ASYMPTOTIC METHODS IN THE HYDRODYNAMIC THEORY OF STABILITY*

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I. INTRODUCTION

The use of asymptotic methods in the linear hydrodynamic theory of stability is well known, e.g. in connection with the problem of stability of Poiseuille flow (for a more detailed account seen reference [1]). The main point is that it is necessary to reach solutions and find the eigenvalues $\omega = \omega(k)$ for the given boundary conditions of the following equation:

$$\alpha \frac{d^4\varphi}{d\xi^4} - \overline{U}_2(\xi, k, \omega) \frac{d^2\varphi}{d\xi^2} + \overline{U}_1(\xi, k, \omega)\varphi = 0, \tag{1}$$

where α is a small parameter, ξ is a co-ordinate (in the case of Poiseuille flow α is proportional to the viscosity). The presence of the small parameter α makes it possible to construct a formal asymptotic series which will give a solution for a correctly chosen power of α.

Recently, a large number of studies have appeared on the subject of the stability of a slightly non-uniform plasma. In those cases where a detailed analysis was made, the problem reduced to the following equation:

$$\frac{d^2\varphi}{d\xi^2} - \overline{U}(\xi, k, \omega)\varphi = 0. \tag{2}$$

In order that (2) may include an explicitly small parameter β, characterizing a slight non-uniformity, we introduce a non-dimensional co-ordinate $x = \xi/L$ (L being the characteristic dimension of the problem). We assume $\overline{U} = k_0^2 U$, where U is always approximately unity, except near point x_0, where $U(x_0) = 0$. We then have instead of Eq. (2):

$$\beta \frac{d^2\varphi}{dx^2} - U(x, k, \omega)\varphi = 0, \tag{3}$$

$$\beta = \frac{1}{k_0^2 L^2} \ll 1. \tag{4}$$

In [2] it was proposed that a small β should be used in finding the asymptotic solutions, which are well known in quantum mechanics under the name

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* Work performed by ZASLAVSKY, G. M., MOISEEV, S. S. and SAGDEEV, R. Z.

"quasi-classical". (For a detailed survey of work in this direction, see [10] and [11].)

In a number of cases the following situation arises: in the region considered there exists a point at which U becomes ∞. This fact has been studied in connection with the problem of wave transformation in a plasma [3]. In the cases examined in Ref. [3] the pole U was imaginary and vanished when account was taken of the higher derivative with the small α-type parameter*. In problems on wave transformation in a plasma, we used the method of successive approximations [4].

An asymptotic method similar to that used in both Refs. [4] and [1] was employed in Ref. [5] for an equation of the type (1) in a study of the stability of a non-uniform plasma, with account being taken of finite conductivity. It will be clear from what follows that this method is of very limited applicability.

The present study aims to show that there is a simple asymptotic approach to the analysis of the equation:

$$\alpha\beta^2 \varphi^{IV} - \beta U_2(x,k,\omega)\varphi'' + U_1(x,k,\omega)\varphi = 0. \tag{5}$$

Equation (5) models the above - mentioned problems for the conditions of a slightly non-uniform medium.

II. STATEMENT OF THE PROBLEM

The physical considerations discussed in the introduction make necessary an analysis of the following equation:

$$\alpha\beta^2 \frac{d^4\varphi}{dx^4} - \beta U_2(x,k,\omega)\frac{d^2\varphi}{dx^2} + U_1(x,k,\omega)\varphi = 0, \tag{6}$$

where x is a non-dimensional co-ordinate; k and ω are the parameters of the problem; α and β are small parameters:

$$\alpha, \beta \ll 1. \tag{7}$$

Usually, in the physical statement of the problem, the parameter α is involved in calculation of a slight dissipative process, and β is a "quasi-classical" parameter, equal to the ratio between the characteristic length of change of φ and the characteristic length of change of U_1, U_2. In (6) U_1 and U_2 are non-dimensional parameters and

$$U_1,\ U_2 \simeq 1, \tag{8}$$

except for the points where they become zero.

* These remarks are a rather rough representation of the situation studied in Ref. [3].

Solutions tending to zero at ± ∞ will be referred to below as finite or local, otherwise they will be termed non-local.

For $\beta = 1$ in (6), an analysis of the equation has been made in studies by C.C. Lin [1] and Wasow [6] in connection has been made in studies stability of Poiseuille flow. For $\alpha = 0$ the equation becomes a second-order equation, which has been the subject of detailed study in numerous works, especially in connection with the quasi-classical approximation in quantum mechanics (e.g. see reference [7]).

We are looking for a solution to Eq. (6) in the following form:

$$\varphi = C \exp\left\{\frac{1}{\sqrt{\beta}}\int^{x} q(x)\,dx\right\}, \tag{9}$$

$$q(x) = q^{(0)}(x) + \sqrt{\beta}\, q^{(1)}(x) + \dots . \tag{10}$$

Substituting (9) and (10) in (6), and taking into account (7) and (8), we get:

$$q^{(0)4} - \frac{U_2}{\alpha} q^{(0)2} + \frac{U_1}{\alpha} = 0, \tag{11}$$

$$4 q^{(1)} q^{(0)3} + 6 q^{(0)2}\frac{dq^{(0)}}{dx} - \frac{U_2}{\alpha}\left(\frac{dq^{(0)}}{dx} + 2 q^{(1)} q^{(0)}\right) = 0. \tag{12}$$

From equation (11) we find $q^{(0)}$:

$$q^{(0)} = \pm\left[\frac{U_2}{2\alpha} \pm \left(\frac{U_2^2}{4\alpha^2} - \frac{U_1}{\alpha}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

or, taking into account small α, we get the following two pairs of values:

$$q_i^{(0)} = \pm\left(\frac{U_1}{U_2}\right)^{\frac{1}{2}} \qquad (i = 1, 2),$$
$$q_i^{(0)} = \pm\left(\frac{U_2}{\alpha}\right)^{\frac{1}{2}} \qquad (i = 3, 4). \tag{13}$$

Similarly, from (12) we get:

$$q_i^{(1)} = -\frac{1}{2}\frac{dq_i^{(0)}}{dx}\frac{1}{q_i^{(0)}} \qquad (i = 1, 2),$$
$$q_i^{(1)} = -\frac{5}{2}\frac{dq_i^{(0)}}{dx}\frac{1}{q_i^{(0)}} \qquad (i = 3, 4). \tag{14}$$

Formulae (13) and (14) allow us to write the solution (9) in the following form (to within the following terms in the expansion (10)):

$$\varphi_i = \frac{C}{\sqrt{p_i}} \exp \int^x p_i \, dx \qquad (i = 1, 2), \tag{15}$$

$$\varphi_i = \frac{C}{\sqrt{p_i^5}} \exp \int^x p_i \, dx \qquad (i = 3, 4), \tag{16}$$

where $p_i = q_i^{(0)}/\sqrt{\beta}$.

The solution obtained in (15) is asymptotic and its accuracy is limited to the region of applicability of the expansion (10), which region we refer to below as external. Clearly, the solution (15), (16) is not applicable to regions near the points where U_1 and U_2 become zero, referred to below as internal regions. The solution for the internal regions can be sought separately. Accordingly, the solution of Eq. (6) for the given boundary conditions reduces to the following three procedures: (1) finding solutions in the internal and external regions; (2) showing what each of the solutions for a particular region becomes in some other region (this question arises owing to the presence of Stokes lines when an asymptotic expression is used); and (3) satisfying the boundary conditions (in addition to everything else, this also gives an equation for the eigenvalues of the problem).

It should be noted that the point where $U_2 = 0$ has no special importance, since in the vicinity of this point the role of the term with φ^{IV} in (6) is unimportant and the behaviour of the solution in the vicinity of this point is determined by the theory developed for Eq. (6) with $\alpha = 0$.

In what follows, without limiting the generality of the method developed below, and for the sake of convenience, we shall select the specific form $U_1(x)$ and $U_2(x)$ (Fig. 1). In Fig. 2 regions I, II are external, and region III is internal.

The above considerations conclude the statement of the problem whose solution will be worked out in sections III-V.

III. WEAK CASE

For the selected form of $U_1(x)$ and $U_2(x)$ (Fig. 1) the values U_1, U_2 become zero at the points A, B and 0_1, 0_2*, respectively. We will assume that the distance between B and 0_2 is greater than unity. In the vicinity of the point 0_2 we may represent

$$U_2 = Ux, \quad x < 1, \quad U \simeq 1, \tag{17}$$

and regard U_1 as of constant value. For purposes of visualization, the regions in which various approximations are applicable are shown in Fig. 3. The expansion (17) holds good in section $(0_2, 1)$; solutions (15), (16) hold good for sections 1 and 3 respectively.

* The statements made will be symmetrical relative to the substitution $(0_1, A) \rightleftarrows (0_2, B)$; we shall refer only to points $(0_2, B)$.

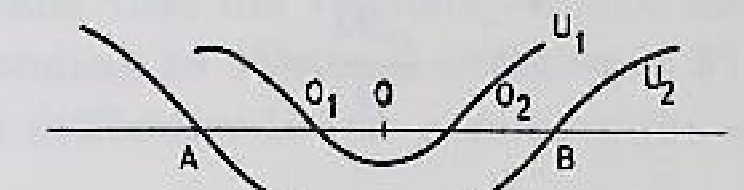

Fig. 1

Specific forms for $U_1(x)$ and $U_2(x)$.

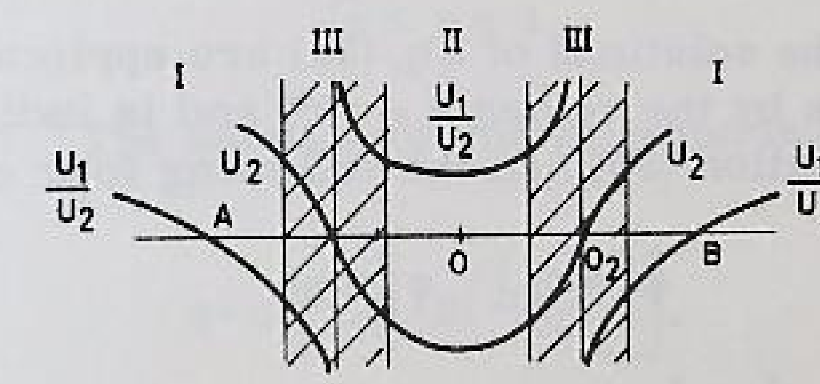

Fig. 2

Case for which regions I and II are external and region III is internal.

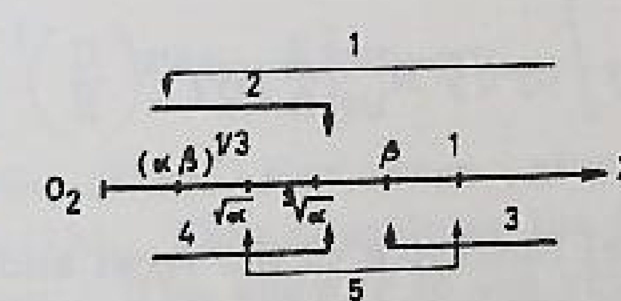

Fig. 3

Possibility of coupling due to the fact that sections 1 and 2 have a common part.

For $x < 1$ equation (6) takes the form:

$$\alpha\beta^2\varphi^{IV} - \beta Ux\varphi'' + U_1\varphi = 0. \tag{18}$$

As in [1] we make the substitution

$$x = \alpha^{1/3}y \tag{19}$$

and consider the solution of the equation obtained:

$$\beta\frac{d^4\varphi}{dy^4} - Uy\frac{d^2\varphi}{dy^2} + \frac{\alpha^{1/3}}{\beta}U_1\varphi = 0, \tag{20}$$

in the vicinity of $y \simeq 1$.

In this section of the paper we shall limit ourselves to the case

$$\frac{\alpha^{1/3}}{\beta} < 1, \tag{21}$$

where a solution can be found similar to that of Ref. [1] in the form of an asymptotic series:

$$\varphi = \varphi^{(0)} + \frac{\alpha^{1/3}}{\beta}\varphi^{(1)} + \ldots . \tag{22}$$

Substituting (22) in (20) we get:

$$\beta \frac{d^4\varphi^{(0)}}{dy^4} - Uy\frac{d^2\varphi^{(0)}}{dy^2} = 0. \tag{23}$$

The region in which the solutions of Eq. (23) are applicable is determined on the right-hand side by the values $x \simeq \alpha^{1/3}$ and is indicated in Fig. 3 by section 2 (or 4). Equation (23) has the following four solutions [1]:

$$\varphi_1 = 1; \qquad \varphi_2 = x; \tag{23a}$$

$$\varphi_3 = \int^y dy \int^y dy\sqrt{y}\, H^{(1)}_{1/3}\left[\frac{2}{3}(iy)^{3/2}\left(\frac{U}{\beta}\right)^{\frac{1}{2}}\right];$$

$$\varphi_4 = \int^y dy \int^y dy\sqrt{y}\, H^{(2)}_{1/3}\left[\frac{2}{3}(iy)^{3/2}\left(\frac{U}{\beta}\right)^{\frac{1}{2}}\right], \tag{24}$$

where $H^{(1)}$, $H^{(2)}$ are Hankel functions of the first and second type, respectively. Considering that the argument of the Hankel functions in (24) is large, we can write for the solutions of φ_3 and φ_4, which become zero at $+\infty$,

$$\varphi \simeq x^{-5/4}\exp\left[-\frac{2}{3}\left(\frac{U}{\alpha\beta}\right)^{\frac{1}{2}} x^{3/2}\right] \qquad (x > 0)$$

$$\varphi \simeq |x|^{-5/4}\sin\left[\frac{2}{3}\left(\frac{U}{\alpha\beta}\right)^{\frac{1}{2}} |x|^{3/2} + \frac{\pi}{4}\right] \qquad (x < 0). \tag{25}$$

If φ may not become zero for $+\infty$, then for $x > 0$ the solution also consists of a growing exponential and the solution for $x < 0$ is determined by the normal rules, which take into account that $x = 0$ is a turning point [7]. The solutions of (25) become the solutions determined by Eq. (16) and they can therefore be coupled. This possibility of coupling is due to the fact that sections 1 and 2 in Fig. 3 have a common part within which coupling in fact takes place. The picture is entirely different in the case of the solutions of (23a) and (15), which do not coincide with one another and cannot therefore be directly coupled. This is due to the following circumstance. The pair of solutions in (23a) have, in principle, no quasi-classical form and for them we have

$$k_y^2 \equiv \frac{1}{\varphi}\frac{d^2\varphi}{dy^2} = 0. \tag{26}$$

The equality (26) is in fact determined to an accuracy of:

$$\frac{\alpha^{1/3}}{\beta} \ll 1. \tag{27}$$

The inequality (27) means that the regions, where the solutions of (15) and (23a) are fit (corresponding to sections 3 and 4 in Fig. 3), do not overlap.

To overcome this difficulty let us consider the equation:

$$\beta Ux\varphi'' - U_1\varphi = 0, \tag{28}$$

which is true as a zero approximation in the region

$$\sqrt{\alpha} < x < 1, \tag{29}$$

(section 5 in Fig. 3). The solution of this equation is:

$$\varphi = \sqrt{x}\, Z_1\left[-2\left(\frac{U_1 x}{U\beta}\right)^{\frac{1}{2}}\right], \tag{30}$$

where Z_1 is one of two linearly independent cylindrical functions (for example, I_1 and N_1). For small values of the argument in (30) we have:

$$\varphi_1 = 1; \ \varphi_2 = x, \tag{31}$$

i.e. Eq. (30) becomes Eq. (23a). For large values of the asymptotic argument Z_1 coincides with (15). This completes the first procedure mentioned in section II of this paper. The answer to the second part of the problem, also mentioned in that section, is given by the theorems of Wasow [6] which retain their force in the present case.

Equations for eigenvalues in the case of local solutions can be written down immediately, proceeding as in the quasi-classical approximation for a second-order equation [7] in deriving the "rules of quantization":

$$\int_{0_1}^{0_2}\left(\frac{U_2}{\alpha\beta}\right)^{\frac{1}{2}} dx = (n + \tfrac{1}{2})\pi, \tag{32}$$

$$\int_{0_2}^{A}\left(\frac{U_1}{\beta U_2}\right)^{\frac{1}{2}} dx = (n + \tfrac{1}{2})\pi. \tag{33}$$

The expressions (32) and (33) give two independent solutions for the eigenvalues. This corresponds to the fact that for $+\infty$ (or $-\infty$) we have two linearly independent solutions, determined by (15) or (16), which are then separately "extended" at $-\infty$ (or $+\infty$) (a connection only arises for α/β^2).

IV. CLASSIFICATION

In Eq. (18), which is true for $x < 1$, we make the substitution

$$x = \beta y, \tag{34}$$

which gives

$$\frac{\alpha}{\beta^2}\frac{d^4\varphi}{dy^4} - Uy\frac{d^2\varphi}{dy^2} + U_1\varphi = 0. \tag{35}$$

The solution of this equation is obtained by the method of Laplace:

$$\varphi(y) = \int \frac{1}{t^2}\exp\left(yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right)dt, \tag{36}$$

where the integration is performed in the plane of the complex variable t along a contour at the ends of which the function:

$$\exp\left(yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right)$$

becomes zero. The solution (36), in accordance with (34) and (17), is true in the region $y < 1/\beta \gg 1$. We will limit our discussion to the following region:

$$1 < y < \frac{1}{\beta} \gg 1, \tag{37}$$

or

$$\beta < x < 1. \tag{37a}$$

Since $y > 1$ in the range considered, the "saddle-point" method can be made use of in determining the integral in (36). We have the following four "saddle points":

$$t_0 \equiv \overline{q}_i(y) = \pm\left(\frac{\beta^2}{2\alpha}U\right)^{\frac{1}{2}}\sqrt{y \pm \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}}. \tag{38}$$

This determines four contours, integration along which gives four linearly independent solutions. By appropriately selecting the contours we get the following solutions:

$$\varphi_i(y) \simeq \left[\frac{\pi}{y\left(\frac{U_1}{U}\frac{1}{\overline{q}_i^3} - \frac{\alpha}{\beta^2}\frac{\overline{q}_i}{U}\right)}\right]^{\frac{1}{2}}\frac{1}{\overline{q}_i^2}\exp\int^y \overline{q}_i(y)\,dy \qquad (i = 1, 2, 3, 4). \tag{39}$$

Determining:

$$\overline{q}_i = \pm\left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\sqrt{y - \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}} \qquad (i = 1, 2),$$

$$\overline{q}_i = \pm\left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\sqrt{y + \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}} \qquad (i = 3, 4), \tag{40}$$

we get from (39):

$$\varphi_i(y) \simeq (\overline{q}_i)^{-1/2}\exp\int^y \overline{q}_i(y)\,dy \qquad (i = 1, 2),$$

$$\varphi_i(y) \simeq (\overline{q}_i)^{-5/2}\exp\int^y \overline{q}_i(y)\,dy \qquad (i = 3, 4). \tag{41}$$

For large values of y, it is not difficult to see that the solutions of (41) become the corresponding solutions in the internal region (15), (16).

Let us consider the value of y, for which the inner root of (40) becomes zero:

$$y_0 \equiv -ia = \pm\left(\frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}. \tag{42}$$

(For the type of functions considered $U_1(x)$ and $U_2(x)$, at the points y_0, the value $U_1 < 0$ and y_0 is purely imaginary.) The points y_0 will be referred to below as branching points. Taking (8) and (34) into account, we see that the value at the branching points is $x \simeq \sqrt{\alpha}$, and the distance between the branching points is approximately $\sqrt{\alpha}$. From (42) it immediately follows that:

$$a < 1 \qquad (\alpha/\beta^2 < 1), \tag{43}$$

$$a > 1 \qquad (\alpha/\beta^2 > 1). \tag{44}$$

In the case (43) the branching points do not fall within the region of (37), where solution (41) holds good, and they may be disregarded.

In case (44) the situation is different and, as we shall see from what follows, by taking the branching points into account, we make an essential change in the entire treatment and this may lead to a qualitatively different physical picture of the process. We shall call case (43) the weak case and (44) the strong case.

Since the solution of the problem stated in section III was true for $\alpha/\beta^2 < \beta < 1$, it is correct to consider it as the weak case.

It should be noted that if we introduce the concept of wavelength, $\lambda \simeq \varphi(d\varphi/dx)^{-1}$, then, in the strong case, many wavelengths can be "fitted in" between branching points, which is not so in weak case. Thus, classification is made in accordance with the number of wavelengths lying between the branching points (i. e. in accordance with the ratio between the parameters α and β), although the distance between the branching points is the same in both cases ($\simeq \sqrt{\alpha}$).

V. STRONG CASE

As has already been noted, the need to take account of the branching points a_1, a_2, b_1 and b_2 (Fig. 4) entirely changes the rules for the transition from, say, the region $x < 0_1$ to the region $x > 0_1$.

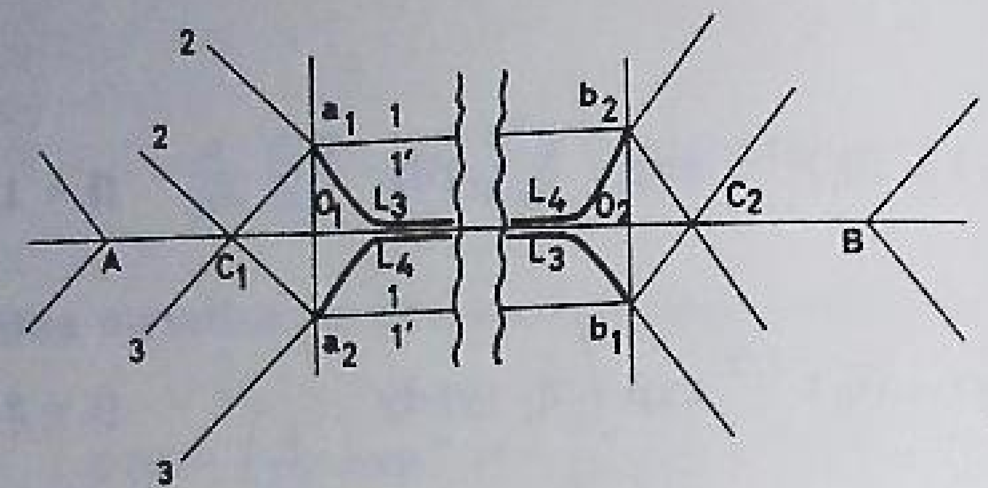

Fig. 4

Pattern for the level lines with respect to each of the branching points separately.

To obtain these rules for (41) we shall use the following form of the roots in (40) (in the vicinity of the point 0_1):

$$\pm\sqrt{y-(y^2+a^2)^{\frac{1}{2}}} = \pm\frac{1}{\sqrt{2}}(\sqrt{y+ia}-\sqrt{y-ia}); \qquad (y>0)$$
$$= \pm\frac{i}{\sqrt{2}}(\sqrt{|y|+ia}-\sqrt{|y|-ia}); \qquad (y<0), \tag{45}$$

$$\pm\sqrt{y+(y^2+a^2)^{\frac{1}{2}}} = \pm\frac{1}{\sqrt{2}}(\sqrt{y+ia}+\sqrt{y-ia}); \qquad (y>0)$$
$$= \pm\frac{i}{\sqrt{2}}(\sqrt{|y|+ia}+\sqrt{|y|-ia}); \qquad (y<0) \tag{46}$$

We then write down the solutions of (41) in the form:

$$\varphi_{1,2} = (\overline{q}_1)^{-1/2}\exp\left[\pm\int^y (w_1(y)-w_2(y))dy\right]; \qquad (y>0)$$
$$= (\overline{q}_1)^{-1/2}\exp\left[\pm i\int^y (w_1(|y|)-w_2(|y|))dy\right]; \qquad (y<0), \tag{47}$$

$$\varphi_{3,4} = (\overline{q}_3)^{-5/2}\exp\left[\pm\int^y (w_1(y)+w_2(y))\,dy\right]; \qquad (y>0)$$
$$= (\overline{q}_3)^{-5/2}\exp\left[\pm\int^y (w_1(|y|)+w_2(|y|))dy\right]; \qquad (y<0). \tag{48}$$

where

$$w_1 = \left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\times\sqrt{y-ia},$$
$$w_2 = \left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\times\sqrt{y+ia}. \tag{49}$$

In expressions (47) and (48) the exponential is factorized for both branching points, which then makes it possible to employ rules of the type in [8].

On the left-hand side of point A* we write down an arbitrary solution, which becomes zero at $-\infty$,

$$\varphi = |\overline{q}_1|^{-1/2}\exp\left[-i\int_A^y w_1(y)dy + i\int_A^y w_2(y)dy\right] + D\,|q_3|^{-5/2}\left[\exp\int^y w_1(y)dy + \int^y w_2(y)dy\right]. \tag{50}$$

On the right-hand side of point A, the second term of (50) does not change, but the first is transformed in accordance with the rules of (45). This gives:

$$\varphi(y) = |\overline{q}_1|^{-1/2}\left[\exp\left(-\frac{i\pi}{4}+i\int_A^y \overline{q}_1 dy\right) + \exp\left(\frac{i\pi}{4}-i\int_A^y \overline{q}_1\,dy\right)\right]$$
$$+ D\,|\overline{q}_3|^{-5/2}\left[\exp\int^y w_1(y)dy + \int^y w_2(y)\,dy\right]. \tag{51}$$

By taking into account formulae (47) to (49) we can establish a pattern for the level lines with respect to each of the branching points separately (Fig. 4). The level lines from two adjacent branching points intersect on the real axis at points C_1, C_2. The solution of (51) can then be written in the following form, for $A < y < C_1$:

$$\varphi(y) = \frac{1}{|\overline{q}_1|^{1/2}}e^{i\phi_1}\exp\left[-\int_y^{a_1} w_1(y)dy + \int_y^{a_2} w_2(y)\,dy\right]$$
$$+ \frac{1}{|\overline{q}_1|^{1/2}}e^{-i\phi_1}\exp\left[\int_y^{a_1} w_1(y)\,dy - \int_y^{a_2} w_2(y)\,dy\right]$$
$$+ D\,|\overline{q}_3|^{-5/2}\exp\left[\int_y^{a_1} w_1(y)\,dy + \int_y^{a_2} w_2(y)\,dy\right], \tag{52}$$

where

$$i\phi_1 = \left(\frac{\beta^2}{2\alpha}\right)^{\frac{1}{2}}\left[\int_{L_1}\sqrt{U_2(z)-(4U_1(z)\alpha/\beta^2)^{\frac{1}{2}}}\,dz - \int_{L_2}\sqrt{U_2(z)+(4U_1(z)\alpha/\beta^2)^{\frac{1}{2}}}dz\right] - i\frac{\pi}{4}. \tag{53}$$

The integrals in (52) are taken from the branching points a_1 (a_2) along a line where ω_1 (ω_2) is purely imaginary, to the point C_1 and then along the real axis. The contours L_1 and L_2 in (53) start at point A, go along the real axis to point C_1 and then along the lines, where ω_1 and ω_2, respectively, are purely imaginary, up to the branching points a_1, a_2. In writing (53)

* The meaning of all the symbols and letters used in this section and not discussed in the text is clear from Fig. 4.

account is also taken of the fact that in the vicinity of point 0_1, where $U_2 = 0$, the expressions under the integral sign are transformed into

$$\overline{p}_1 = \left[\frac{\beta^2 U}{2\alpha}(z - ia)\right]^{\frac{1}{2}}$$

and

$$\overline{p}_2 = \left[\frac{\beta^2 U}{2\alpha}(z + ia)\right]^{\frac{1}{2}},$$

respectively. Sections are selected along the lines 1. It is not difficult to see that the value of ϕ_1 determined by expression (53) is purely real.

In order to write the solution of (52) (determined on the left-hand side of point 0_1) for the right-hand side of 0_1, use is made of the rules of rotation given in the Appendix. It is convenient to introduce the following designations:

$$p_1 = \sqrt{\frac{\beta^2}{2\alpha}[U_2 - (4U_1\alpha/\beta^2)^{\frac{1}{2}}]},$$
$$p_2 = \sqrt{\frac{\beta^2}{2\alpha}[U_2 + (4U_1\alpha/\beta^2)^{\frac{1}{2}}]}. \qquad (54)$$

Then, for $0_1 < y < 0_2$, and to within the constant factor, we have:

$$\varphi(j) = ie^{i\varphi_2}|p_1 + p_2|^{-5/2}\exp\left[i\int_{a_1}^{y} p_1(z)\,dz + i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$-(ie^{i\varphi_1} + D)|p_1 + p_2|^{-5/2}\exp\left[-i\int_{a_1}^{y} p_1(z)\,dz - i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$+e^{i\varphi_1}|p_1 - p_2|^{-1/2}\exp\left[i\int_{a_1}^{y} p_1(z)\,dz - i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$+(2\cos\phi_1 - iD)|p_1 - p_2|^{-1/2}\exp\left[-i\int_{a_1}^{y} p_1(z)dz + i\int_{a_2}^{y} p_2(z)dt\right] \qquad (55)$$

where integration is performed from a_1, a_2 respectively along the lines L_3, L_4 (Fig. 4), continuing down to the real axis and proceeding along it to the point y. Transferring the solution of (55) to point 0_2, we can re-write it in the form:

$$\varphi(y) = ie^{i\phi_1}\frac{\Phi}{|p_1 + p_2|^{5/2}}\exp\left[-i\int_y^{b_1} p_1(z)dz - i\int_y^{b_2} p_2(z)dz\right]$$
$$-\frac{ie^{i\phi_1} + D}{\Phi}|p_1 + p_2|^{-5/2}\exp\left[i\int_y^{b_1} p_1(z)\,dz + \int_y^{b_2} p_2(z)\,dz\right]$$
$$+\frac{e^{i\phi_1}\Psi}{|p_1 - p_2|^{1/2}}\exp\left[-i\int_y^{b_1} p_1(z)\,dz + i\int_y^{b_2} p_2(z)\,dz\right]$$
$$+\frac{2\cos\phi_1 - iD}{\Psi}|p_1 - p_2|^{-1/2}\exp\left[i\int_y^{b_1} p_1(z)\,dz - i\int_y^{b_2} p_2(z)dz\right] \qquad (56)$$

The integrals from y to b are interpreted in the same way as those from y to a. In addition we have assumed:

$$\Phi \equiv e^{i\phi_2} = \exp\left[i\int_{L_3} p_1(z)dz + i\int_{L_4} p_2(z)dz\right], \qquad (57)$$

$$\Psi = \exp\left[-i\int_{L_3} p_1(z)\,dz + i\int_{L_4} p_2(z)\,dz\right]. \qquad (58)$$

The contours L_3 and L_4 are shown in Fig. 4. It can easily be seen that the argument of the exponential in (58) is purely real, and ϕ_2 in (57) is purely imaginary. For this, contour L_3 is curved in such a way that it goes from a_1 to 0_1, thence along the real axis and subsequently from 0_2 to b_1. We follow a similar procedure with L_4. Then in accordance with (54), on the real axis, where L_3 and L_4 coincide, we have $\int_{0_1}^{0_2}(p_1 - p_2)dy$, which is purely imaginary; $\int_{0_1}^{0_2}(p_1 + p_2)dy$, which is purely real. Taking into account the fact that:

$$\int_{a_2}^{0_1}(z + ia)^{1/2}dz = (ia)^{3/2},$$

$$\int_{a_1}^{0_1}(z - ia)\;\;dz = (-ia)^{3/2},$$

we come directly to the above-mentioned statement.

To the right of point 0_2, using again the rules of rotation indicated in the Appendix, we get:

$$\varphi(y) = |p_1 + p_2|^{-5/2}\left(i\Phi e^{i\varphi_1} - i\frac{2\cos\phi_1 - iD}{\Psi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz + \int_{b_2}^{y} p_2(z)dz\right]$$

$$+ |p_1 + p_2|^{-5/2}\left(\frac{2\cos\phi_1 - iD}{\Psi} - \frac{ie^{i\varphi_1} + D}{\Phi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz + \int_{b_2}^{y} p_2(z)dz\right]$$

$$+ |p_1 - p_2|^{-1/2}\left(-e^{i\varphi_1}\Phi + e^{i\varphi_1}\Psi - \frac{e^{i\varphi_1} - iD}{\Phi} + \frac{2\cos\phi_1 - iD}{\Psi}\right)$$

$$\times \exp\left[-\int_{b_1}^{y} p_1(z)dz + \int_{b_2}^{y} p_2(z)dz\right]$$

$$+ |p_1 - p_2|^{-1/2}\left(\frac{2\cos\phi_1 - iD}{\Psi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz - \int_{b_2}^{y} p_2(z)dz\right]. \qquad (59)$$

The condition of the absence of increasing solutions at $+\infty$ is given by

$$e^{i\varphi_1}\Phi\Psi = 2\cos\phi_1 - iD$$

and

$$(\Psi - \Phi)\,e^{i\varphi_1} - \frac{e^{i\varphi_1} - iD}{\Phi} + \frac{2\cos\phi_1 - iD}{\Psi} = \frac{e^{2i\varphi_3}(2\cos\phi_1 - iD)}{\Psi}, \qquad (60)$$

where

$$i\phi_3 = -\int_{L_1'} p_1(z)dz + \int_{L_2'} p_2(z)dz, \qquad (61)$$

and contours L_1' and L_2' are similar to contours L_1, L_2 and are obtained, respectively, from b_1, b_2 through point C_2 to B. It should be noted that ϕ_3, like ϕ_2, is purely real and positive.

Solving the system (60), we find:

$$e^{i(\varphi_1 + \varphi_2 + \varphi_3)} = \pm 1 \qquad (62)$$

from which

$$\phi_1 + \phi_2 + \phi_3 = n\pi,$$

or

$$i\int_{L_1} p_1 dz - i\int_{L_2} p_2 dz + \int_{L_3} p_1 dz + \int_{L_4} p_2 dz - i\int_{L_1'} p_1 dz + i\int_{L_2'} p_2 dz = \left(n + \frac{1}{2}\right)\pi \qquad (63)$$

Equation (63) is a generalized "quantization rule" for the strong case. The left-hand side of (63) represents the total incidence of a phase consisting of three parts:

(1) Phase incidence in the region $A0_1$, where the wavelength

$$\lambda_x \simeq \left[\int (U_1/\beta U_2)^{\frac{1}{2}}\,dx\right]^{-1};$$

(2) Phase incidence in the region 0_1, 0_2, where the wavelength

$$\lambda_x \simeq \left[\int (U_2/\alpha\beta)^{\frac{1}{2}}\,dx\right]^{-1};$$

(3) Phase incidence in the region 0_2B which is of the same type as that in region $A0_1$.

This "strong coupling" of the oscillations is characteristic of the strong case and to that extent condition (63) expresses this fact.

VI. REMARKS

1. In accordance with the classification given in section IV, for $\alpha/\beta^2 < 1$ we have the weak case. The solution given in section III is true for $\alpha/\beta^2 < \beta$. Thus, for the weak case there remains a region $\beta \leqslant d/\beta^2 < 1$ not yet considered. The solution given in section III, as has already been stated, is a generalization of a known solution [1] for the quasi-classical case. However, a solution can be found for $\alpha/\beta^2 < 1$, including $\alpha/\beta^2 < \beta$ as a particular case. For this, we refer to the formulas (39) - (42). The solutions of (41) are unknown for $\alpha/\beta^2 < 1$. The coupling rules are the same for them as in section III, since the branching points of (42) do not lie in the region in which (41) is valid. The "quantization rules" of (32) and (33) remain the same.

The statements which have been made represent a unification of the method developed in sections IV and V. A purely technical difference arises in connection with the fact that the asymptotic solutions found in (41) have a different structure for the Stokes lines, depending on whether or not the branching points of (42) fall in the region in which solution (41) is valid.

2. The asymptotic method presented can be easily generalized for the case in which the behaviour of $U_2(x)$, in the vicinity of U_2 = zero, has the form: $U_2 \simeq Ux^m$, it being quite natural that the conditions for coupling the solutions should change, although equations (32), (33) and (62) remain the same. The case of m = 2 for weak coupling was investigated in [5]. For example, the "gravitational mode" found in [5] can be obtained directly from condition (33).

3. It is known [3, 4, 9] that the existence of a non-uniformity in the medium results in one type of oscillation in a certain range giving rise to another type of oscillation (wave "transformation" effect).

A detailed physical picture of this phenomenon is given in [3]. The method developed above can be applied to this phenomenon. The transformation effect is already contained in the solution. Thus, for example, in the strong case (section V), the presence of an oscillating solution $\varphi_{3,4}$ in the region $0_1 0_2$ leads to the appearance of an oscillatory solution $\varphi_{1,2}$ in the region $A0_1$. It can be said that the points leading to transformation are branching points. The coefficient of transformation is obtained as the ratio of the amplitudes φ_1 and φ_3. Of course, in the weak case the transformation effect is small, since the "birth" of a new solution takes place successively, according to the small parameter α/β^2. The essential factor in the strong case is the strong transformation, where the coefficient of transformation may be approximately unity.

VII. SOME CHARACTERISTIC FEATURES OF A PLASMA INSTABILITY IN THE FIELD OF GRAVITY

By way of illustrating the theory set out above, let us consider the question of the characteristic features of the stabilization of a so-called "flute" instability of a plasma, taking into account the finiteness of the ion Larmor radius [12]. The differential equation for the perturbed values in the case with which we are concerned has, as we know from [10], the form:

$$\beta\varphi'' - \left[1 - \frac{G}{r(r-1)}\right]\varphi = 0 \tag{64}$$

where $\beta = (k_y L)^{-2}$ (L is a characteristic dimension), $r = \omega/\omega_i$; $\omega_i = (cT/eH_0) \times k_y n_0'/n_0$; $G = (g/\omega_i^2)\, n_0'/n_0$; [g is the gravity acceleration, n_0 is the density, T is the temperature, H_0 is the intensity of the magnetic field, $n_0' = (g/\omega_i^2) n_0'/n_0$]. The instability becomes stabilized if $G \lesssim 1$, and the treatment may be considered correct if finite solutions can be shown to exist. However, as seen from (64), the coefficient of φ breaks down at the point where $r = 1$ and the existence of finite solutions has to be substantiated. We shall therefore proceed on the basis of broader assumptions in deriving the equation for the perturbed values and shall take into account the perturbation of the temperature T, which in a quasi-classical approximation satisfies the equation:

$$\frac{3}{2} n_0\left(\frac{\partial T}{\partial t} + (\vec{V}_{0i}\cdot\vec{\nabla})T\right) + n_0 T_0 \operatorname{div}\vec{V}_i = -\operatorname{div}\vec{q}_i$$

$$\vec{q}_i = \frac{5}{2}\frac{c n_0 T_0}{eH_0}(\vec{h}\times\vec{\nabla}T), \quad \left(\vec{h} = \frac{\vec{H}_0}{H}\right). \tag{65}$$

$\vec{V}_{0i}$ and T_0 are the unperturbed ion velocity and temperature, respectively. For simplicity, we shall consider the electrons to be cold. Selecting the perturbations in the form $\varphi(x)\exp(iyk_y + i\omega t)$ and making standard, simple calculations, differing from the derivation of (64) only in that account is taken of temperature perturbation, we obtain the following equation for φ in a system of co-ordinates in which the ions are at rest:

$$\varphi^{IV} - \left\{2\beta^{-1} + (r-1)\left[3rR^2 - \frac{G}{\beta(r-1)^2}\right]\right\}\varphi''$$

$$+ \left\{\beta^{-2} + \beta^{-1}(r-1)\left[3rR^2 - \frac{G}{\beta(r-1)^2}\right] - 3\beta^{-1}R^2 G\right\}\varphi = 0, \tag{66}$$

$R = L/r_i$ (r_i being the Larmor radius of the ions).

For simplicity, let us consider the case of a weak connection, corresponding to two separate equations for finding the eigenfrequencies (32) and (33). Here, (33) corresponds to the case of ordinary flute perturbations, the role of the second "turning point" being played by the point where $U_2 = 0$, and finite solutions exist if, outside the interval between the "turning points", the potential U_1/U_2 leads to damping solutions. If

$$rR^2 \gg \frac{G}{\beta(r-1)^2} \qquad (G \lesssim 1),$$

we obtain from (66) a result corresponding to (64), i.e. a stabilization of the instability.

Let us now consider what is the result of the second equation for eigenfrequencies, i.e.

$$\int_{0_1}^{0_2} \sqrt{U_2}\, dx = \left(n + \frac{1}{2}\right)\pi .$$

A qualitatively correct result can already be obtained from the condition $U_2 \approx 0$. Using the form of U_2 from (66) in the case where $r = 1 + r_1$ ($r_1 \ll 1$) we find that

$$r_1 = \frac{2}{3}(k_y r_i)^2 \pm \left[\frac{4}{9}(k_y r_i)^4 + (k_y r_i)^2 G\right]^{\frac{1}{2}}. \tag{67}$$

From (67) we see that taking account of the temperature perturbations leads to an instability if

$$|G| > k_y^2 r_i^2. \tag{68}$$

As we can see, the stabilization of this instability imposes more rigid conditions on the ion Larmor radius than would be required according to analysis of equation (64). The distance between the points of "intersection" of the solutions in this case is:

$$x \simeq k_y r_i L \tag{69}$$

and the treatment used is correct if

$$\frac{L}{r_i}(k_y r_i)^{3/2} \ll 1 \tag{70}$$

APPENDIX

In this Appendix we shall derive the rules for coupling the solutions of φ_i in the case of rotation around the branching points (more specifically, let us take the points a_1 and a_2 in Fig. 4). We start with line 1, proceeding from point a_1. On this line, we write the solution in the form:

$$\varphi(y) = A_1\Pi_1 \exp\left\{i\int^{y}[w_1(z) - w_2(z)]\,dz\right\} + B_1\Pi_2 \exp\left\{i\int^{y}[w_1(z) - w_2(z)]\,dz\right\}$$

$$+ C_1\Pi_3 \exp\left\{i\int^{y}[w_1(z) + w_2(z)]\,dz\right\} + D_1\Pi_4 \exp\left\{i\int^{y}[w_1(z) + w_2(z)]\,dz\right\} \quad (A1)$$

where the quantity Π_i is determined from (39). We get:

$$\Pi_1 = \Pi_2 = \exp\left[-\frac{1}{2}\ln(w_1 - w_2)\right],$$

$$\Pi_3 = \Pi_4 = \exp\left[-\frac{5}{2}\ln(w_1 + w_2)\right], \quad (A2)$$

for rotation around point a_1, and

$$\Pi_1 = \Pi_2 = \exp\left[-\frac{1}{2}\ln(w_2 - w_1)\right],$$

$$\Pi_3 = \Pi_4 = \exp\left[-\frac{5}{2}\ln(w_2 + w_1)\right], \quad (A3)$$

for rotation around point a_2.

It will be seen from (A1) and (A4) that rotation can take place around points a_1 and a_2 separately, the same coupling rules obtaining in the case of rotation around each branching point separately as in the case of [8]. It should be noted that the pair of solutions at A_1 and D_1 rotate around point a_1 independently. Similarly, the pair of solutions at A_1 and C_1 and the pair at B_1 and D_1 rotate around point a_2 independently. A_i, B_i, C_i and D_i define a system of coefficients for a solution in the vicinity of the lines with index i, from points a_1 and a_2. The result of simultaneous rotation around a_1 and a_2 then leads to the following:

$$\left\{\begin{aligned} A_2 &= A_1 + iD_1 \\ B_2 &= B_1 + iD_1 \\ C_2 &= iA_1 + iB_1 + C_1 - D_1 \\ D_2 &= D_1 \end{aligned}\right. \qquad \left\{\begin{aligned} A_3 &= A_2 + iC_2 \\ B_3 &= B_2 + iC_2 \\ C_3 &= C_2 \\ D_3 &= iA_2 + iB_2 - C_2 + D_2 \end{aligned}\right. \quad (A4)$$

$$\left\{\begin{aligned} A_1' &= A_3 + iD_3 \\ B_1' &= B_3 + iD_3 \\ C_1' &= iA_3 + iB_3 + C_3 - D_3 \\ D_1' &= D_3 \end{aligned}\right. \qquad \left\{\begin{aligned} A_1' &= -B_1 \\ B_1' &= -A_1 \\ C_1' &= -D_1 \\ D_1' &= -C_1 \end{aligned}\right.$$

These are the coupling formulas which are being sought. When we write the last column in (A4) we take account of the fact that the solution of Eq. (6) must be analytical in the complex plane y.

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* A translation or transliteration of each Cyrillic reference is given, set in italics, to aid the reader.