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On the Theory of the Pion Electromagnetic Form Factor.

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Summary. — An analysis of the behaviour of the pion form factor at $t \to \pm \infty$ has been carried out under some definite assumptions on the property of the form factor as an analytic function in a complex plane of the momentum transfer t. Bounds on the modulus of the form factor and the electromagnetic radius of the pion have been found. The method has been formulated which permits one to obtain the integral representations of the form factor, and some definite representations have been obtained. The sum rules for the form factor have been found and analysed, and some questions connected with the usage of the sum rules have been considered. The experimental situation of to-day for the pion form factor has been discussed.

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1. – The cross-section of the process $\ell^+ + \ell^- \to \pi^+ + \pi^-$ has been recently rather accurately determined from the colliding beam experiments in the region of ρ^0 resonance [1-3].

Hence the modulus of the electromagnetic form factor of the pion $F(t)$ can be found for the timelike momentum transfer t (for details concerning the results obtained, see Sect. **7**).

For the spacelike region of the momentum transfer, the form factor is determined from measurements of the cross-section of electroproduction of the pion on a proton under the kinematic conditions when contribution from the

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(1) V. L. AUSLANDER, G. I. BUDKER, YU. N. PESTOV, V. A. SIDEROV, A. N. SKRINSKY and A. G. KHABAKPASHEV: *Phys. Lett.*, **25** B, 433 (1967).

(2) V. L. AUSLENDER, G. I. BUDKER, E. V. PAKHTUSOVA, YU. N. PESTOV, V. A, SIDOROV, A. N. SKRINSKIĬ and A. G. KHABAKHPASHEV: *Yadern. Fiz.*, **9**, 114 (1969).

(3) J. AUGUSTIN, J. BIZOT, J. BUON, J. HAISSINSKI, D. LALANNE, F. LAPLANCHE, J. LEFRANÇOIS, P. LEHMANN, P. MARIN, F. RUMPF and E. SILVA: *Phys. Lett.*, **28** B, 508 (1969).

pion-pole diagram dominates ([4]) (for other experiments, see ref. ([4])). The degree of reliability of these results is essentially lower than that for the time-like region because of the uncertainities present in the theoretical analysis and also because of considerable experimental errors. The averaged experimental data yield

$$F(t) = \left(1 + \frac{|t|}{m^2}\right)^{-1}, \tag{1}$$

where $m^2 = (0.56 \pm 0.08)\,(\text{GeV})^2$, $t < 0$, $0 < |t| < 0.4\,(\text{GeV}/c)^2$. From (1) it follows that the charge radius of the pion equals to $r_\pi = (0.86 \pm 0.14)$ fm. The results obtained from some different experiments in this field have been less definite ($r_\pi < 3$ fm from the data concerning π-ℓ scattering and $r_\pi < 1$ fm from the experimental data concerning π-α scattering).

The properties of the form factor resulting from the experimental information available (in particular, restrictions on the speed of decrease of the form factor as well as the asymptotic behaviour at $t \to \pm \infty$) are of great interest, if one proceeds from the reliably established properties of the form factor disconnected with certain models. We shall assume that the form factor has the following properties:

1) The form factor $F(t)$ is an analytic function in the complex t-plane with a cut (*) from $t = 1$ to ∞.

2) The function $F^*(t) = F(t^*)$, so that $F(t)$ is a real function on the real axis when $t < 1$.

3) $|F(t)| < A \exp\left[\varepsilon/|\sqrt{t}|\right]$ holds for any $\varepsilon > 0$ in the whole complex t-plane. The assertion exists that this inequality results from a local field theory (see, for example, ref. ([5])).

4) The normalization condition $F(0) = 1$ is fulfilled.

2. – The relation between the asymptotic behaviour of the form factor at $t \to \pm \infty$ is of interest. For comparison of the behaviour of the form factor at $t \to -\infty$ and at $t \to \infty$ one may use the following assertion. For any function $A(t)$ which is analytic in the upper half-plane satisfying the condition $|A(t)| < C \exp[\varepsilon |t|]$ for any $\varepsilon > 0$ at $\operatorname{Im} t \geqslant 0$ such that $F(t)A(t) \to a$ at $t \to -\infty$, there holds $|a| \leqslant a_0$, with a_0 being the superior limit of $|F(t)A(t)|$ at $t \to +\infty$.

([4]) C. Mistretta, D. Imrie, J. A. Appel, R. Budnitz, L. Carroll, M. Goitein, K. Hanson and R. Wilson: *Phys. Rev. Lett.*, **20**, 1523 (1968).

(*) In investigating the analytic properties of the form factor, the authors will make use of the momentum in the units of $4m_\pi^2$.

([5]) N. N. Meiman: *Žurn. Èksp. Teor. Fiz.*, **46**, 1502 (1964).

This follows from the Phragmén-Lindelöf theorem ([6]), and also from the theorem on sets of the limit points of the analytic function $w = f(z)$ when z tends to reach some boundary point of the region along the boundary arcs (ref. ([7])). For instance, this assertion implies that when

$$F(-t) \xrightarrow[t\to+\infty]{} \frac{a \exp[-bt^{\alpha_1}]}{t^{\alpha_2}[\ln t]^{\alpha_3}}, \qquad 0 \leqslant \alpha_1 < \tfrac{1}{2}, \tag{2}$$

then in the case when $|F(t)|$ tends to some definite limit at $t \to +\infty$

$$|F(t)| \geqslant \frac{|a| \exp[-bt^{\alpha_1} \cos \pi\alpha_1]}{t^{\alpha_2}(\ln t)^{\alpha_3}}. \tag{3}$$

The possibility of the exponential decrease of $F(-t) \xrightarrow[t\to+\infty]{} a \exp\left[-b\sqrt{t}\right]$ $(b > 0)$, may be of particular interest. The point is that the modern experimental data for the form factor of the nucleon (it is quite probable that the qualitative peculiarities in the behaviour of the form factors of the nucleon and the pion are the same) seem to indicate such a decrease. Then at $t \to +\infty$, $|F(t)| \geqslant |a|$.

3. – The form factor $F(t)$ satisfying the above requirements cited in Sect. **1** cannot arbitrarily decrease at $t \to \pm\infty$. The restriction for the rate of decrease can be obtained from a theorem on the two constants (ref. ([6])). This theorem implies that if the function $f(z)$ is analytic and bounded in the domain D whose boundary C consists of the two sets α_1 and α_2, each of the latter containing a finite number of arcs, where

$$\overline{\lim_{z\to\xi\in\alpha_1}} |f(z)| \leqslant m_1, \qquad \overline{\lim_{z\to\xi\in\alpha_2}} |f(z)| \leqslant m_2 \tag{4}$$

then for any $z \in \mathscr{D}$

$$|f(z)| \leqslant m_1^{\omega(z,\alpha_1,\mathscr{D})} m_2^{\omega(z,\alpha_2,\mathscr{D})}, \tag{5}$$

where $\omega(z, \alpha, \mathscr{D})$ is the harmonic measure of the set α with respect to the domain $\mathscr{D}$ at the point z. By definition, $\omega(z, \alpha, \mathscr{D})$ is the harmonic and bounded function in the domain D such that

$$\left|\begin{array}{l} \omega(z, \alpha_1, \mathscr{D}) = \begin{cases} 1, & z \in \alpha_1, \\ 0, & z \in \alpha_2, \end{cases} \\ \\ \omega(z, \alpha_1, \mathscr{D}) + \omega(z, \alpha_2, \mathscr{D}) = 1. \end{array}\right. \tag{6}$$

([6]) S. Stoilow: *Teoria Functiilor de a Variabila Complexa* (Bucharest, 1954).

([7]) R. Nevanlinna: *Eindeutige analytische Funktionen* (Berlin, Göttingen, Heidelberg, 1953).

If the domain $\mathscr{D}$ is conformally mapped into the circle so that point z goes into the circle centre, then the harmonic measure $\omega(z, \alpha_1, \mathscr{D})$ will be equal to the angle of the arc into which α_1 goes divided by 2π.

As implied by the Phragmén-Lindelöf theorem, the form factor $F(t)$ because of condition 3 is bounded in the complex t-plane if it is bounded on the cut. Hence the theorem on the two constants may be applied to the form factor $F(t)$ bounded on the cut. Let

$$|F(t)| \leqslant m_1\,, \quad 1 \leqslant t \leqslant t_1\,; \quad |F(t)| \leqslant m_2\,, \quad t_1 < t < \infty; \tag{7}$$

then on calculating, in accordance with the above-mentioned reasonings, the harmonic measure, we obtain for $(-t_0) < 1$

$$|F(-t_0)| \leqslant m_1^{\omega(-t_0,\alpha_1)} m_2^{\omega(-t_0,\alpha_2)}\,, \tag{8}$$

where

$$\omega(-t_0, \alpha_1) = \frac{2}{\pi} \operatorname{arctg} \frac{\sqrt{t_1 - 1}}{\sqrt{t_0 + 1}}\,. \tag{9}$$

Hence in passing to the large t we obtain

$$\max_{t > t_1} |F(t)| \geqslant m_1 [|F(-t_0)| m_1^{-1}]^{(\pi/2)(\sqrt{t_1}/\sqrt{t_0+1})}\,. \tag{10}$$

The theorem on the two constants admits generalization for the case when restriction for the modulus of the function is given more than on the two parts of the boundary, *i.e.* when

$$\lim_{z \to \xi \in \alpha_k} |f(z)| \leqslant m_k \qquad (k = 1, 2, \ldots, n)\,.$$

The generalized theorem (on n constants) asserts that for any z from $\mathscr{D}$

$$|f(z)| \leqslant \exp\left[\sum_{k=1}^{n} \ln m_k\, \omega(z, \alpha_k, \mathscr{D})\right], \tag{11a}$$

where the harmonic measure is now

$$\left\{\begin{aligned} &\omega(z, \alpha_1, \mathscr{D}) = \begin{cases} 1\,, & z \in \alpha_1\,, \\ 0\,, & z \in \alpha_2, \ldots, \alpha_n\,, \end{cases} \\ &\sum_{k=1}^{n} \omega(z, \alpha_k, \mathscr{D}) = 1\,. \end{aligned}\right. \tag{11b}$$

Let us apply this generalization to the form factor $F(t)$. Taking into account that

$$\omega(-t_0, \alpha_k) = \frac{\pi}{2}\left[\operatorname{arctg}\frac{\sqrt{t_k-1}}{\sqrt{t_0+1}} - \operatorname{arctg}\frac{\sqrt{t_{k-1}-1}}{\sqrt{t_0+1}}\right], \tag{12}$$

where α_k is the part of the cut between the points t_{k-1} and t_k ($t_k > t_{k-1}$), so by passing over the limit $t_k - t_{k-1} \to 0$ ($n \to \infty$) we obtain

$$|F(-t_0)| \leqslant \exp\left[\frac{\sqrt{1+t_0}}{\pi}\int\limits_0^\infty \frac{\ln|F(t)|\,dt}{\sqrt{t-1}(t+t_0)}\right]. \tag{13}$$

In this case inequality (10) becomes more definite:

$$\max_{t>t_1} |F(t)| \geqslant \left[|F(-t_0)| \exp\left[-\frac{\sqrt{t_0+1}}{\pi}\int\limits_1^{t_1} \frac{\ln|F(t')|\,dt'}{\sqrt{t'-1}(t'+t_0)}\right]\right]^{\frac{\pi}{2\operatorname{arctg}(\sqrt{t_0+1}/\sqrt{t_1-1})}}. \tag{14}$$

If one replaces here $\ln|F(t')|$ by its maximal value within the integration range, then one comes back to (10). From the condition $F(0) = 1$ and (13) it follows that (cf. [8,9])

$$\int\limits_1^\infty \frac{\ln|F(t)|\,dt}{t\sqrt{t-1}} \geqslant 0 \tag{15}$$

and from the condition $F(0) = 1$ and (14)

$$\max_{t>t_1} |F(t)| \geqslant \exp\left[\frac{-1}{2\operatorname{arctg}(1/\sqrt{t_1-1})}\int\limits_1^{t_1} \frac{\ln|F(t')|\,dt'}{t'\sqrt{t'-1}}\right]. \tag{16}$$

Note that from (13) it follows that the form factor $F(t)$ at $t \to +\infty$ cannot fall down faster than the exponent of the type

$$\exp\left[-\varepsilon\frac{\sqrt{t}}{\ln t \ln\ln t \dots}\right]$$

for any $\varepsilon > 0$. Formula (14) at $t_0 \gg t_1$ gives

$$\max_{t>t_1} |F(t)| \geqslant |F(-t_0)|, \tag{17}$$

(8) B. V. Geshkenbein and B. L. Yoffe: *Žurn. Èksp. Teor. Fiz.*, **46**, 903 (1964).
(9) B. V. Geshkenbein: *Yadern. Fiz.*, **9**, 1232 (1969).

where one does not require that this function should tend to some definite limit at $t_0 \to \infty$ (cf. Sect. 2).

If the function $F(t)$ has no zeros in the complex plane, $|F(t)| > A \exp[-\varepsilon|\sqrt{t}|]$ at any $\varepsilon > 0$ and $1/F(t)$ is bounded on the cut, then the same arguments can be attributed to the function $1/F(t)$ which had led us to formula (13), *i.e.*

$$|1/F(t)| \leqslant \exp\left[\frac{-\sqrt{1-t}}{\pi}\int\limits_1^\infty \frac{\ln|F(t')|\,\mathrm{d}t'}{\sqrt{t'-1}(t'-t)}\right], \qquad t<1. \tag{18}$$

From (13) and (18) it immediately follows that when the above conditions are fulfilled

$$F(t) = \exp\left[\frac{\sqrt{1-t}}{\pi}\int\limits_1^\infty \frac{\ln|F(t')|\,\mathrm{d}t'}{\sqrt{t'-1}(t'-t)}\right]. \tag{19}$$

It is evident that this formula can be applied for any t; below we shall arrive at this integral representation from much more simple considerations.

If one applies the theorem on the n constants (11) to the function $F(t)B(t)$ (the function $B(t)$ satisfies the properties of (1)-(3)), then we obtain an assertion similar to the theorem cited in Sect. 2. Note that here one does not require that $F(t)B(t)$ should tend to some definite limit at $t \to -\infty$, whereas the quantity a is the superior limit of the values $|F(t)B(t)|$ at $t \to -\infty$.

4. – Consider the restrictions for the rate of the change of $F(t)$ at $t < 1$. For this purpose, we map the plane with the cut $1 \leqslant t < \infty$ into the unit circle with the centre $z = 0$ so that the point $t_0 < 1$ would go into the circle centre

$$t = \frac{4z + t_0(1-z)^2}{(1+z)^2}. \tag{20}$$

Taking into account that for the function $\varphi(z)$, which is analytic in the unit circle, such that $|\varphi(z)| \leqslant M$ in the circle, there holds the inequality

$$M|\varphi(0)|\left|\frac{\varphi(z)-\varphi(0)}{M^2\varphi(0)-\varphi(z)|\varphi(0)|^2}\right| \leqslant 1, \tag{21}$$

which is checked directly, and applying Schwarz lemma to the function in the left-hand side of (21) (ref. [6]), we obtain the restriction for the rate of the change of the form factor:

$$|F'(t_0)| \leqslant \frac{M^2 - F^2(t_0)}{4M(1-t_0)}, \tag{22}$$

where M is the maximum modulus value of the form factor on the cut. The electromagnetic radius of the pion is defined from

$$\tau_\pi^2 = 6F'(0) \qquad \left(\tau_\pi^2 = \frac{3}{2m_\pi^2}\, F'(0)\right). \tag{23}$$

This and (22) imply (the same restriction has been obtained in [9] using a different method with use made of the explicit representation)

$$\tau_\pi^2 \leqslant \frac{3}{8m_\pi^2}\left(M - \frac{1}{M}\right). \tag{24}$$

If $F(t)$ has no zeros in the complex plane, then restrictions on the rate of change may be obtained by applying Caratheodory inequality (ref. [6]) to the function $\ln\big(F(t)/F(t_0)\big)$:

$$|F'(t_0)| \leqslant \frac{|F(t_0)|\ln\big(M/|F(t_0)|\big)}{2(1-t_0)}. \tag{25}$$

Here we have a stronger restriction for the electromagnetic radius of the pion than in (24):

$$\tau_\pi^2 \leqslant \frac{3}{4m_\pi^2}\ln M. \tag{26}$$

The restrictions on the behaviour of the form factor obtained in the Sect. **2-4** are stronger than those found in ref. [10,11]. We note that in these papers (in particular, in applying the theorem on two constants) an additional cut was actually introduced at $t<0$, so that the results obtained may be referred to a wider class of functions and may be made more restrictive.

5. – Let us consider the problem of integral representations of the form factor $F(t$)proceeding from a different (simpler) viewpoint [*]. The measurements of the form factor of the pion in the timelike region of the momentum transfer give $F(t)$ values on the cut so that, in view of the properties of the form factor adopted by us, one can in principle, using the Cauchy theorem, express the function inside the region by measured values; for instance, one can

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([10]) NGYEN VAN HIEU: *Dokl. Akad. Nauk SSSR*, **182**, 1303 (1968).

([11]) V. BALUNI, NGUYEN VAN HIEU and V. SULEYMANOV: *Yadern. Fiz.*, **9**, 635 (1969).

([*]) The main contents of Sect. **5** and **6** was a part of the report of the present authors (E 860) at the *XIV International Conference on High-Energy Physics* (Vienna, 1968).

obtain the well-known expression of the form factor $F(t)$ in terms of $\operatorname{Im} F(t)$ on the cut. However, 1) the modulus $F(t)$ is measured in modern experiments in the timelike region, 2) when one writes down the representation $F(t)$ by $\operatorname{Im} F(t)$ the question always remains about the number of subtractions required. Therefore one usually passes to considering the function $\ln F(t) = \ln |F(t)| + i\varphi(t)$, though in this case a rather unpleasant problem does arise about the zeros of the function $F(t)$ in the complex plane. So far we shall consider the case of absence of zeros. Applying from the beginning the Cauchy theorem to the function $\ln F(t)$ we obtain the well-known representation

$$F(t) = \exp\left[\frac{t}{\pi}\int\limits_1^\infty \frac{\varphi(t')\,\mathrm{d}t'}{t'(t'-t)}\right], \tag{27}$$

where in accordance with the adopted properties of $F(t)$ one subtraction has been performed. However, in the experiments the phase of the form factor is not measured. To obtain the representation of $F(t)$ in terms of $|F(t)|$ on the cut, it is sufficient to consider the function $\sqrt{1-t}\ln F(t)$ or $\ln F(t)/\sqrt{1-t}$; it is necessary to make two subtractions in the first case (so that in the representation of the form factor, for instance, the pion radius will enter) and, in the second case, one may restrict oneself to one subtraction only, with more rigid restrictions on the function $F(t)$ at the point $t = 1$. If one applies the Cauchy theorem to the function $\ln F(t)/\sqrt{1-t}$, then we obtain formula (19) (without subtraction as implied from the conditions imposed on $F(t)$). If one considers functions of the form

$$\sqrt{\frac{(a_1-t)\dots(a_n-t)}{(a_{n+1}-t)\dots(a_m-t)}}\ln F(t),$$

then one can obtain different representations for $\ln F(t)$ such that the form factor is expressed by the modulus and the phase of the form factor at various parts of the cut, if $a_k \geqslant 1$. If one would choose some $a_k < 1$, then representations of the form factor will contain additional integrals (over the additional cuts introduced, both finite and infinite). The case when one has three two-terms $(a_k - t)$ under the root has been considered in papers [12,13] at $a_1 = 1$ and a_2, $a_3 < 0$.

Let us give the examples of the representation $F(t)$ (together with (19)) which may be obtained in the above-mentioned way. If one applies the Cauchy

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([12]) L. A. KHALFIN and JU. P. SHCHERBIN: *Sov. Phys. JETP Lett.*, **8**, 588 (1968).
([13]) L. A. KHALFIN and JU. P. SHCHERBIN: *Sov. Phys. JETP Lett.*, **8**, 642 (1968).

theorem to the function $\left(\sqrt{t_1 - t}/t\sqrt{1-t}\right)\ln F(t)$ for $t_1 > 1$, then

$$\ln F(t) = \frac{t\sqrt{1-t}}{\pi\sqrt{t_1 - t}}\left[\int_1^{t_1} \frac{\sqrt{t_1 - t'}\ln|F(t')|\,\mathrm{d}t'}{\sqrt{t'-1}\,t'(t'-t)} + \int_{t_1}^{\infty} \frac{\varphi(t')\sqrt{t'-t_1}\,\mathrm{d}t'}{t'\sqrt{t'-1}(t'-t)}\right]. \tag{28}$$

This representation may appear to be useful if the phase of the form factor $\varphi(t)$ which in the elastic region is connected with the phase of $\pi\pi$ scattering at $t\to\infty$ tends to a constant (if there exists only the ρ-resonance, then it is quite possible that $\varphi(t)\to\pi$). Then on taking t_1 to be the boundary of the region where $|F(t)|$ is measured (and already $\varphi \simeq \pi$) one can obtain $F(t)$ in spite of knowing nothing of the behaviour of $|F(t)|$ at $t\to\infty$.

If one applies the Cauchy theorem to the function $(1/t^2)\sqrt{1-t}\ln F(t)$ then one obtains for the electromagnetic radius of the pion r_π

$$r_\pi^2 = \frac{6\sqrt{1-t}}{t}\left[\ln F(t) + \frac{t^2}{\pi\sqrt{1-t}}\int_1^{\infty} \frac{\sqrt{t'-1}\ln|F(t')|\,\mathrm{d}t'}{t'^2(t'-t)}\right]. \tag{29}$$

If one makes use of most accurate measurements of $F(t)$ in the spacelike region and if due to satisfactory convergence of the integral in the right-hand side the main contribution is given by a narrow region (for which $|F(t)|$ has been measured), then from (29) we shall define the electromagnetic pion radius (the right-hand side of (29) must be constant).

We have discussed the case when $F(t)$ has no zeros in the complex region. If such zeros are present, then the direct application of the Cauchy theorem to the function $\ln F(t)$ is difficult to do due to additional cuts introduced by these zeros. To eliminate the zeros from consideration one should introduce the function $\tilde{F}(t)$ which contains no zeros and is such that no characteristics are changed through which this function is expressed. If we express $F(t)$ in terms of the values $|F(t)|$ on the cut, then $\tilde{F}(t) = \chi^{-1}(t)F(t)$, where $|\chi| = 1$ on the cut, $\chi(t)$ is real at $t < 1$ and $\chi(0) > 0$. The explicit form of (19) with subtraction and consideration of zeros can be written as follows (cf. ([9])):

$$F(t) = \frac{\chi(t)}{[\chi(0)]^{\sqrt{1-t}}}\exp\left[\frac{t\sqrt{1-t}}{\pi}\int_1^{\infty} \frac{\ln|F(t')|\,\mathrm{d}t'}{t'\sqrt{t'-1}(t'-t)}\right], \tag{30}$$

where

$$\chi(t) = (-1)^n \prod_k \frac{\sqrt{1-t} - \sqrt{1-t_k}}{\sqrt{1-t} + \sqrt{1-t_k^*}}, \tag{31}$$

where n is the number of zeros on the real axis at $t<0$. If we express $F(t)$ in terms of the phase on the cut (27), then $\tilde{F}(t)=\psi^{-1}(t)F(t)$, where $\psi_+(t)/\psi_-(t)=1$ on the cut ($\psi_\pm(t)$ are respectively the values of the function on the upper and lower edges of the cut). For the case when $F(t)$ is expressed in terms of $|F(t)|$ and the phase on the cut (as, for example, (28)), then one must introduce $\tilde{F}(t)=\eta^{-1}(t)F(t)$, where $\eta(t)$ satisfies the above-mentioned requirements on the corresponding parts of the cut.

6. – To analyse the experimental data some of the sum rules following from the integral representations for the form factor $F(t)$ may be helpful.

We have written down the sum rules (15) implied from condition $F(0)=1$. The same sum rule is derived from the representation (30) and condition $|F(t)|<A\exp[\varepsilon|\sqrt{t}|]$, $\varepsilon>0$. From formulae (19) and (30) it follows that the equality sign in (15) holds when the following conditions have been fulfilled:

a) $F(t)$ has no zeros in the complex plane with a cut,

b) $|F(t)|>A\exp[-\varepsilon|\sqrt{t}|]$, $\varepsilon>0$.

If these conditions are not fulfilled, then we have an inequality sign in the sum rule (15).

Another sum rule containing only $|F(t)|$ on the cut may be obtained from formula (19) (or (30)), if one takes into account that in the vicinity of $t=1$ the phase of $\pi\pi$ scattering $\delta_{\pi\pi}(t)\sim(t-1)^{\frac{3}{2}}$ and the phase $\varphi(t)=\delta_{\pi\pi}(t)+(1-(-1)^n)(\pi/2)$, where n is the number of zeros of $F(t)$ in the interval $(0, 1)$. If one puts $t>1$, then (19) implies

$$\frac{\varphi(t)}{\sqrt{t-1}}=-\frac{P}{\pi}\int_1^\infty\frac{\ln|F(t')|\,\mathrm{d}t'}{\sqrt{t'-1}(t'-t)}=-\frac{1}{\pi}\int_1^\infty\frac{\ln|F(t')/F(t)|\,\mathrm{d}t'}{\sqrt{t'-1}(t'-t)}\,. \tag{32}$$

Such sort of consideration leads to the sum rule

$$\int_1^\infty\frac{\ln|F(t)/F(1)|\,\mathrm{d}t}{(t-1)^{\frac{3}{2}}}\geqslant 0\,, \tag{33}$$

where the equality sign stands for the same conditions as in formula (15).

There exist various sum rules besides this one, which contain integrals of $|F(t)|$ not only on the cut or an integral on the cut contains $|F(t)|$ in some intervals, while it contains $\varphi(t)$ in some other intervals. Some rules of the 1st type have been discussed in ref. [12,13].

We consider a general scheme of derivation of the sum rules of this sort. Let us apply the Cauchy theorem, for instance to the function $\ln F(t)/\sqrt{(1-t)(t+t_0)}$

for $F(t)$ which has no zeros, $(-t_0 < 1)$,

$$(34)\qquad \ln F(t) = \frac{\sqrt{(1-t)(t+t_0)}}{\pi}\cdot\left[\int_{t_0}^{\infty}\frac{\ln|F(-t')|\,\mathrm{d}t'}{\sqrt{(t'+1)(t'-t_0)}(t+t')}+\int_{1}^{\infty}\frac{\ln|F(t')|\,\mathrm{d}t'}{\sqrt{(t'-1)(t'+t_0)}(t'-t)}\right].$$

From the condition $F(0)=1$ we have the sum rule. In order to introduce zeros of $F(t)$ in consideration let us define $\tilde{F}(t)=\tau^{-1}(t)F(t)$, where $\tilde{F}(t)$ has no zeros in the plane with the cuts $(-\infty,-t_0)$, $(1,+\infty)$, $|\tau(t)|=1$ on the cuts; $\tau(t)$ is real in the interval $(-t_0,1)$, $\tau(0)>0$.

For this case (cf. (31))

$$(35)\qquad \tau(t)=(-1)^n\prod_k\frac{\sqrt{1-t}-\sqrt{(1-t_k)/(t_0+t_k)}\sqrt{t+t_0}}{\sqrt{1-t}+\sqrt{(1-t_k^*)/(t_0+t_k^*)}\sqrt{t+t_0}},$$

where n is the number of zeros in the interval $(0,-t_0)$.

In accordance with the principle of maximum we have $\tau(0)\leqslant 1$ and, hence, $\tilde{F}(0)\geqslant 1$, *i.e.* the sum rule for the general case has the form

$$(36)\qquad \int_{t_0}^{\infty}\frac{\ln|F(-t)|\,\mathrm{d}t}{t\sqrt{(t+1)(t-t_0)}}+\int_{1}^{\infty}\frac{\ln|F(t)|\,\mathrm{d}t}{t\sqrt{(t-1)(t+t_0)}}\geqslant 0\,.$$

One can also obtain, besides this sum rule, the analogue of (33) being an analogue of (15).

Let us apply the Cauchy theorem to the function $\ln\tilde{F}(t)/\sqrt{(t_1-t)(1-t)}$ at $t_1>1$ here $\tilde{F}(t)=\omega^{-1}(t)F(t)$, where $\omega(t)$ is the analytic function with the cut $(1,t_1)$, and $|\omega(t)|=1$ on the cut and $\omega(t)$ is a real function on the real axis outside the cut, $\omega(0)>0$ and $\tilde{F}(t)$ has no zeros in the plane with the cut $(1,t_1)$. The explicit form of the function $\omega(t)$ (also of all such functions given above (31), (35)) is derived from

$$(37)\qquad \omega(t)=\pm\prod_k\frac{z-z_k}{1-z_k^*z},$$

where $z(t)$ is a conformal mapping of the plane with the cut $(1,t_1)$ into a unit circle so that the real axis goes into the real axis. Since in accordance with the principle of maximum in the plane $|\omega(t)|\leqslant 1$, then we obtain the sum rule

$$(38)\qquad \int_{1}^{t_1}\frac{\ln|F(t)|\,\mathrm{d}t}{t\sqrt{(t-1)(t_1-t)}}-\int_{t_1}^{\infty}\frac{\varphi(t)\,\mathrm{d}t}{t\sqrt{(t-1)(t-t_1)}}\geqslant 0\,.$$

Another sum rule may be obtained from condition

$$\delta_{\pi\pi}(t) \sim (t-1)^{\frac{3}{2}} .$$

In a similar way one can obtain other sum rules.

7. – The experimental information on the electromagnetic pion form factor $F(t)$ is at present very restricted (measurements of the form factor have been conducted within narrow intervals of the momentum transfer, the accuracy of experiments is not, so far, very high). Therefore, to compare the integral representation obtained in the preceding Sections with the experiments and also the sum rules containing integrals of modulus or phase of the form factor along the whole cut one is obliged to make use of the simple analytic approximation. In the timelike region there are expressions of a resonance type (the process proceeds through the ρ-meson), where the parameters entering in them are determined from experiment. The most simple formula of Breit-Wigner type (in the usual units) has the form

$$|F(t)|^2 = \frac{F_0^2 m_\rho^2 \Gamma^2}{(t-m_\rho^2)^2 + m_\rho^2 \Gamma^2} , \tag{39}$$

where

$$F_0^2 = 42 \pm 8 \;\; (55.6 \pm 6.2) ,$$
$$m_\rho = (754 \pm 9)\ \text{MeV} \;\; \big((760 \pm 5.5)\ \text{MeV}\big) ,$$
$$\Gamma = (105 \pm 20)\ \text{MeV} \;\; \big(\Gamma = (112 \pm 11.5)\ \text{MeV}\big)$$

have been defined in ([2]) (([3])). In the spacelike region in which the accuracy of measurements is essentially lower (cf. formula (1)) for an approximation to the form factor, the model of vector dominance is also used.

One must take into account that formulae of type (39) are valid only in a narrow region $\sqrt{t} \simeq m_\rho$. Naturally, formulae which are valid in a wider region of the momentum transfer are interesting (at best at all the momentum transfers) having at $\sqrt{t} \simeq m_\rho$ the Breit-Wigner form (39). Formulae of this sort must satisfy obviously the requirements 1)-4). To obtain them one can use several methods:

a) If the phase of $\pi\pi$ scattering in the state $I = J = 1$ of the resonance form is given (ρ-meson), then by substitution in formula (21) we shall obtain an expression for the form factor satisfying all the requirements (in this case by neglecting all the inelastic channels).

b) If one assumes that the modulus of the form factor is everywhere rather well given by formula (39), then substituting in (30) we shall also get the representation of the form factor. In this case the modulus of the form factor must satisfy the sum rule (15), (33) otherwise we shall arrive at the inner contradictions (for greater detail, see below). One can make use of formula (28) as well, etc. All the representations of the form factor obtained in this way possess the Breit-Wigner behaviour at $\sqrt{t} \simeq m_\rho$. At the same time they somewhat differ from each other in the resonance region. The parameters of the resonance (ρ-meson), defined by various formulae are also different as well as the width of the resonance peak of the form factor and its position differs from those of the resonance in $\pi\pi$ scattering. All these distinctions are of the order of Γ/m_ρ; this is connected with the fact that Breit-Wigner approximation is valid up to this accuracy. Far away from the resonance the form factors may be quite different. At present, there are no theoretical arguments, in general, that would permit to separate some definite representation for the form factor, or, at least, a class of representations. To do this one must go beyond the framework of the Breit-Wigner approximation to describe wide resonances (unstable particles). At the time being various authors place additional « reasonable requirements » on the form of the form factor. Thus, in ref. [14] some definite (still not unique) form of the phase of $\pi\pi$ scattering was used, which permitted one to obtain some concrete representation for the form factor, and in the authors' opinion, to calculate the corrections Γ/m_ρ. This representation was then used in ref. [3] for fitting the experiments and it turns out to be quite suitable because the calculated correction $(1 + 0.48\,\Gamma/m_\rho)$ coincides numerically with $F_0\Gamma/m_\rho$ which were obtained in ref. [3]. However it is too early to attach any importance to this coincidence. In ref. [15] an analysis of various representations of the form factor has been carried out, in which the author gives preference to representations for which the phase of $\pi\pi$ scattering depends in a more complex way on the momentum, since they lead to the desired result ($\Gamma \simeq 120$ MeV). Note that treatment of the experimental data by a formula of the same type [14] carried out in ref. [3,15] yields $\Gamma = (111 \pm 6)$ MeV and $\Gamma = (123 \pm 7)$ MeV, respectively. All this information supports that the spread of the parameter values is objective, and cannot be eliminated within the framework of the Breit-Wigner approximation.

In the subsequent analysis of the form factor it is desirable to establish the qualitative peculiarities in its form (the extent of asymmetry, the rate of decrease at $\sqrt{t} > m_\rho$, the behaviour near the threshold of the reaction). For this purpose measurements must be made far away from the resonance (*). By the above-cited reasons this seems to be much more interesting than the subsequent more accurate definition of the experimental information on the ρ peak. Thus this treatment finally permits one to have a better understanding of the properties

(14) G. Gounaris and J. Sakurai: *Phys. Rev. Lett.*, **21**, 244 (1968).

(15) M. Roos and J. Pisut: *Nucl. Phys.*, B **10**, 563 (1969).

(*) We shall not discuss the problem of ρ-ω interpretation, the study of which is highly interesting.

of the $\pi\pi$ interaction. Independent measurements of the phase of the form factor are also desirable though this would require more complicated experiments (for example, measurements of photon polarization in the reaction $e^+e^- \to \pi^+\pi^-\gamma$).

8. – We make one more note on the use of the sum rules. Their direct use is so far impossible, since the experimental data are not sufficient so that, in fact, a check is made of the consistency of the analytic approximations to the form factor at the experimental parameter values only. Let the representation

$$|F(t)|^2 = \frac{P_n(t)}{Q_m(t)}\,, \qquad t \geqslant 1, \tag{40}$$

hold, in which $P_n(t)$, $Q_m(t)$ are polynomials of corresponding ranks. Hence one can find the form factor (assuming that zeros are lacking)

$$F(t) = R(t)\,\exp\left[-\sqrt{1-t}\,\ln R(0)\right], \tag{41}$$

where

$$R(t) = |F(1)|\,\frac{\prod\limits_{k=1}^{n}\left(1+\sqrt{(1-t)/(1-t_k)}\right)}{\prod\limits_{l=1}^{m}\left(1+\sqrt{(1-t)/(1-t_l)}\right)}\,. \tag{42}$$

Here t_i are the roots of corresponding polynomials. From formula (41) it follows that at $R(0)<1$ the form factor increases exponentially at $t<0$, but this is forbidden. Therefore we must have $R(0)\geqslant 1$ (*) (exponential decrease may generally be absent if the form factor has zeros). For the form factor (39) (if one applied it over the whole region) proceeding from these reasonings, we have that at $k = F_0\Gamma/m_\rho < 1 + 2m_\pi\Gamma/m_\rho^2$, $(R(0)<1)$ so we have come to a contradiction.

In ref. ([12]) the sum rules derived by the authors were checked with the aid of numerical calculations. It has been found that the sum rules are poorly satisfied. It must be stressed that the authors used representation (39) with $k = 0.77$ at which, as it was shown above, the sum rules cannot be satisfied at all.

(*) This condition follows directly from the sum rule (15).

RIASSUNTO (*)

Si è condotta un'analisi del comportamento del fattore di forma del pione per $t \to \pm\infty$ in definite ipotesi sulla proprietà del fattore di forma come funzione analitica in un piano complesso dell'impulso trasferito t. Si sono trovati i limiti del modulo di fattore di forma e del raggio elettromagnetico del pione. Si è formulato il metodo che permette di ottenere le rappresentazioni integrali del fattore di forma e si sono ottenute alcune rappresentazioni definite. Si sono trovate ed analizzate le regole di somma per il fattore di forma e si sono considerate alcune questioni collegate all'uso delle regole di somma. Si è discussa la situazione sperimentale odierna del fattore di forma del pione.

(*) *Traduzione a cura della Redazione.*

О теории электромагнитного форм-фактора пионов.

Резюме (*). — Был проведен анализ поведения пионного форм-фактора при $t \to \pm\infty$ при некоторых определенных предположениях на свойство форм-фактора, как аналитической функции в комплексной плоскости передаваемого импульса t. Были найдены границы для модуля форм-фактора и электромагнитный радиус пиона. Был сформулирован метод, который позволяет получить интегральные представления форм-фактора, и были получены некоторые определенные представления. Были найдены и проанализированы правила сумм для форм-фактора, и были рассмотрены некоторые вопросы, связанные с использованием правил сумм. Была обсуждена экспериментальная ситуация для пионного форм-фактора на сегодняшний день.

(*) *Переведено редакцией.*