(\boldsymbol{p}\cdot\boldsymbol{k})^2}+\frac{1}{E^2\omega^2-(\boldsymbol{q}\cdot\boldsymbol{k})^2}\right]. \tag{11}$$

Then we have

$$\int dk\, W_k(e^{-i\omega\eta}-1) = -\frac{4\alpha L}{\pi}(\ln 2E|\eta| - \tfrac{1}{2}i\pi \operatorname{sign}\eta), \tag{12}$$

$$L \equiv \ln\frac{4E^2}{m^2}.$$

The integral sine and cosine $\mathrm{Ci}(E\eta)$, $\mathrm{Si}(E\eta)$ originating in the integration can be omitted since the argument $E\eta \gg 1$ for characteristic $\eta \approx (2E-Q_0)^{-1}$. Finally, we obtain the normalized energy distribution function in the form

$$df_E = \frac{4\alpha L}{\pi}\left(\frac{E-Q_0}{E}\right)^{-1+4\alpha L/\pi}\Gamma\left(1-\frac{4\alpha L}{\pi}\right)\frac{dQ_0}{E}, \tag{13}$$

$$0 < Q_0 < E.$$

4. Angular Distribution

The angular distribution is obtained from eq. (10) by integration over Q_0. Let us first find the distribution in the scattering plane, for which purpose the expression obtained has to be integrated over $dQ_2(Q_2 \perp Q_p;\ Q_2 \perp \boldsymbol{n}_0;\ Q_p$ is the projection of $Q_\perp$ on the scattering plane). Simplifying the expression obtained we can, just as in the case of the energy distribution, assume that $\eta = \xi_r = 0$ and use eq. (11) for W_k. Whereupon we have the following integral in the exponent index:

$$\mathscr{I} = \frac{\alpha}{\pi^2}\int\frac{dk}{\omega}(e^{ik\xi_p}-1)\left(\frac{1}{\omega^2-(\boldsymbol{v}_p k)^2}+\frac{1}{\omega^2-(\boldsymbol{v}_q k)^2}\right), \tag{14}$$

$$\xi_p p = \xi, \qquad \xi_p v_q = 0.$$

Eq. (14) easily reduces to the integral

$$\mathscr{I} = -\frac{4\alpha}{\pi}\int_0^1 dx[\ln E\xi x - \mathrm{Ci}\, E\xi x]\left(\frac{1}{1-v^2x^2}+\frac{1}{\sqrt{(m/E)^2+x^2}}\right).$$

The estimate of the latter when $E\xi \gg 1$ leads to the expression

$$\mathscr{I} = -\frac{2\alpha}{\pi}(L\ln E\xi + \ln^2 E\xi). \tag{15}$$

Using eq. (15) we obtain the normalized distribution function

$$df_p = \frac{dQ_p}{|Q_p|}\frac{\alpha}{\pi}\left(L+\ln\left|\frac{E}{Q_p}\right|\right)\exp\left\{-\frac{2\alpha L}{\pi}\ln\left|\frac{E}{Q_p}\right| - \frac{\alpha}{\pi}\ln^2\left|\frac{E}{Q_p}\right|\right\}, \tag{16}$$

$$-E < Q_p < E; \quad |Q_p| > m.$$

The distribution (17) can be rewritten as a function of the angle θ between scattered electrons since at small θ we have $|Q_p| = E\theta$. When $\theta \gg m/E$ we obtain

$$df_p = \frac{2\alpha L}{\pi}\frac{d\theta}{\theta^{1-2\alpha L/\pi}}. \tag{17}$$

The distribution (17) is normalized across the interval $0 < \theta < 1$. It is noteworthy that the terms proportional to a large logarithm L have originated in eq. (16) from the term $[\omega^2-(\boldsymbol{v}_p\cdot\boldsymbol{k})^2]^{-1}$ in eq. (14), i.e., they are due to the radiation of initial electrons. This can be expected since the radiation is directed in the main along the motion of the electrons and hence a large momentum transfer with respect to the motion of scattered electrons can only be carried by the photons radiated by the initial electrons. For the same reason the angular distribution in the plane perpendicular to the scattering plane must be very narrow since the momentum carried in this direction by the quanta radiated by the initial and final electrons is very small and is by order of magnitude $k_\perp \approx \omega\theta_k \approx \omega m/E$. To obtain the explicit form of this distribution let us integrate eq. (10) over $dQ_0 dQ_p$ and simplify the expression as before. The integral originating in the exponent index has the form (14) but instead of ξ_p we have ξ_r; $\xi_r\cdot \boldsymbol{v}_p = 0$; $\xi_r\cdot\boldsymbol{v}_q = 0$. Its estimate leads to the expressions

$$\mathscr{I} = \frac{4\alpha}{\pi}\ln^2 E\xi \qquad \text{when } 1 \ll E\xi \ll E/m,$$

$$\mathscr{I} = -\frac{4\alpha}{\pi}\left(\ln\frac{E^2}{m^2}\ln E\xi - \ln^2\frac{E}{m}\right) \qquad \text{when } E\xi \gg E/m.$$

This gives the following angular distribution functions in the plane normal to the scattering plane:

$$df_2 = \frac{dQ_2}{|Q_2|}\frac{4\alpha}{\pi}\ln\frac{E}{m}\exp\left\{-\frac{4\alpha}{\pi}\left[2\ln(E/m)\ln\left|\frac{E}{Q_2}\right| - \ln^2\frac{E}{m}\right]\right\}, \qquad |Q_2| < m$$

$$df_2 = \frac{dQ_2}{|Q_2|}\frac{4\alpha}{\pi}\ln\left|\frac{E}{Q_2}\right|\exp\left\{-\frac{4\alpha}{\pi}\ln^2\left|\frac{E}{Q_2}\right|\right\}, \qquad |Q_2| > m. \tag{18}$$

For small angles we have $|Q_2| = E\theta$. The distribution (18) proves normalized to unity. It can readily be determined from this distribution that 77% of the electrons are scattered through an angle corresponding to Q_2 smaller than m and 90% are scattered through an angle smaller than 5 m/E or 1.5° (the values are given for an initial electron energy of 100 MeV). In the scattering plane the distribution is much more elongated. Let us introduce a quantity q denoting the fraction of electrons scattered through an angle than θ_q. From eq. (17) we easily find that

$$\theta_q = (q)^{\frac{1}{4}\pi\alpha L}. \tag{19}$$

Using this formula we can directly find that for 100 MeV electron energy 50% of the distribution ($q = 0.5$) lies within 1″ and 90% within 10°. For 500 MeV energy the relevant figures are 10″ and 13.5°. The dependence of $df_p/d\theta$ on the angle between scattered electrons compiled according to eq. (17) is given in table 1. The initial electron energy is assumed to be 100 MeV.

TABLE 1

Angular distribution of electrons in the scattering plane

θ	20′	40′	1°	5°	10°	15°	20°
$df/d\theta$	4.5	2.4	1.56	0.30	0.15	0.10	0.07

Eqs. (13), (17) and (18) for the energy and angular distributions have been obtained for the scattering of electrons through $\frac{1}{2}\pi$. If the scattering angle $\theta_{scat} \neq \frac{1}{2}\pi$ they conserve their structure, but $L = \ln(4E^2/m^2)$ is replaced by

$$L = \ln\frac{4E^2\sin^2\theta_{scat}}{m^2}.$$

This statement remains valid up to scattering angles $\theta_{scat} \gg m/E$.

5. Contribution from Hard Quanta

Let us now see what contribution the non-classical (hard) quanta can yield. Since the terms having the form $d\omega/\omega$ in eq. (6) cancel out the contribution of hard quanta may lead to additions to the energy distribution function $(\lambda\alpha/\pi)dQ_0$, where λ is a constant. Comparison with the distribution function (13) due to the radiation of soft quanta shows that the latter is always decisive at small $2E-Q_0$. The correction to eq. (13) from hard quanta depends on the constant λ. To determine it we can use the precise formula derived by Garibyan [1]) for the electron-electron scattering cross section with one quantum radiation. The estimates thus obtained show that the above distribution (13) must be valid accurately to within 30% up to $2E-Q_0 \approx \frac{1}{2}E$. The same applied to the angular distribution (17). We give below by way of illustration the data for the energy distribution in the quantity $k = 2E-Q_0$ obtained by eq. (13) and with the aid of the numerical integration of the Garibyan formula (scattering through $\frac{1}{2}\pi$ and the initial electron energy 100 MeV). The quantity $\ln[137(df/d(Q_0/E))]$ is calculated.

TABLE 2

Comparison of energy distributions

	k/E	$5\cdot10^{-3}$	10^{-2}	0.1	0.15	0.2	0.3	0.4	0.5
$\ln\left(\frac{1}{\alpha}\frac{df}{d(Q_0/E)}\right)$	eq. (13)	7.35	6.73	4.68	4.33	4.07	3.73	3.46	3.26
	numerical calculation	7.38	6.68	4.39	4.00	3.73	3.3	3.14	3.12

The functions in table 2 differ on the average by a factor of 1.35, which roughly corresponds to the expected accuracy. It is clear from table 2 that eq. (13) determines sufficiently well the distribution function for large values of the argument as well.

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