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Collective Excitations and Their Microscopic Models.

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The topic of these lectures is a critical analysis of existing models of collective excitations (mostly microscopic ones) and ways of their improvement. The lectures do not claim either to make a full survey of the subject or to present a complete solution of the problem. The author was led mostly by his personal interests and stresses, not by existing results and achievements but rather by problems which remain to be solved.

I. - Phenomenological models.

The low-lying excited states of even-even nuclei (below $(2,5 \div 3)$ MeV), have mostly a collective nature. But the character and structure of this «collectivity» cannot be interpreted unambiguously. The great number of $0^+, 1^+$ states, and the large static quadrupole moments in «spherical» nuclei, are not the only examples which cannot find any satisfactory explanation within the framework of the traditional models.

In fact, all of our phenomenological models are based on the two simplest models: namely, the rigid rotator and the harmonic vibrator.

One can only conditionally speak of harmonic vibrations in nuclei. This model allows only a very rough description of experimental trends for the first one or two excited states, and one could hardly speak of well-defined vibrational bands.

As for the rotational model, the situation is much more satisfactory. The predictions of the model are rather well confirmed for «deformed» nuclei. But even in this region we are facing essential deviations from an ideal model.

This becomes more evident as new data are collected and as the experimental accuracy is increased.

Vibrations and rotations are only two ideal limiting cases. The structure of the real nuclear excitations are more complex and very likely to be of some intermediate structure between vibrations and rotations. Thus the search for new phenomenological models is going today along the path either of generalization of the ideal models or of a construction of some interpolation formula.

Let's consider as an example some modifications of the rigid rotor model. It is well known that for not very high angular momenta good approximations to experimental data are achieved with the series

$$(1.1) \quad E(I) = AI(I+1) + BI^2(I+1)^2 + CI^3(I+1)^3 + \dots$$

But discrepancy grows when higher angular momenta come into play and more sophisticated formulae for the energy levels $E(I)$ are needed. Several more or less successful approximations are known [1]. It is very convenient to represent their results in terms of a variable moment of inertia (VMI) [2]. We write down the energy as

$$(1.2) \quad E(I) = I(I+1)/2\mathcal{J} + \mathcal{V}(\mathcal{J})$$

and assume the moment of inertia \mathcal{J} to be determined from the extremum condition

$$(1.3) \quad \left(\frac{\partial E}{\partial \mathcal{J}} \right)_{I=\text{const}} = 0.$$

The function $\mathcal{V}(\mathcal{J})$ is to be chosen and parametrized to fit experimental data. We restrict ourselves to the two-parameter approximations

$$(1.4) \quad \begin{cases} a) & \mathcal{V}(\mathcal{J}) = \frac{1}{2\mathcal{J}_0 k} \left(\frac{\mathcal{J}}{\mathcal{J}_0} - 1 \right)^2 & \text{(Harris),} \\ b) & \mathcal{V}(\mathcal{J}) = \frac{1}{2\mathcal{J} k} \left(\frac{\mathcal{J}}{\mathcal{J}_0} - 1 \right)^2 & \text{(Holmberg, Lipas).} \end{cases}$$

For small values of the parameter k ($kI(I+1) \ll 1$) eqs. *a*) and *b*) are equivalent:

$$(1.5) \quad \mathcal{J}(I) \approx \mathcal{J}_0 + \frac{1}{2} k I(I+1).$$

In the opposite limiting case ($kI(I+1) \gg 1$) eqs. *a*) and *b*) lead to different asymptotic behaviour, namely

$$(1.6) \quad a) \quad \mathcal{J} \propto I^{\frac{3}{2}}, \quad b) \quad \mathcal{J} \propto I.$$

Experimental data available today do not allow one to make an unambiguous choice between the two formulae, but the «oscillatorlike» asymptotic relation

$$E \propto I, \quad \mathcal{J} \propto I$$

seems to be more favourable (see Prof. Kienle's lectures).

Now, it is a good place to make one general remark concerning phenomenological models. If one has in mind these models as the only goal to fit experimental data, unlimited possibility for speculation becomes open. In fact, the models must not be in contradiction to the consequences of more basic and fundamental microscopic models, which comprise much broader sets of phenomena. Two examples for illustration are:

1) A rigid static γ -deformation was postulated in the Davydov-Filippov model. But all attempts to find a microscopic confirmation of this assumption failed. Calculations gave too soft γ -deformations if any.

2) Different modifications of the original Bohr-Mottelson rotation-vibrational Hamiltonian have been used with hydrodynamical expressions for inertial parameters. On the other side, realistic microscopic calculations give inertial parameters which have nothing to do with hydrodynamical form.

In this sense the VMI model is not at all a model since it does not contain any cause of variation of the moment of inertia. It is rather an empirical formula for the interpretation of experimental data.

The microscopic calculation of the moment of inertia \mathcal{J} is possible within the framework of the cranking model. In this model one considers rotation induced from the outside with a definite angular velocity Ω . Therefore this model defines in fact the function $\mathcal{J}(\Omega)$. The angular momentum I appears only at the last stage, in the relation

$$(1.7) \quad I = \Omega \mathcal{J}(\Omega).$$

It is worth-while to stress that the asymptotic solution $E \propto I$, $\mathcal{J} \propto I$ (and correspondingly $\Omega \approx \Omega_c = \text{const}$ (*)) can be consistent with the cranking model relation (1.7) only if the function $\mathcal{J}(\Omega)$ has a singularity for Ω near Ω_c , e.g.,

$$\mathcal{J}(\Omega) \sim \mathcal{J}_0 \frac{\Omega_c^2}{\Omega^2 - \Omega_c^2} \quad \text{and} \quad \Omega^2(I) \underset{I \rightarrow \infty}{\approx} \Omega_c^2 \left(1 + \frac{\mathcal{J}_0}{I} \right).$$

Whether this is an indication of an inconsistency in the usually made expansion in Ω when solving the cranking-model equation, or of an inconsistency in the model itself, remains to be investigated.

2. - Microscopic models.

The more profound regularities of the collective states and their nucleon structure can be found only within the framework of microscopic models.

(*) Interrelations among the quantities E , \mathcal{J} , Ω and angular momentum I are determined by the general equations $\Omega(I) = dE(I)/dI = I/\mathcal{J}$.

The starting postulate of the models is the following. Nucleons move in a potential well (its self-consistency is assumed but not always proved explicitly) and interact via residual (effective) forces. *The final goal* is to find collective types of excitations in this system and determine their characteristics.

Every particular realization of the microscopic model has at least two weak points:

- a) choice of the initial effective interaction and
- b) validity of the approximations made.

Correspondingly, there are two ways of improving the model: to vary and correct the type of interaction (unfortunately, it is often done by just increasing the number of free parameters), or to correct the scheme of calculations (usually by expansion and subsequent complication).

A few words concerning the choice of effective interaction. We do not know the effective interaction in every detail but we do know parts of it. First of all we believe that there is some type of quadrupole-quadrupole interaction. The existence of deformed nuclei, the low-lying 2^+ -states and some other phenomena are their indications.

Then, we have an evidence of pairing forces which manifest themselves in odd-even binding energy differences, energy gap in excitation spectra of even-even nuclei, etc.

But the pairing and quadrupole forces alone cannot explain even qualitatively the whole variety of collective excitations in real nuclei.

So, we face the problem of how to search for additional parts of the effective interaction without so manysided promptings from nature as in the case of pairing and quadrupole forces. One has to realize that to put simply an additional term in the effective interaction (usually in a separable form with free parameters) and to fit restricted experimental data of a collective state is a very simple but rather dangerous way to solve the problem. One should always be sure that the choice made is consistent with more fundamental facts and laws; and, even better, to try to use these «first principles» as a search tool. It is useful to give a few examples for illustration.

i) The simplified version of pairing forces usually used is not consistent with general principles. It takes into account only matrix elements of a very specific form, namely

$$\langle \nu, \tilde{\nu} | V | \nu', \tilde{\nu}' \rangle,$$

where ν and $\tilde{\nu}$ are time-conjugated single-particle states (*).

(*) In infinite media it corresponds to the interaction matrix elements $\langle \mathbf{p}, -\mathbf{p} | V | \mathbf{p}', -\mathbf{p}' \rangle$. It means that a pair of nucleons interact only if their total momentum equals zero. When the pair starts to move as a whole the interaction is switched off.

Such an interaction violates gauge invariance. But it turns out that starting from the given pairing matrix elements one can complete the interaction by additional elements exploiting just the principle of gauge invariance without an introduction of any new parameters. The additional terms in the interaction generate a new branch of excitations, «pairing fluctuations» [3]. From the general point of view this is an example of the «Goldstone theorem» which states that when a symmetry is broken in the process of formation of single-particle excitations, the symmetry must be restored by the proper collective excitations. Therefore, from a knowledge of the single-particle structure (pairing forces) we can unambiguously predict the collective excitations («pairing fluctuations») needed to restore gauge invariance.

ii) Effective forces and self-consistent potentials have the same origin, namely, the initial nucleon-nucleon interaction. On the other hand, from the observed single-particle spectra we know many characteristics of the nuclear potential, one of them being a spin-orbit splitting which can be described as

$$(2.1) \quad \Delta\varepsilon \propto (\mathbf{l} \cdot \mathbf{s}) \frac{1}{r} \frac{dV}{dr}.$$

It is very natural to assume that (2.1) is an image of the corresponding term in the two-nucleon effective interaction, say of the form

$$(2.2) \quad V_{ls} = -\frac{1}{2}k(\mathbf{s}_1 + \mathbf{s}_2) \cdot (\mathbf{p}_1 - \mathbf{p}_2) \times \nabla_1 \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

(\mathbf{r} , \mathbf{p} , \mathbf{s} are position, momentum and spin of the nucleon). One can easily see that eq. (2.2) leads to the self-consistent potential (2.1) and it fixes the coupling constant k completely. This allows one to use the term (2.2) in order to find the new collective excitations, «spin-orbit vibrations» [4].

Now, we shall come to the second «weak point» of microscopic models, namely, *the validity of the approximations made*.

Any microscopic model deals with the many-body problem and therefore needs some approximations. Numerous approximations used for treating collective excitations are in fact equivalent to the random-phase approximation (RPA). This can be confirmed to some extent for the excitations which are sufficiently close to harmonic vibrations. As for the more complicated cases the problem has to be investigated. The simple models which allow exact solutions are very useful for this purpose.

We consider *two simplified models* of nucleon pairing. The nucleons occupy two degenerate levels with the splitting ε (each with the same degeneracy 2Ω) and interact with pairing forces.

The first model is $T_z = |1|$ (*i.e.* pairing isotopic-noninvariant) with the Hamiltonian

$$H_p = -GP^\dagger P, \quad P = \sum (a_1 a_1 + a_2 a_2).$$

We restrict ourselves to the most interesting case, when the number of nucleons $N=2\Omega$. Then in the absence of interaction ($G=0$) the lower level is fully occupied and the upper one is completely empty. The excitations of the system occur when two nucleons jump from the lower to the upper level (*) with the energy increase $\Delta W=2\varepsilon$. As the pairing constant G increases these two-nucleon excitations become more and more collective (« pairing vibrations »). At the critical point where $G=G_c \equiv \varepsilon/2\Omega$ the permanent pairing becomes advantageous (« pairing deformation »).

It is very instructive to compare this « phase transition » in exact and approximate solutions. The results for the first two excited states are shown in Fig. 1 (extracted from [5]), which indicates that RPA gives satisfactory

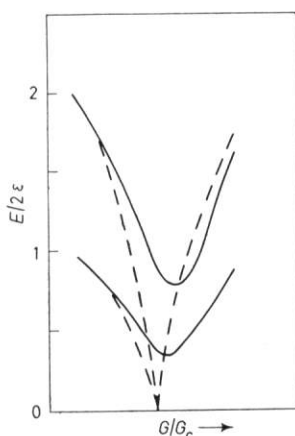


Fig. 1. - Comparison of the exact solution and RPA for the two first excited states in the two-level model with $|T_z|=1$ pairing (from ref. [5]): — exact, - - - RPA.

results for $G/G_c \ll 1$ as well as for $G/G_c \gg 1$ but completely fails near the $G/G_c = 1$ region.

Now let us consider a model with isotopic-invariant $T=1$ pairing with the Hamiltonian

$$H = -G \sum_{\tau=0,\pm 1} P_{\tau}^{\dagger} P_{\tau},$$

where P_{τ} corresponds to a nucleon pair with isotope-spin projection τ . The excitation spectrum for this case (which is represented in Fig. 2 [6]) is more complicated. Being pure vibrational near $G=0$ it turns into the set of isotopic « rotational bands » ($W \propto T(T+1)$) for the limit $G/G_c \gg 1$. So, in this

(*) We are not interested here in odd-nucleon transitions.

model one can test the applicability of the RPA not only for vibrationlike excitations but for the rotational ones as well.

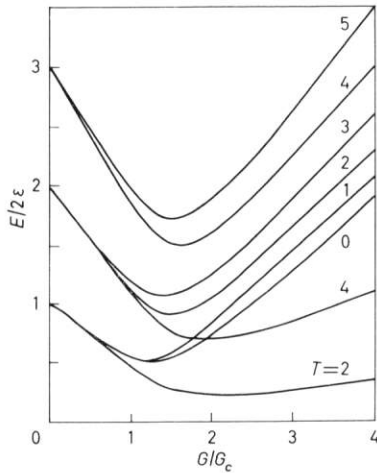


Fig. 2.

Fig. 2. - The level structure of exact solution in the two-level model with $T=1$ pairing (from ref. [6]).

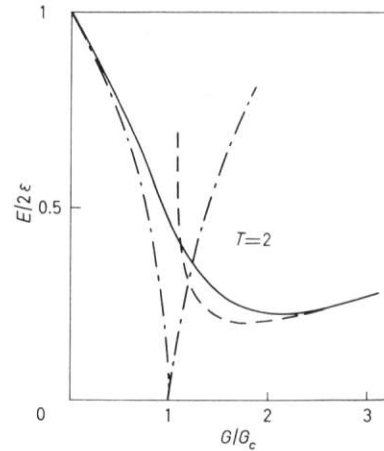


Fig. 3.

Fig. 3. - Comparison of the exact solution for the first excited state (one of those plotted in Fig. 2 with the results of the cranking model) and RPA (from ref. [6]): — exact, - - - cranking model, - · - · - RPA.

Fig. 3 shows the energy of the first excited state ($T=2$) given by exact solution and RPA (built on a « spherical » as well as a « deformed » basis). It is seen that RPA is justified only in the case $G/G_c \ll 1$ and completely fails in describing the « rotational » case. In the same plot the cranking-model result is also represented which gives a rather good description of the « rotational » limit but deviates strongly when coming nearer to the transition point.

One can hardly expect any constructive prescription from such simple models but two conclusions are evident:

- 1) RPA cannot be applied for a description of the rotational states,
- 2) neither RPA nor the cranking model can be effective in the transition region.

From the first view the method for the proper description of the transition region can be obtained by making a marriage of the RPA and the cranking model.

The union (although put into practice as projection methods) can hardly be fruitful as the cranking model is too inconsistent to be a good starting point in breeding new microscopic methods. This point needs to be investigated in more detail.

3. - Equations of motion for pair operators.

Pair operators are the fundamental quantities in the microscopic theory, the quadrupole moment Q_μ being such an example:

$$(3.1) \quad Q_\mu = \sum_{1,2} \langle 1 | q_\mu | 2 \rangle a_1^\dagger a_2.$$

Here a_1, a_1^\dagger are the destruction and creation operators of a particle in state $|1\rangle$. Later, we shall consider the corresponding time-conjugated state $|\tilde{1}\rangle$.

In the spherical quantum number representation (with definite angular momentum j , its projection m , and additional quantum numbers γ)

$$(3.2) \quad \begin{cases} |1\rangle = |\gamma_1 j_1 m_1\rangle, \\ |\tilde{1}\rangle = (-1)^{j_1 - m_1} |\gamma_1 j_1 - m_1\rangle. \end{cases}$$

Note that under two-fold T -conjugation:

$$(3.2') \quad |\tilde{\tilde{1}}\rangle = -|1\rangle, \quad \tilde{\tilde{a}}_1 = -a_1.$$

Condensing notation, we rewrite (3.1) in the form of the trace over single-particle states:

$$Q_\mu = - \sum_{1,2} \langle 1 | q_\mu | 2 \rangle a_2 a_1^\dagger = - \text{Tr} (q_\mu a a^\dagger),$$

where aa^\dagger is understood to be

$$a a^\dagger = \sum_{1,2} |1\rangle a_1 a_2^\dagger \langle 2|.$$

In addition to operators of the Q -type (« particle-hole » pair operator), we need another kind of operator (« particle-particle » and « hole-hole » type) in order to treat pairing phenomena:

$$P = - \sum_{1,2} \langle 1 | p | 2 \rangle a_2 a_1 = - \text{Tr} (p a \tilde{a}).$$

In order to condense our expressions we introduce the following *matrix*

notation ($\alpha = 1, 2$):

$$(3.3) \quad \begin{cases} \Psi_\alpha = \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} = \sum_{\mathbf{1}} |1\rangle \begin{pmatrix} a_{\mathbf{1}} \\ a_{\mathbf{1}}^\dagger \end{pmatrix} = \sum_{\mathbf{1}} |1\rangle \Psi_\alpha(1), \\ \Psi_\alpha^\dagger = (a^\dagger, \tilde{a}) = \sum_{\mathbf{1}} (a_{\mathbf{1}}^\dagger, a_{\mathbf{1}}) \langle 1| = \sum_{\mathbf{1}} \Psi_\alpha^\dagger(1) \langle 1|. \end{cases}$$

In matrix notation, an arbitrary pair operator can be written as

$$(3.4) \quad X = -\frac{1}{2} \text{Tr}(\hat{x} \Psi \Psi^\dagger),$$

where the trace is taken over single-particle states as well as over the matrix indices $\alpha, \beta = 1, 2$:

$$\sum_{\mathbf{1}, \mathbf{2}; \alpha, \beta} \langle 1\alpha | \hat{x} | 2\beta \rangle \langle 2\beta | \Psi \Psi^\dagger | 1\alpha \rangle \equiv \sum_{\mathbf{1}, \mathbf{2}; \alpha, \beta} \hat{x}_{\alpha\beta}(\mathbf{1}, \mathbf{2}) \Psi_\beta(2) \Psi_\alpha^\dagger(1)$$

(later, we assume repeated indices are summed).

Note. — A transformation from operators a to Ψ is ambiguous because of the evident relations

$$(3.5) \quad \begin{cases} \Psi_\alpha(1) = \Psi_\beta^\dagger(\tilde{1}) i\tau_{\beta\alpha}^2, \\ \Psi_\alpha^\dagger(1) = -i\tau_{\alpha\beta}^2 \Psi_\beta(\tilde{1}) \end{cases}$$

(τ^1, τ^2, τ^3 are the Pauli matrices). Using (3.5) we can write the trace (3.4) in two equivalent forms:

$$(3.6) \quad \begin{aligned} \text{Tr}(\hat{x} \Psi \Psi^\dagger) &= \hat{x}_{\beta\alpha}(21) \Psi_\alpha(1) \Psi_\beta^\dagger(2) = \\ &= -\hat{x}_{\beta\alpha}(\tilde{2}\tilde{1}) \tau_{\alpha'\alpha}^2 \tau_{\beta\beta'}^2 \Psi_{\beta'}(2) \Psi_{\alpha'}^\dagger(1) + \hat{x}_{\alpha\alpha}(11). \end{aligned}$$

It is convenient to eliminate the ambiguity by always defining \hat{x} invariantly with respect to the transformation (3.5), *i.e.* by taking the half-sum of the two equivalent right-hand sides of equation (3.6). Then, the new \hat{x} defined as

$$\hat{x}_{\alpha\beta}(12)_{\text{new}} = \frac{1}{2} (\hat{x}_{\alpha\beta}(12) - \tau_{\beta\beta'}^2 \hat{x}_{\beta'\alpha'}(\tilde{2}\tilde{1}) \tau_{\alpha'\alpha}^2)_{\text{old}}$$

has the symmetry property

$$(3.7) \quad \hat{x}_{\alpha\beta}(12) = -\tau_{\beta\beta'}^2 \hat{x}_{\beta'\alpha'}(\tilde{2}\tilde{1}) \tau_{\alpha'\alpha}^2$$

With this convention any pair operator X can be unambiguously written as

$$(3.4') \quad X = -\frac{1}{2} \text{Tr}(\hat{x} \Psi \Psi^\dagger) + (\text{c-number term}).$$

The expression (3.4') is specified below for some physical operators:

Quadrupole moment

$$(3.8a) \quad E_\mu: \quad \hat{q}_\mu = \tau^3 q_\mu .$$

Pairing operator

$$(3.8b) \quad P \equiv \Delta/G = - \sum_i a_i a_i^\dagger: \quad \hat{p} = \tau^1 - i\tau^2 .$$

Angular momentum operator

$$(3.8c) \quad J_\mu: \quad j = \tau^0 j_\mu = j_\mu .$$

Particle-Number operator

$$(3.8d) \quad N = \sum_i a_i^\dagger a_i = \frac{1}{4} \text{Tr} (1) - \frac{1}{2} \text{Tr} (\tau^3 \Psi \Psi^\dagger): \quad \hat{n} = \tau^3 .$$

Commutation relations.

From the definition (3.3) it follows

$$(3.9) \quad \begin{cases} \Psi_\alpha(1) \Psi_\beta^\dagger(2) + \Psi_\beta^\dagger(2) \Psi_\alpha(1) = \delta_{\alpha\beta} \delta_{12} , \\ \Psi_\alpha(1) \Psi_\beta(2) + \Psi_\beta(2) \Psi_\alpha(1) = i\tau_{\alpha\beta}^2 \delta_{1\bar{2}} = i\tau_{\beta\alpha}^2 \delta_{2\bar{1}} . \end{cases}$$

(Note. — In accordance with (3.2') $\delta_{1\bar{2}} = -\delta_{\bar{2}1}$). Using eq. (3.9) one can obtain convenient commutation relations of Ψ and Ψ^\dagger with a pair operator (3.4')

$$(3.9') \quad \begin{cases} [\Psi_\alpha(1), \frac{1}{2} \text{Tr} (\hat{x} \Psi \Psi^\dagger)] = -\hat{x}_{\alpha\beta}(1, 2) \Psi_\beta(2) , \\ [\Psi_\alpha^\dagger(1), \frac{1}{2} \text{Tr} (\hat{x} \Psi \Psi^\dagger)] = \Psi_\alpha^\dagger(2) \hat{x}(2, 1) . \end{cases}$$

As a consequence one obtains

$$(3.10) \quad [\Psi \Psi^\dagger, \frac{1}{2} \text{Tr} (\hat{x} \Psi \Psi^\dagger)] = [\Psi \Psi^\dagger, \hat{x}] ,$$

where the right-hand side (or rather its matrix element $\langle 1\alpha | \dots | 2\beta \rangle$) has the following meaning:

$$\Psi_\alpha(1) \Psi_\gamma^\dagger(3) \hat{x}_{\gamma\beta}(3, 2) - \hat{x}_{\alpha\gamma}(1, 3) \Psi_\gamma(3) \Psi_\beta^\dagger(2) .$$

Hamiltonian. — For simplicity we consider separable forces when the interaction Hamiltonian can be represented as a bilinear combination of pair oper-

ators:

$$(3.11) \quad H = H_0 + \frac{1}{2} \sum_{\sigma} g_{\sigma} \bar{X}^{\sigma} X^{\sigma},$$

where

$$H_0 = -\frac{1}{2} \text{Tr} (\hat{\varepsilon}^0 \Psi \Psi^{\dagger}), \quad \hat{\varepsilon}^0 = \tau^3 \varepsilon^0.$$

(Here \bar{X}^{σ} and X^{σ} are pair operators of type (3.4) and the g_{σ} are the coupling constants.)

Equations of motion. - Using equations (3.10) and (3.11) we obtain

$$(3.12) \quad [\Psi \Psi^{\dagger}, H] = [\hat{\varepsilon}^0, \Psi \Psi^{\dagger}] - \frac{1}{4} \sum_{\sigma} g_{\sigma} \text{Tr} (\hat{x}^{\sigma} \Psi \Psi^{\dagger}) [\hat{x}^{\sigma}, \Psi \Psi^{\dagger}] - \\ - \frac{1}{4} \sum_{\sigma} g_{\sigma} [\hat{x}^{\sigma}, \Psi \Psi^{\dagger}] \text{Tr} (\hat{x}^{\sigma} \Psi \Psi^{\dagger}).$$

If we further neglect the commutators of X^{σ} with $\Psi \Psi^{\dagger}$ (they result in a non essential renormalization of $\hat{\varepsilon}$ only), eq. (3.12) can be written as

$$(3.13) \quad [\Psi \Psi^{\dagger}, H] = [S^{\text{op}}, \Psi \Psi^{\dagger}],$$

where

$$(3.14) \quad S^{\text{op}} = \hat{\varepsilon}^0 - \frac{1}{4} \sum_{\sigma} \hat{x}^{\sigma} g_{\sigma} \text{Tr} (\hat{x}^{\sigma} \Psi \Psi^{\dagger}) - \frac{1}{4} \sum_{\sigma} \hat{x}^{\sigma} g_{\sigma} \text{Tr} (\hat{x}^{\sigma} \Psi \Psi^{\dagger}).$$

The case of pairing and quadrupole interaction. - The corresponding interaction Hamiltonian in Eq. (3.11) has the form

$$\frac{1}{2} \sum_{\sigma} g_{\sigma} \bar{X}^{\sigma} X^{\sigma} \rightarrow -\frac{1}{2} \varkappa \sum_{\mu} Q_{\mu}^{\dagger} Q_{\mu} - \frac{1}{2} G P^{\dagger} P,$$

hence

$$S^{\text{op}} = \hat{\varepsilon}^0 + \frac{1}{2} \varkappa \sum_{\mu} \hat{q}_{\mu}^{\dagger} \text{Tr} (\hat{q}_{\mu} \Psi \Psi^{\dagger}) + \frac{1}{4} G \hat{p}^{\dagger} \text{Tr} (\hat{p} \Psi \Psi^{\dagger}) + \frac{1}{4} G \hat{p} \text{Tr} (\hat{p}^{\dagger} \Psi \Psi^{\dagger}).$$

Using eq. (3.8) one can write this as

$$(3.15) \quad S^{\text{op}} = \varepsilon \tau^3 - \Delta^{(+)} \tau^1 - \Delta^{(-)} \tau^2,$$

where

$$(3.16) \quad \left\{ \begin{array}{l} \varepsilon = \varepsilon^0 - \varkappa \sum_{\mu} \hat{q}_{\mu}^{\dagger} Q_{\mu}, \quad Q_{\mu} = -\frac{1}{2} \text{Tr} (\tau^3 q_{\mu} \Psi \Psi^{\dagger}), \\ \Delta^{(+)} = G P^{(+)} = -\frac{1}{2} G \text{Tr} (\tau^1 \Psi \Psi^{\dagger}) = \frac{1}{2} (\Delta + \Delta^{\dagger}), \\ \Delta^{(-)} = G P^{(-)} = +\frac{1}{2} G \text{Tr} (i \tau^2 \Psi \Psi^{\dagger}) = \frac{1}{2} (\Delta - \Delta^{\dagger}). \end{array} \right.$$

In explicit matrix notation, eq. (3.15) has the form

$$(3.17) \quad S^{\text{op}} = \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta^\dagger & -\varepsilon \end{pmatrix}$$

and the equation of motion (3.13) is equivalent to four equations following from:

$$(3.18) \quad \left[\begin{pmatrix} a & a^\dagger \\ \tilde{a}^\dagger & \tilde{a} \end{pmatrix}, H \right] = \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta^\dagger & -\varepsilon \end{pmatrix} \begin{pmatrix} a & a^\dagger \\ \tilde{a}^\dagger & \tilde{a} \end{pmatrix} - \\ - \begin{pmatrix} a & a^\dagger \\ \tilde{a}^\dagger & \tilde{a} \end{pmatrix} \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta^\dagger & -\varepsilon \end{pmatrix}.$$

4. - Methods of approximation for equations of motion: Hartree-Fock approximation.

The eq. (3.13)

$$(4.1) \quad [\Psi_\alpha(1)\Psi_\beta^\dagger(2), H] = S_{\alpha\gamma}^{\text{op}}(1, 3)\Psi_\gamma(3)\Psi_\beta^\dagger(2) - \Psi_\alpha(1)\Psi_\gamma^\dagger(3)S_{\gamma\beta}^{\text{op}}(3, 2)$$

is the operator relation containing the destruction and creation operators of nucleons. Let $|w\rangle$ be the exact states of a nucleus (eigenfunctions of H):

$$(4.2) \quad H|w\rangle = W|w\rangle.$$

Upon taking the matrix element of (4.1) over the ground state of the nucleus $|0\rangle$, the left-hand side vanishes, and for the right-hand side we make the following *fundamental assumption*:

$$(4.3) \quad \langle 0|S^{\text{op}}\Psi\Psi^\dagger|0\rangle = \sum_w \langle 0|S^{\text{op}}|w\rangle \langle w|\Psi\Psi^\dagger|0\rangle \approx \langle 0|S^{\text{op}}|0\rangle \langle 0|\Psi\Psi^\dagger|0\rangle,$$

which is justified if the nondiagonal matrix elements $\langle 0|S|w\rangle$ are small compared to the diagonal elements (moreover, they are not summed coherently in (4.3)).

Using approximation (4.3), eq. (4.1) takes the form

$$(4.4) \quad [S, R] = 0,$$

where

$$(4.4') \quad S_{\alpha\beta}(1, 2) = \langle 0|S_{\alpha\beta}^{\text{op}}(1, 2)|0\rangle \quad \text{and} \quad R_{\alpha\beta}(1, 2) = \langle 0|\Psi_\alpha(1)\Psi_\beta^\dagger(2)|0\rangle.$$

Equation (4.4) defines the single-particle density matrix $R_{\alpha\beta}(1, 2)$. The quantity S plays the role of the single-particle Hamiltonian, and has a self-consistent nature due to its feedback dependence on R (through Q_μ and Δ).

The natural way of solving eq. (4.4) is as follows: Let $\varphi^{(\lambda)}$ and E_λ be the eigenfunctions and corresponding eigenvalues of the Hermitian operator S

$$(4.5) \quad \begin{cases} S_{\alpha\beta}(12) \varphi_\beta^{(\lambda)}(2) = \varphi_\alpha^{(\lambda)}(1) E_\lambda, \\ \varphi_\beta^{+(\lambda)}(2) S_{\beta\alpha}(21) = E_\lambda \varphi_\alpha^{+(\lambda)}(1) \end{cases}$$

(or symbolically $S\varphi^{(\lambda)} = \varphi^{(\lambda)}E_\lambda$; $\varphi^{+(\lambda)}S = E_\lambda\varphi^{+(\lambda)}$).

Then the combination

$$(4.6) \quad R_{\alpha\beta}(12) = \sum_\lambda \varphi_\alpha^{(\lambda)}(1) \vartheta_\lambda \varphi_\beta^{+(\lambda)}(2)$$

satisfies eq. (4.4) for any set of numbers ϑ_λ . The quantities ϑ_λ have the meaning of particle occupation numbers, *i.e.* the probabilities to find the corresponding level in the self-consistent well occupied by a particle (or rather by a quasi-particle).

If we consider « pure configurations » only then we have to put all ϑ_λ equal to 1 or 0. (The pure configurations with Slater determinants as wave functions proved to be the most stable solutions of the HF equation. They can also be obtained from the variational principle.) This is equivalent to the following condition imposed upon the density matrix:

$$(4.7) \quad R^2 = R$$

(which is evidently valid for any matrix R with eigenvalues 1 or 0 only).

Thus, eq. (4.4) allows a variety of solutions which is connected, in fact, with the freedom of the choice of the state $|0\rangle$. The additional condition (4.7) restricts the choice of possible states only to those with « pure configurations ».

To extract the real ground state of the nucleus one should chose a specific set of ϑ_λ ($= 0, 1$) describing a Fermi-sea completely filled.

Random phase approximation (RPA). – In the Hartree-Fock method the interaction between nucleons is accounted for only in the self-consistent potential. The nucleon motion inside the potential well is assumed to be completely independent and uncorrelated.

In fact, to describe the nucleon-nucleon dynamical correlation, a single particle density matrix R is insufficient and higher (correlational) matrices should be considered, *e.g.*

$$(4.8) \quad \langle 0 | \Psi \Psi^\dagger \Psi \Psi^\dagger | 0 \rangle = \sum_v \langle 0 | \Psi \Psi^\dagger | v \rangle \langle v | \Psi \Psi^\dagger | 0 \rangle.$$

Usually we can distinguish two different kinds of excited states $|v\rangle$, those where the correlation in nucleon motion is weak (single-particle states $|v_{s.p.}\rangle$) and those where the correlation is comparatively strong. The latter are designated as collective states $|v_{coll}\rangle$. Naturally, this distinction is only qualitative. Semi-quantitative estimates can be obtained as follows.

Suppose we have found a pair operator, say Y , such that the matrix element from the ground state into the collective state under consideration is large as compared to all other transitions

$$(4.9) \quad \langle v_{coll} | Y | 0 \rangle \gg \langle v_{s.p.} | Y | 0 \rangle .$$

Then the ratio of these two matrix elements gives some indication of the « degree of collectivity » for $|v_{coll}\rangle$.

In the case when correlations are strong enough to provide collective behaviour, but not so strong as to be compared with the « ground state correlation » (*i.e.* a correlation already taken into account in the self-consistent field) one can make use of the corresponding inequality

$$(4.10) \quad \langle v_{coll} | Y | 0 \rangle \ll \langle 0 | Y | 0 \rangle .$$

Then, one may linearize the equation of motion for the nondiagonal elements $\langle v_{coll} | \Psi \Psi^\dagger | 0 \rangle$ and use the Hartree-Fock results for the diagonal elements

$$\langle 0 | \Psi \Psi^\dagger | 0 \rangle \approx \langle v_{coll} | \Psi \Psi^\dagger | v_{coll} \rangle .$$

This is the main idea of the random phase approximation (RPA).

In the case of strong correlations when the inequality (4.10) is violated, the problem becomes very complicated. However in two important cases the approximate solution may be obtained with the help of the generalization of the Hartree-Fock method.

Hartree-Fock-Bogoliubov approximation. Cranking model. — The pairing effect of nucleons is related to large nondiagonal elements of the operators Δ , Δ^\dagger (see eq. (3.8b) and (3.16) between the ground states of nuclei differing by two nucleons, *i.e.*

$$(4.11) \quad \langle N | \Delta | N+2 \rangle, \quad \langle N+2 | \Delta^\dagger | N \rangle .$$

If states $|w\rangle$ considered in the Hartree-Fock method have a strictly determined number of nucleons, then the quantities (4.11) should be neglected, if only the diagonal averaged values are considered. However one may deviate from the strict conservation of nucleon number and assume that $|w\rangle$ is the superposition of states with a different number of particles. Then we can admit the nonvanishing diagonal elements $\langle w | \Delta | w \rangle$. The states $|w\rangle$ now can

be characterized by the average nucleon number which can be controlled by the additional term in the Hamiltonian $-\lambda N$ with the Lagrangian factor λ (chemical potential).

Formally the case of rotational states is similar. In this case the quadrupole moment has large nondiagonal elements (of the order of diagonal) between the states of the same rotational band ($J|Q|J'$). We can include them in our considerations if we permit the violation of angular momentum conservation, and assume $|w\rangle$ to be a superposition of the states with different J . The average quantity J is restricted by the additional term $-\Omega J$ in the Hamiltonian, where the Lagrangian factor Ω now has the meaning of the rotational angular velocity.

One can summarize the technical essence of the methods (which are known as Hartree-Fock-Bogoliubov in the case of pairing and cranking model in the case of rotation) described above as follows.

Instead of looking for the eigenstates of the original Hamiltonian H which have definite values N or J :

$$(4.12) \quad \begin{cases} H|N\rangle = W(N)|N\rangle, \\ H|J\rangle = W(J)|J\rangle, \end{cases}$$

one looks for the ground state of a modified Hamiltonian

$$(4.13) \quad \begin{cases} (H - \lambda N)|\lambda\rangle = W(\lambda)|\lambda\rangle & \text{(pairing)}, \\ (H - \Omega J)|\Omega\rangle = W(\Omega)|\Omega\rangle & \text{(rotation)}. \end{cases}$$

The parameters λ or Ω have to be chosen in order to ensure the correct average values of N or J :

$$(4.14) \quad \begin{cases} \langle \lambda | N_{op} | \lambda \rangle = N & \text{(pairing)}, \\ \langle \lambda | J_{op} | \lambda \rangle = J & \text{(rotation)}. \end{cases}$$

The methods considered permit an approximate description of pairing and rotation but suffer from a violation of the conservation laws. The methods themselves are rather simple and give satisfactory results in many cases but the accuracy of the approximations is rather difficult to estimate and numerous suggestions for improvement of the methods (mostly by some projection procedure) are very complicated and not very consistent.

5. - Density matrix in collective space.

To describe pairing and rotations more accurately (without violation of the conservation laws) one must consider simultaneously the whole band of states with fixed N or J . In the space of these states we define a generalized

density matrix (cf. (4.4')). In the case of pairing

$$(5.1) \quad (N|\Psi_\alpha(v)\Psi_{\alpha'}^\dagger(v')|N') \equiv \langle N, v\alpha | R | N', v'\alpha' \rangle.$$

In the case of rotations the matrix is defined in the space of the rotational band $|JM\rangle$ or more generally in the group of states $|KJM\rangle$, *i.e.*

$$(5.2) \quad (KJM|\Psi_\alpha(v)\Psi_{\alpha'}^\dagger(v')|K'J'M') \equiv \langle KJM, v\alpha | R | K'J'M', v'\alpha' \rangle.$$

The generalization in eqs. (5.1) and (5.2) as compared with (4.4') consists in the following. R is now the matrix not only in the «inner» (single-particle) space but in the «outer» (collective) one as well. The expressions on the right-hand sides of (5.1) and (5.2) emphasize the formal equivalence of these spaces.

From (3.13) we obtain an equation for R restricting ourselves only to the states which belong to the singled out «collective» space

$$(5.3) \quad [R, H] = [S, R],$$

where all the quantities are matrices in the combined space, for example

$$SR \rightarrow \sum_{N_1 v_1 \alpha_1} \langle N v \alpha | S | N_1 v_1 \alpha_1 \rangle \langle N_1 v_1 \alpha_1 | R | N' v' \alpha' \rangle.$$

The matrix elements of the Hamiltonian H depend naturally only on the collective quantum numbers

$$(5.4) \quad \begin{cases} \langle N v \alpha | H | N' v' \alpha' \rangle = \delta_{vv'} \delta_{\alpha\alpha'} \langle N | H | N' \rangle = \delta_{vv'} \delta_{\alpha\alpha'} \delta_{NN'} H(N), \\ \langle KJM v \alpha | H | K'J'M' v' \alpha' \rangle = \delta_{vv'} \delta_{\alpha\alpha'} \delta_{JJ'} \delta_{MM'} \langle KJ | H | K'J' \rangle, \end{cases}$$

so the quantity H in (5.3) plays the role of a collective Hamiltonian which describes the collective band in the usual sense (the eigenstates $|w\rangle$ and eigenenergies of H are the states and energies of the band under consideration). The eq. (5.3) can be rewritten

$$(5.3') \quad [S + H, R] = 0,$$

where the sum $S + H$ has the meaning of a complete Hamiltonian containing an «intrinsic» part S and a collective part H . The fact that the intrinsic and collective «degrees of freedom» are coupled manifests itself in the dependence of the intrinsic part S on the collective variables.

In addition to (5.3) we shall assume that the condition (4.7) remains valid

$$(5.5) \quad R^2 = R$$

but in the combined space.

The eqs. (5.3) and (5.15) formally almost coincide with the equations for the usual density matrix. The generalization consists only in the broadening of the space. However this «only» leads to large complications.

Below we discuss one of the possible ways for the simplification of the problem.

Operators in collective space. — The collective space—a set of strongly connected states of the system—corresponds usually to a small number of degrees of freedom, and the appropriate variables and operators can be introduced explicitly. To make the presentation clearer we consider below the cases of pairing and rotations simultaneously. For pairing the operator of the number of particles N and its conjugate phase ϕ can be chosen as collective variables. For rotation we choose Eulerian angles θ_i and usual angular-momentum operators in the laboratory J_μ and body-fixed $I_\lambda = \mathbf{n}_\lambda \cdot \mathbf{J}$ frames.

$$(5.6) \quad \begin{array}{cc} \textit{Pairing} & \textit{Rotation} \\ \phi; N = -i \frac{\partial}{\partial \phi}, & \theta_i; J_\mu, I_\lambda. \end{array}$$

Note. — These operators are formally introduced in the outer «collective» space and have the meaning only as matrix elements between the states $(N|\dots|N')$ and $(KJM|\dots|K'J'M')$. In particular, one must not mix N, J_μ with the corresponding single-particle operators (3.8).

The single-particle Hamiltonian S (which contains Δ and Q_μ) is not diagonal in the collective space, *i.e.* it does not commute with N and J_μ . However, there is an operator in the combined (single-particle + collective) space commuting with S .

In our cases

$$(5.7) \quad [S, N + \tau^3]_- = 0, \quad [S, J_\mu + j_\mu] = 0.$$

For rotations this results is evident: the simultaneous rotation both of the «inner» (single-particle) space and of the «outer» one cannot change S . For pairing (5.7) can be checked directly. From (3.17) we find

$$\begin{aligned} \langle N'|[S, N]|N\rangle &= (N - N') \langle N'|S|N\rangle = \\ &= (N - N') \begin{pmatrix} \varepsilon_N \delta_{NN'} & -\delta_{N', N-2} \langle N-2|\Delta|N\rangle \\ -\delta_{N', N+2} \langle N+2|\Delta^\dagger|N\rangle & -\varepsilon_N \delta_{N'N} \end{pmatrix} = \\ &= 2 \langle N'| \begin{pmatrix} 0 & -\Delta \\ \Delta^\dagger & 0 \end{pmatrix} |N\rangle. \end{aligned}$$

On the other hand

$$[S, \tau^3]_- = -2 \begin{pmatrix} 0 & -\Delta \\ \Delta^\dagger & 0 \end{pmatrix},$$

whence (5.7) follows.

Let D be such a unitary operator in the inner space that

$$(5.8) \quad D(N + \tau^3)D^{-1} = N, \quad D(J_\mu + j_\mu)D^{-1} = J_\mu.$$

It is evident that (5.8) is possible only when D depends on the collective variables. One can easily check the existence of these operators, namely

$$(5.9) \quad D(\phi) = \exp[i\phi\tau^3], \quad D(\theta_i) \rightarrow D_{mm'}^i(\theta_i)$$

($D(\theta_i)$ in $|jm\rangle$ representation coincides with the Wigner's D -function.)

After the unitary transformation (5.9) eqs. (5.3) and (5.5) take the form

$$(5.10) \quad [s + \hbar, r] = 0, \quad r^2 = r,$$

where

$$r = DRD^{-1}, \quad \hbar = DHD^{-1}, \quad s = DSD^{-1}.$$

Note that due to eqs. (5.7) and (5.8) we have

$$(5.11) \quad [s, N] = 0, \quad [s, J_\mu] = 0,$$

which means that s does not depend on phase ϕ (in the pairing case) and may depend only on the «intrinsic» angular momenta $I_\lambda = \mathbf{n}_\lambda(\theta_i)\mathbf{J}$ (the only combination of J_μ and θ_i commuting with J_μ) in the rotational case.

On the other hand, the «collective Hamiltonian» H (which by definition depends only on the collective variables N or \mathbf{I}) becomes after transformation dependent on intrinsic variables:

$$(5.12) \quad \begin{cases} \hbar = DH(N)D^{-1} = H(DND^{-1}) = H(N - \tau^3), \\ \hbar = DH(I_\lambda)D^{-1} = H(DI_\lambda D^{-1}) = H(I_\lambda - j_\lambda). \end{cases}$$

In the rotational case we have used the relation

$$(5.13) \quad DI_\lambda D^{-1} = I_\lambda - j_\lambda,$$

which can be proved by explicit calculation using the definitions of Wigner's D -function and I_λ .

For the case of pairing the dependence of h on intrinsic variables (represented in eq. (5.12) by the operator τ^3) is in fact very simple. Due to the properties of Pauli matrices one has for any function $H(N)$

$$(5.14) \quad h = H(N - \tau^3) \equiv h_0(N) - \tau^3 \lambda(N),$$

where

$$(5.14') \quad \begin{cases} h_0(N) = \frac{1}{2}\{H(N+1) + H(N-1)\}, \\ \lambda(N) = \frac{1}{2}\{H(N+1) - H(N-1)\}, \end{cases}$$

which indicates that $\lambda(N)$ is just by definition the chemical potential.

The transformed eqs. (5.10) are considerably simpler than the initial eqs. (5.3), (5.5) due to elimination of ϕ - and θ -dependence from r and s . But still collective and intrinsic variables are mixed. This is not only due to remaining N - and I_λ -dependence of s , r but also in addition, due to some complication of the collective Hamiltonian h as compared to H . To proceed further it is convenient to consider the pairing and rotational cases separately.

6. - Pairing problem (treatment with fixed particle number).

For the pairing case eqs. (5.10) contain only one collective operator, N . Therefore it can be considered as a c -number parameter. This fortunate fact simplifies the problem to a great extent and allows one to find an exact solution of eqs. (5.10) (in contrast to the rotational case, where eqs. (5.10) contain three noncommuting collective operators I_λ).

To get an idea of the structure of the Hamiltonian $s+h$ lets perform the transformation (5.9) explicitly. From (5.9) and (3.10) we easily obtain

$$(6.1) \quad s = \exp[i\phi\tau^3] \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta^\dagger & -\varepsilon \end{pmatrix} \exp[-i\phi\tau^3] = \begin{pmatrix} \varepsilon & -\bar{\Delta} \\ -\bar{\Delta}^\dagger & -\varepsilon \end{pmatrix},$$

where we have introduced

$$(6.2) \quad \bar{\Delta} = \exp[i\phi] \Delta \exp[i\phi], \quad \bar{\Delta}^\dagger = \exp[-i\phi] \Delta^\dagger \exp[-i\phi].$$

The quantities $\bar{\Delta}$ and $\bar{\Delta}^\dagger$ are in fact independent of the phase ϕ (and correspondingly are diagonal in the N -representation) which follows directly from the evident selection rule

$$(6.3) \quad \langle N' | \exp[i\phi] | N \rangle = \delta_{N', N+1}, \quad \langle N' | \exp[-i\phi] | N \rangle = \delta_{N', N-1}$$

In what follows we assume the quantities $\bar{A}(N)$ and $\bar{A}^\dagger(N)$ are real and hence equal ($\bar{A} = \bar{A}^\dagger$). Then it follows from (6.1) that

$$(6.4) \quad s = \varepsilon\tau^3 - \bar{A}\tau^1$$

The first term $h_0(N)$ in eq. (5.14) is independent of intrinsic variables and hence commutes with r . The second term $-\tau^3\lambda(N)$ has the same structure as the « kinetic energy » term in (6.4). Therefore, eq. (5.10) is reduced to the form

$$(6.5) \quad [s', r] = 0,$$

where in s' (in contrast to s) the single-particle energies are measured from the chemical potential λ :

$$(6.6) \quad s' = s - \tau^3\lambda = \begin{pmatrix} \varepsilon - \lambda & -\bar{A} \\ -\bar{A} & -\varepsilon + \lambda \end{pmatrix}.$$

The eq. (6.5) formally coincides with the corresponding equation in the HFB approach. Its solution with the additional condition $r^2 = r$ is well known, namely

$$(6.7) \quad r_\nu = \begin{pmatrix} 1 - v_\nu^2 & -u_\nu v_\nu \\ -u_\nu v_\nu & v_\nu^2 \end{pmatrix},$$

where ν denotes the single-particle eigenstates of ε and the quantities u_ν , v_ν are defined from

$$(6.8) \quad v_\nu^2 = \frac{1}{2} \left(1 - \frac{\varepsilon_\nu - \lambda}{E_\nu} \right), \quad 2u_\nu v_\nu = \frac{\bar{A}}{E_\nu}, \quad E_\nu = \sqrt{(\varepsilon_\nu - \lambda)^2 + \bar{A}^2},$$

(time-conjugate states ν , $\bar{\nu}$ are degenerate, $\varepsilon_\nu = \varepsilon_{\bar{\nu}}$).

In our case the quantities λ , \bar{A} in eq. (6.8) depend on N as on the parameter.

The density matrix R can be obtained by the inverse D -transformation

$$(6.9) \quad R_\nu = D^{-1} r_\nu D = \begin{pmatrix} 1 - v_\nu^2(N+1) & -\exp[-i\phi] u_\nu(N) v_\nu(N) \exp[-i\phi] \\ -\exp[i\phi] u_\nu(N) v_\nu(N) \exp[i\phi] & v_\nu^2(N-1) \end{pmatrix},$$

where the displacement of the argument in v_ν^2 is due to the equalities

$$\exp[-i\phi] v^2(N) \exp[i\phi] = v^2(N+1),$$

$$\exp[i\phi] v^2(N) \exp[-i\phi] = v^2(N-1).$$

So far we have considered the quantity $\bar{\Delta}$ as a free parameter. Now we have to choose it in order to fulfil the self-consistency condition. From eqs. (3.16) and (6.9) we find

$$(6.10) \quad \Delta = \exp[-i\phi] \bar{\Delta} \exp[-i\phi] = -G \operatorname{Tr} \left(\frac{\tau^1 - i\tau^2}{2} R \right) = \\ = \exp[-i\phi] \sum_{\nu} u_{\nu} v_{\nu} \exp[-i\phi]$$

and after using eq. (6.8) we obtain the equation for $\bar{\Delta}$ (which also coincides with that of the HFB approach)

$$(6.11) \quad 1 = G \sum_{\nu} \frac{1}{2E_{\nu}}.$$

The chemical potential $\lambda(N)$ (and consequently the energies $H(N)$) still remains to be found. For this purpose we shall use the consistency condition for the collective variable N , namely

« N must be equal to the total number of particles given by eq. (3.8d)».

As a result one gets

$$N = \frac{1}{4} \operatorname{Tr} (1) - \frac{1}{2} \operatorname{Tr} (\tau^3 R),$$

or after using eq. (6.9)

$$(6.12) \quad N = \frac{1}{2} \sum_{\nu} \{v_{\nu}^2(N+1) + v_{\nu}^2(N-1)\}.$$

Equation (6.12) differs from its counterpart of the HFB theory by displaced arguments in $v^2(N)$. Hence in principle one cannot solve the problem separately for one nucleus but one has to consider the recurrent eq. (6.12) for the whole chain of nuclei with ... $N-2$, N , $N+2$... nucleons.

But in fact the argument displacement in eq. (6.12) modifies the HFB results for $\lambda(N)$ and $\Delta(N)$ only in small terms of the order of $(1/N^2)$. These terms are of no importance in calculating the energies of the ground states $H(N)$, or rather the consequent differences

$$H(N+1) - H(N-1) = 2\lambda(N),$$

$$H(N+2) + H(N-2) - 2H(N) = 2\{\lambda(N+1) - \lambda(N-1)\} \approx 4(\partial\lambda/\partial N).$$

7. - Rotational problem (scheme of solution).

Before going into details we repeat in a schematic instructional way the sequence of operations needed to approach the final result.

I) We start with equations

$$(7.1) \quad [S + H, R] = 0, \quad R^2 = R,$$

where S and R are matrices in the combined intrinsic + collective space $\langle KJM, \nu\alpha | \dots | K'J'M', \nu'\alpha' \rangle$ and H , the collective Hamiltonian, is a matrix in the collective space $\langle KJ | H | K'J \rangle$.

II) We then perform a unitary transformation $DRD^{-1} = r, \dots$, and obtain the transformed equations

$$(7.2) \quad [s + h, r] = 0, \quad r^2 = r,$$

where the dependence on collective variables may be as indicated

$$s(I_\lambda), \quad r(I_\lambda), \quad h = H(I_\lambda - j_\lambda).$$

III) Solve equations for the matrix r with h as given.

IV) Perform an inverse transformation, obtaining

$$R = D^{-1}rD, \quad H = D^{-1}hD.$$

V) To find H we use the consistency condition

$$(7.3) \quad J_\mu = -\frac{1}{2} \text{Tr} (j_\mu R(I_\lambda))$$

A few additional comments.

a) Physical meaning of D -transformation. The formal manipulation with Wigner's D -functions gives, *e.g.*

$$(7.4) \quad Dj_\mu D^{-1} = D_{\mu\lambda}^{1*}(\theta_i) j_\lambda, \quad Dq_\mu^\dagger D^{-1} = D_{\mu\lambda}^{2*}(\theta_i) q_\lambda^\dagger.$$

If we compare these formulae to those which relate the angular momentum operators J_μ and I_λ , namely

$$(7.5) \quad J_\mu = D_{\mu\lambda}^{1*}(\theta_i) I_\lambda,$$

then it becomes evident, that the D -transformation corresponds to a rotation from the laboratory frame of reference to the intrinsic (« body-fixed ») frame.

Consider, for example, the usual unified model expression for the quadrupole moment

$$Q_\mu(\theta_i) = \sum_\lambda D_{\mu\lambda}^2(\theta_i) \alpha_\lambda,$$

where

$$\alpha_0 = \beta \cos \gamma, \quad \alpha_2 = \alpha_{-2} = (1/\sqrt{2}) \beta \sin \gamma, \quad \alpha_1 = \alpha_{-1} = 0.$$

Then the single-particle quadrupole interaction transforms as follows:

$$D \sum_{\mu} q_{\mu}^{\dagger} Q_{\mu}(\theta_i) D^{-1} = \sum_{\mu} D q_{\mu}^{\dagger} D^{-1} Q_{\mu}(\theta_i) = \sum_{\lambda \lambda'} q_{\lambda}^{\dagger} \alpha_{\lambda'} \sum_{\mu} D_{\mu \lambda}^{2*}(\theta_i) D_{\mu \lambda'}^2(\theta_i) = \sum_{\lambda} q_{\lambda}^{\dagger} \alpha_{\lambda},$$

i.e. transforms into the quadrupole interaction in the body-fixed frames which is (as it should be) independent of θ_i .

b) The consistency condition (1.3) can be written in the form

$$(7.6) \quad I_{\lambda} = D_{\mu \lambda}^1 J_{\mu} = -\frac{1}{2} D_{\mu \lambda}^1(\theta_i) \text{Tr} (j_{\mu} D^{-1}(\theta_i) r(\mathbf{I}) D(\theta_i)),$$

which shows that all dependence on the angles θ_i in the right-hand side must disappear in the correct solution.

c) The D -transformation helps us to simplify the initial equations. The new « total Hamiltonian » $s + h$ does not contain the angles θ_i . But the problem is still complicated and one has to look for some approximation.

8. - Adiabatic approximation.

The rotational energy, as a rule, is considerably smaller than the single-particle one. Therefore we can use an adiabatic approximation.

In *zeroth order* we can neglect in eq. (7.2) the term containing h (which gives the rotational splitting). Then eq. (7.2) is reduced to the usual equation for the single-particle density matrix $[s, r] = 0$ and one can look for a solution which is independent of the collective variables.

In the next orders there appears a dependence also on the collective variables I_{λ} . Therefore it is reasonable to look for the solution of eq. (7.2) in the form of a series of terms with increasing powers of I_{λ} , (or $I_{\lambda} - j_{\lambda}$):

$$(8.1) \quad \begin{cases} r = r^0 + r^{(1)} + \dots = r^0 + r_{\lambda}(I_{\lambda} - j_{\lambda}) + \dots, \\ s = s^0 + s^{(1)} + \dots = s^0 + s_{\lambda}(I_{\lambda} - j_{\lambda}) + \dots \end{cases}$$

In the *first order* in I_{λ} one has to solve the linear equation

$$(8.2) \quad [s^0, r^{(1)}] + [s^{(1)}, r^0] + [h^{(1)}, r^0] = 0,$$

where for $h^{(1)}$ one needs a quadratic term. When $I_0 = K$ is a good quantum

number the rotational energy $H(\mathbf{I})$ is a function of \mathbf{I}^2 and therefore

$$(8.3) \quad h^{(1)}(\mathbf{I}) = H^{(1)}(\mathbf{I} - \mathbf{j}) = \frac{1}{2\mathcal{F}} (\mathbf{I} - \mathbf{j})^2.$$

This results in the linear term after commutation in eq. (8.2). When we put (8.1) and (8.3) in eq. (8.2) two kinds of terms appear, namely, those linear in I_λ and those independent of I_λ . One obtains, therefore, two equations. Equating the coefficients in front of I_λ we obtain

$$(8.4) \quad [s, r_\lambda] + [s_\lambda, r] - \frac{1}{\mathcal{F}} [j_\lambda^\dagger, r] = 0.$$

This equation defines the first order correction to the density matrix which is linear in I_λ . (The second equation gives a correction to the I_λ -independent term and is therefore less important.) An additional equation for r_λ , originated from $r^2 = r$, has the form

$$(8.4') \quad \overset{\circ}{r} r_\lambda + r_\lambda \overset{\circ}{r} = r_\lambda.$$

The moment of inertia \mathcal{F} as introduced in eq. (8.3) as a free parameter, has to be obtained from the consistency condition (7.3). Making use of (8.1) one proceeds in the following way:

$$J_\mu = -\frac{1}{2} \text{Tr} (j_\mu D^{-1} r_\lambda (I_\lambda - j_\lambda) D) = -\frac{1}{2} \text{Tr} (j_\mu D^{-1} r_\lambda D D^{-1} (I_\lambda - j_\lambda) D)$$

(then using the relation $D^{-1}(I_\lambda - j_\lambda)D = I_\lambda$),

$$= -\frac{1}{2} \text{Tr} (j_\mu D^{-1} r_\lambda D) I_\lambda$$

(after rearrangement inside the trace),

$$= -\frac{1}{2} \text{Tr} (D j_\mu D^{-1} r_\lambda) I_\lambda$$

(and due to (7.4)),

$$J_\mu = -\frac{1}{2} \text{Tr} (j_\lambda, r_\lambda) D_{\mu\lambda}^{1*} I_\lambda.$$

This equation is equivalent to

$$(8.5) \quad I_{\lambda'} = -\frac{1}{2} \text{Tr} (j_{\lambda'}, r_\lambda) I_\lambda,$$

which results in the following equation for the moment of inertia \mathcal{F} :

$$(8.6) \quad \frac{1}{2} \text{Tr} (j_{\lambda'}, r_\lambda) = -\delta_{\lambda\lambda'}.$$

Equations (8.4) and (8.6) coincide with the corresponding equations of the cranking model which in fact proves the equivalence of two approaches in the approximation considered. Nevertheless, it is worth-while to obtain an explicit expression for the moment of inertia using eqs. (8.4) and (8.6).

Consider eq. (8.4) in the representation when \hat{s} and \hat{r} are diagonal

$$(8.7) \quad \langle \nu\alpha | \hat{s} | \nu'\alpha' \rangle = \delta_{\nu\nu'} \delta_{\alpha\alpha'} s(\nu\alpha), \quad \langle \nu\alpha | \hat{r} | \nu'\alpha' \rangle = \delta_{\nu\nu'} \delta_{\alpha\alpha'} r(\nu\alpha).$$

Then, from eq. (8.4) we have

$$(8.8) \quad \langle \nu\alpha | r_\lambda | \nu'\alpha' \rangle = \frac{1}{\mathcal{F}} \frac{r(\nu\alpha) - r(\nu'\alpha')}{s(\nu\alpha) - s(\nu'\alpha')} (\langle \nu\alpha | j_\lambda^\dagger | \nu'\alpha' \rangle - \mathcal{F} \langle \nu\alpha | s_\lambda | \nu'\alpha' \rangle),$$

which, being combined with eq. (8.6), gives

$$(8.9) \quad \mathcal{F} \delta_{\lambda\lambda'} = -\frac{1}{2} \sum_{\nu\nu', \alpha\alpha'} \frac{r(\nu\alpha) - r(\nu'\alpha')}{s(\nu\alpha) - s(\nu'\alpha')} \langle \nu'\alpha' | j_\lambda^\dagger | \nu\alpha \rangle (\langle \nu\alpha | j_\lambda^\dagger | \nu'\alpha' \rangle - \mathcal{F} \langle \nu\alpha | s_\lambda | \nu'\alpha' \rangle).$$

The term containing s_λ describes the variations of the self-consistent single-particle Hamiltonian due to the rotation. It depends on the details of the nucleon-nucleon interaction and is in fact not very important (*) [7].

So we restrict ourselves to the main term in \mathcal{F}

$$(8.10) \quad \mathcal{F} \delta_{\lambda\lambda'} \approx -\frac{1}{2} \sum_{\nu\nu', \alpha\alpha'} \frac{r(\nu\alpha) - r(\nu'\alpha')}{s(\nu\alpha) - s(\nu'\alpha')} \langle \nu'\alpha' | j_\lambda | \nu\alpha \rangle \langle \nu\alpha | j_\lambda^\dagger | \nu'\alpha' \rangle.$$

The representation (8.7) ($\hat{s} = \varepsilon\tau^3 - \bar{1}\tau^1$ is diagonal) is connected with the original one (when $\varepsilon\tau^3$ is diagonal) by the well-known Bogolyubov transformation (**)

$$(8.11) \quad U_\nu = \begin{pmatrix} u_\nu & -v_\nu \\ v_\nu & u_\nu \end{pmatrix},$$

where u_ν , v_ν are given in eq. (8.8). The simple exercise with U_ν leads, to the following explicit expression for the quantities contained in eq. (8.10):

$$(8.12) \quad \begin{cases} s(\nu\alpha) = E_\nu \tau_{\alpha\alpha}^3, \\ r(\nu\alpha) = \frac{1}{2} (1 + \tau^3)_{\alpha\alpha}, \\ \langle \nu\alpha | j_\lambda^\dagger | \nu'\alpha' \rangle = (u_\nu v_{\nu'} - v_\nu u_{\nu'}) i \tau_{\alpha\alpha'}^2 \langle \nu | j_\lambda^\dagger | \nu' \rangle. \end{cases}$$

(*) For the pairing and quadrupole interaction the last term in eq. (8.9) vanishes.

(**) In what follows we consider the ground state rotational band with $I_0 = K = 0$.

From eqs. (8.12) and (8.10) for the moment of inertia we obtain

$$(8.13) \quad \mathcal{F} = \sum_{\nu\nu'} \frac{|\langle \nu | j_1 | \nu' \rangle|^2}{E_\nu + E_{\nu'}} (u_\nu v_{\nu'} - v_\nu u_{\nu'})^2$$

which is just the cranking-model value [8].

9. - A few general remarks in conclusion.

The idea to generalize the HF approximation by simultaneous consideration of the whole rotational band has been utilized by KERMAN and KLEIN [9]. They considered equations of motion for the matrix elements of Fermi operators

$$(9.1) \quad \Psi_i(\nu JM) \equiv \langle i | a_\nu | JM \rangle, \quad \Phi_i(\nu JM) \equiv \langle JM | a_\nu | i \rangle.$$

The quantities $\Psi_i(\nu JM)$ and $\Phi_i(\nu JM)$ have been considered as wave functions of two (unfortunately different) Hamiltonians in the combined space $|\nu JM\rangle$. Two not orthogonal sets of functions Ψ_i and Φ_i are tangled by the anti-commutation relation which has been used in [9] for normalization of Ψ_i and Φ_i . Unfortunately the whole scheme turns out to be too complicated to be used outside the simplest approximations (adiabatic or RPA).

The relative simplicity of our method is due to the use of the normalization condition

$$R^2 - R = 0,$$

which allows to consider a closed set of equations for the density matrix R . Certainly, the relation $R^2 = R$ is an approximation. But the equation of motion in contracted collective space is already an approximation. These two approximations are compatible since the equations of motion (7.1) result in the relation (*)

$$(9.2) \quad [R^2 - R, H + S] = 0.$$

On the other hand the relation $R^2 = R$ seems to be a reasonable approximation from the physical point of view (**).

(*) By the way, eq. (8.2) means that the quantity $R^2 - R$ if not zero has to be a constant of motion in combined space.

(**) The relation $R^2 = R$ holds for the Slater-type state, *i.e.* independent quasiparticle structure. The collective excitation produces a variation of the selfconsistent field and consequently of the quasiparticle wave functions but not of their occupation numbers.

Due to the simplicity of the method suggested (*) it opens a series of possibilities for different approximations. The first one is an asymptotic expansion ($I \gg 1$) for the higher part of a rotational band, where one can use the small values of the commutators among the operators I_λ . The second possibility is to include into the scheme phonon operators in addition to I_λ . Dr. ZELEVINSKY and myself are investigating these and other possibilities.

(*) The first result, though obtained in a slightly different scheme, can be found in the paper by S. T. BELYAEV and V. G. ZELEVINSKY: *Jadernaja Fizika*, **11**, 741 (1970) (English translation in *Sov. Journ. Nucl. Phys.*).

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