NONLINEAR RESONANCES AND BEAM-BEAM EFFECTS FOR ELLIPTICAL BEAMS

A.L.Gerasimov, F.M.Izrailev
Institute of Nuclear Physics
630090, Novosibirsk, U S S R

J.L.Tennyson
Queen Mary College,
London E1 4NS, UK

Abstract

A comparative analysis of different approximate models of the beam-beam interaction field is performed for the case of a large aspect ratio of the beam $\varkappa = \sigma_x/\sigma_z \gg 1$. Applicability limits for known approximate models are found, and new models are suggested, which are convinient for simulation and analytical description of nonlinear resonances. Obtained expressions for isolated resonances are compared with a numerical data.

1. Introduction

Although a large amount of papers have been written, describing simulations of the beam-beam effects in electron-positron storage rings, the problem of maximizing luminosity in a given machine is far from being solved. The difficulties arise not only from the development of the simulation itself, but from the large amount of parameters in optimization problem. To the present moment mostly one-dimensional models were studied, where only one transverse direction is taken into account. Such models can be used for the description of round or very flat beams (see, for example, review papers /1-3/). The motion of particles in this case is governed by one-dimensional nonlinear resonances. These perturb the equilibrium particle distribution, resulting in a deterioration of the beams lifetime and luminosity.

The behavior of simulations based on one-dimensional models of the beam-beam interaction are now well inderstood. Unfortunately, such models do not adequately represent most real machines which typically operate with elliptical beams. With elliptical beams, the beam-beam interaction necessarily involves both transverse dimensions and two-dimensional "coupling" resonances must be taken into account. Systematic studies of two-dimensional models have only recently begin to appear (see, for example /4-6/). The present report investigates, using approximate analytic models of the beam-beam force, the dynamics of particles interacting with an elliptical beam with large aspect ratio ($\varkappa = \sigma_x/\sigma_z \gg 1$). Most of the study is devoted to investigating the influence of isolated nonlinear resonances on the luminosity and lifetimes of elliptical beams.

2. The description of isolated nonlinear resonances for elliptical beams

We consider the two-dimensional transverse oscillations of a storage ring particle which "collides" periodically with bunches of the opposing beam. Using a thin lens approximation for the beam-beam impulse imported by the period collisions, the Hamiltonian has the familiar form

$$H = I_x \nu_{x0} + I_z \nu_{z0} + V(x,z)\,\delta_T(s) \qquad (2.1)$$
$$\delta_T(s) = \sum_{n=-\infty}^{\infty} \delta(s-nT),$$

where the transverse coordinates x, z in the potential $V(x,z)$ can be expressed in terms of the action-angle variables of the linear (single beam) system:

$$x = \sqrt{2I_x\beta_x}\cos\theta_x, \quad z = \sqrt{2I_z\beta_z}\cos\theta_z \qquad (2.2)$$

and ν_{x0}, ν_{z0} are the unperturbed tunes. The longitudinal coordinate s is assumed to the proportional to time.

The actions I_x, I_z can be introduced as functions of the transverse coordinates x, z and the velocities $x' = \frac{dx}{ds}$, $z' = \frac{dz}{ds}$:

$$I_x = \frac{x'^2}{2}\beta_x + \frac{x^2}{2\beta_x}, \quad I_z = \frac{z'^2}{2}\beta_z + \frac{z^2}{2\beta_z}. \qquad (2.3)$$

In equations (2.2)-(2.3) the quantities β_x, β_z are the beta-functions at m_0 interaction points /1/. For (2.1), the period T between interactions is $T = 2\pi/m_0$. The normalized amplitudes A_x, A_z of the betatron oscillation are defined to be

$$A_x = \sqrt{X^2 + P_x^2}, \quad A_z = \sqrt{Z^2 + P_z^2}, \qquad (2.4)$$

where the dimensionless variables X, Z, P_x, P_z are determined by the expressions

$$X = \frac{x}{\sigma_x}; \quad Z = \frac{z}{\sigma_z}; \quad P_x = \frac{x'}{\sigma_x}\beta_x; \quad P_z = \frac{z'}{\sigma_z}\beta_z \qquad (2.5)$$

so that

$$A_x = \frac{\sqrt{2I_x\beta_x}}{\sigma_x}, \quad A_z = \frac{\sqrt{2I_z\beta_z}}{\sigma_z}. \qquad (2.6)$$

The quantities σ_x and σ_z are the r.m.s. values of the beam width and height respectively.

The most important characteristic of a nonlinear oscillator is the dependence of its frequency on amplitude. Since the unperturbed Hamiltonian in our case is linear in I_x and I_z, the principal nonlinearity is determined by the zero harmonic of the potential

$$V_{00} = \frac{1}{(2\pi)^2}\iint_0^{2\pi} V(x,z)\,d\theta_x\,d\theta_z, \qquad (2.7)$$

where the quantitites x, z are given by the expressions (2.2). The nonlinear tune shifts per interaction $\Delta\nu_{x,z}$ are

$$\nu_x = \nu_{x0} + \Delta\nu_x(A_x, A_z); \quad \nu_z = \nu_{z0} + \Delta\nu_z(A_x, A_z); \qquad (2.8)$$
$$\Delta\nu_{x,z} = \frac{m_0}{2\pi}\,\frac{\partial V_{00}}{\partial I_{x,z}}.$$

We now introduce the dimensionless "forces" f_x, f_z:

$$f_x = \frac{1}{\xi_x}\frac{\beta_x}{\sigma_x}\frac{\partial V(x,z)}{\partial x}; \quad f_z = \frac{1}{\xi_z}\frac{\beta_z}{\sigma_z}\frac{\partial V(x,z)}{\partial z}. \qquad (2.9)$$

Here the quantities ξ_x, ξ_z are the linear tune shifts for one interaction point ($\Delta\nu_{x,z} = m_0\xi_{x,z}$ when $A_x, A_z \to 0$). The forces f_x, f_z determined in this way, satisfy the following normalization conditions /7/:

$$f_x = 4\pi X; \quad f_z = 4\pi Z; \quad \text{for } |X|, |Z| \ll 1. \qquad (2.10)$$

To express the tune-shifts in terms of these forces, we substitute the expression (2.7) into the formula (2.8) and interchange the order of integration and differentiation. Taking into account (2.9), the result is

$$\Delta\nu_{x,z}=\frac{m_0\xi_{x,z}}{8\pi^3 A_{x,z}}\int\!\!\int_0^{2\pi} d\theta_x d\theta_z\cdot f_{x,z}(A_x\cos\theta_x, A_z\cos\theta_z)\cos\theta_{x,z}. \qquad (2.11)$$

The functions $\Delta\nu_{x,z}(A_x, A_z)$ determine the position of an arbitrary resonance in the amplitude space through the expression

$$\ell\nu_x + m\nu_z + n = 0, \qquad (2.12)$$

where the values of ν_x, ν_z should be taken from (2.8).

To calculate a resonances width using the "moderate nonlinearity approximation" (see /8/), we first introduce the dimensionless nonlinearity

$$\alpha=\frac{\sigma_x^2}{\xi_x\beta_x}\left\{\ell^2\frac{\partial^2 H_0}{\partial I_x^2}+2\ell m\frac{\partial^2 H_0}{\partial I_x \partial I_z}+m^2\frac{\partial^2 H_0}{\partial I_z^2}\right\}. \qquad (2.13)$$

Here H_0 is defined by $H_0=\frac{m_0}{2\pi}V_{00}$ and the factor $\sigma_x^2/\xi_x\beta_x$ is included to make the quantity dimensionless. The dimensionless nonlinearity α can be expressed in terms of the amplitude dependent tune shifts $\Delta\nu_{x,z}$ by

$$\alpha=\frac{\ell^2}{A_x\xi_x}\frac{\partial(\Delta\nu_x)}{\partial A_x}+\frac{2m\ell}{A_x\xi_x}\frac{\partial(\Delta\nu_z)}{\partial A_x}+\frac{m^2 æ}{A_z\xi_z}\left(\frac{\xi_z}{\xi_x}\right)^2\frac{\partial(\Delta\nu_z)}{\partial A_z}, \qquad (2.14)$$

where $æ=\sigma_x/\sigma_z$. The normalized Fourier coefficient $V_{\ell mn}'=\frac{\beta_x}{\xi_x\sigma_z^2}V_{\ell mn}$ of the potential $V(æ(I_x,\theta_x), z(I_z,\theta_z))$ transformed with respect to θ_x and θ_z is defined by

$$V_{\ell mn}^1=\begin{cases}\dfrac{A_x}{\ell(2\pi)^2}\displaystyle\int\!\!\int_0^{2\pi} d\theta_x d\theta_z \sin\theta_x \sin\ell\theta_x \cos m\theta_z\cdot f_x(A_x\cos\theta_x, A_z\cos\theta_z)\\ \dfrac{A_z æ}{m(2\pi)^2}\displaystyle\int\!\!\int_0^{2\pi} d\theta_x d\theta_z \sin\theta_z \sin m\theta_z \cos\ell\theta_x\cdot f_z(A_x\cos\theta_x, A_z\cos\theta_z)\end{cases} \qquad (2.15)$$

where the upper expression in (2.15) is applicable for $\ell\neq 0$, and the lower for $m\neq 0$. The two expressions coincide for $\ell, m\neq 0$. Since $V_{\ell mn}'$ doesn't depend on n we shall omit the index n in the following. It should also be noted that since the potential $V(æ,z)$ is symmetric with respect to changes in sign of either $æ$, z or both, only the even harmonics ℓ, m are non-zero. With $V_{\ell mn}'$ and α, we can find via the formulas derived in /8/, the width of the resonance ℓ, m, n in the amplitude space

$$\Delta A_x^{\ell m}=\frac{2\ell}{A_x}\sqrt{\frac{V_{\ell m}'}{\alpha}}\sqrt{\frac{m_0}{2\pi}},$$
$$\Delta A_z^{\ell m}=\frac{2m}{æ A_z}\frac{\xi_z}{\xi_x}\sqrt{\frac{V_{\ell m}'}{\alpha}}\sqrt{\frac{m_0}{2\pi}}. \qquad (2.16)$$

The corresponding small amplitude phase frequency is given by

$$\Omega_p=\sqrt{\frac{m_0}{2\pi}}\,\xi_x\sqrt{\alpha V_{\ell m}'}. \qquad (2.17)$$

An example of the graphic representation of a nonlinear resonance in amplitude space is given in fig. 1, where the resonance $4\nu_x-2\nu_z=n$ is shown for several different values of the machine tunes. The quantities $\Delta A_{x,z}$ and $\Delta\nu_{x,z}$, which were used in constructing this plot, were computed by numerically integrating the formulas $V_{\ell m}'$, $\Delta\nu_{x,z}$ given by (2.11), (2.15) and then making use of (2.14) and (2.16).

The positions of the resonance lines in amplitude space are determined by specific natures of the amplitude dependent tune shifts $\Delta\nu_{x,z}(A_x, A_z)$. In particular, if $æ=\sigma_x/\sigma_z\gg 1$, the resonance lines are strongly stretched along the Z axis, as is seen in Fig. 1 where $æ$ = 100.

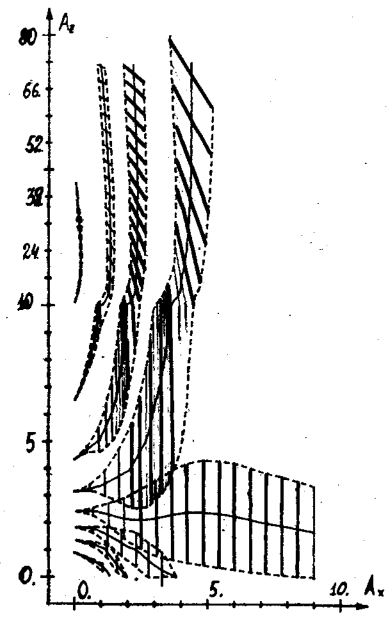

Fig. 1. Possible locations of the resonance $4\nu_x-2\nu_z=n$ in the plane A_x, A_z for different detunings $\delta=\frac{2\nu_{x0}-\nu_{z0}}{\xi_z}$. The quantities $æ$, ξ_z/ξ_x have the values $æ$ = 100, ξ_z/ξ_x =4, while the quantity δ assumes the values δ = 0.1k, where k is an integer.

The same figure also shows that for moderate values of A_z, the oscillations induced by the resonance are almost vertical. As was shown in /9/, this is a consequence of the particular nature of the force field f_x, f_z in the limit $æ\to\infty$. More specifically, the force f_x in this limit doesn't depend on Z, while f_z is a product of two functions which depend separately on X and Z. The exact forms for these assymptotics are

$$f_x = 4\pi\sqrt{2}\,F_D(X/\sqrt{2}),$$
$$f_z = 2\pi\sqrt{2\pi}\,e^{-X^2/2}\,erf(Z/\sqrt{2}), \qquad (2.18)$$

where X and Z are normalized to the beam width and height, respectively, $F_D(y)$ is a Dawson function (see /10/, and $erf(y)$ is an error integral. The assymptotic formulas (2.18) were given earlier in the work /11/. First-order corrections to the expressions (2.18) and their phenomenological generalizations were considered in /9/. It was also shown in /9/ that the model (2.18) can be used to analytically describe nonlinear resonances in a limited region of the amplitude plane defined by $A_z\ll\left|\frac{m}{\ell}\right|\frac{\xi_z}{\xi_x}\sqrt{æ}A_x$ and $A_z\ll æ$. In this region, the resonant oscillations are nearly vertical (see Fig. 1) and the first two terms in the nonlinearity α (2.14) can be neglected. The nonlinearity α then assumes a "one-dimensional" value (see /9/):

$$\mathcal{L} = \frac{m^2 \varkappa}{A_z \xi_z}\left(\frac{\xi_z}{\xi_x}\right)^2 \frac{\partial(\Delta\nu_z)}{\partial A_z}. \qquad (2.19)$$

In addition, in the region $A_z \ll \varkappa$, we can use the approximate force f_z in (2.18) to calculate $\Delta\nu_z$, $\mathcal{L}$ and V_{em} which determine the values of ΔA_z and Ω_p. The situation corresponds to a one-sided coupling of the vertical and horizontal oscillations; the vertical motion depends on the horizontal coordinate while the horizontal motion doesn't depend on the vertical coordinate. Thus the horizontal coordinate appears as an externally-determined time-dependent parameter in the Hamiltonian for the vertical motion. This "one-and-a-half" dimensional approximation is acceptable as long as $q \ll 1$, where q is a dimensionless parameter associated with a particular resonance $\ell\nu_x + m\nu_z + n = 0$ (see /9/).

$$q = \frac{\ell^2}{m^2} \cdot \frac{\xi_x^2}{\xi_z^2 \varkappa}. \qquad (2.20)$$

For typical accelerator parameters the condition $q \ll 1$ is usually fulfilled, so that in the region $A_z \ll A_x/\sqrt{q}$ the width of the vertical resonant oscillation for the resonance $\ell\nu_x + m\nu_z + n = 0$ is given by

$$\Delta A_z^{em} = \sqrt{F_m(A_z)}\ \sqrt{G_\ell(A_x)}\ , \qquad (2.21)$$

where $G_\ell(A_x)$ is

$$G_\ell(A_x) = \frac{I_{\ell/2}(A_x^2/4)}{I_0(A_x^2/4)} \qquad (2.22)$$

and the function $F_m(A_x)$ can be represented in the equivalent forms:

$$F_m = \frac{8}{m(m^2-1)}\ \frac{I_{\frac{m}{2}-1}\left(\frac{A_z^2}{4}\right)(m+1) + 2I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right) - (m-1)I_{\frac{m}{2}+1}\left(\frac{A_z^2}{4}\right)}{I_0\left(\frac{A_z^2}{4}\right) - I_2\left(\frac{A_z^2}{4}\right)} = \qquad (2.23)$$

$$= \frac{A_z^2}{m(m^2-1)}\ \frac{I_{\frac{m}{2}-1}\left(\frac{A_z^2}{4}\right) + I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right) + \frac{2m(m-1)}{A_z^2} I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right)}{I_1\left(\frac{A_z^2}{4}\right)};$$

the functions $I_\nu(y)$ in the expressions (2.22), (2.23) are the modified Bessel functions of ν 'th order. Plots of the functions $\sqrt{F_m(A_z)}$ and $\sqrt{G_\ell(A_x)}$ are shown in Figs. 2-3.

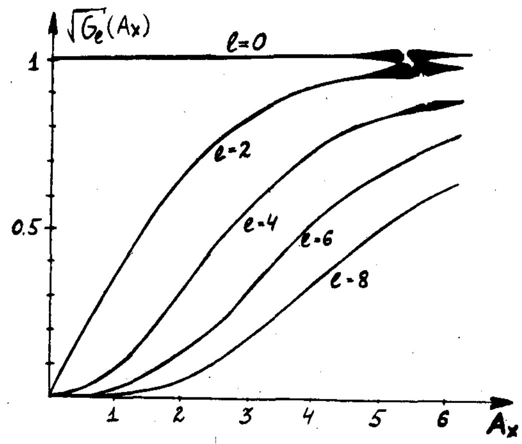

Fig. 2. The dependence $\sqrt{G_\ell(A_x)}$ for different ℓ.

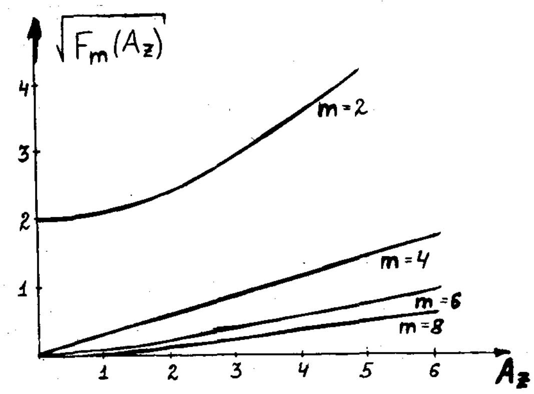

Fig. 3. The dependence $\sqrt{F_m(A_z)}$ for different m.

It's clear that for $\ell = 0$ the quantity ΔA_z doesn't depend on A_x, whereas for $\ell \neq 0$, ΔA_z tends to zero as $A_x \to 0$. The specific case $|m| = 2$ (see Fig. 3) is unusual in that remains nonzero when $A_z \to 0$. This can be explained by the fact that both the zero harmonic V_{00} and the resonance harmonic $V_{\ell\ell}$ are proportional to A_z^2 when $A_z \to 0$. This is series expansion (with respect to $z = A_z \cos\theta_z$) of the potential $V(x,z)$.

The vertical amplitude-dependent tune shift $\Delta\nu_z$, which enters into the expression for $\mathcal{L}$ (2.14), can be calculated via the force (2.18) from (see /9/):

$$\Delta\nu_z = m_0 \xi_z e^{-A_x^2/4} I_0\left(\frac{A_x^2}{4}\right)\left[I_0\left(\frac{A_z^2}{4}\right) + I_1\left(\frac{A_z^2}{4}\right)\right] e^{-A_z^2/4}. \qquad (2.24)$$

The tune shift $\Delta\nu_x$ in the model (2.18) can't be expressed, however, in an explicit form. A simple expression for $\Delta\nu_x$ can be derived for another model /9/ which is also fairly accurate. In this second model, the force f_x is given by

$$f_x = \frac{4\pi X}{(1 + X^2/5.11)^2}. \qquad (2.25)$$

and the resulting horizontal tune shift $\Delta\nu_x$ is

$$\Delta\nu_x = \frac{m_0 \xi_x}{(1 + A_x^2/5.11)^{3/2}}. \qquad (2.26)$$

The resonant oscillation frequency Ω_p for the model (2.18) is

$$\Omega_p = m_0 \xi_z\ B_\ell(A_x) \cdot D_m(A_z); \qquad (2.27)$$

the function $B_\ell(A_x)$ is defined to be

$$B_\ell(A_x) = e^{-A_x^2/4}\sqrt{I_0\left(\frac{A_x^2}{4}\right) \cdot I_{\ell/2}\left(\frac{A_x^2}{4}\right)} \qquad (2.28)$$

and the function $D_m(A_z)$ can be represented in one of the equivalent forms by

$$D_m(A_z) = \frac{m A_z e^{-A_z^2/4}}{\sqrt{8|m|(m^2-1)}}\sqrt{I_{\frac{m}{2}-1}\left(\frac{A_z^2}{4}\right)(m+1) + 2I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right) - (m-1)I_{\frac{m}{2}+1}\left(\frac{A_z^2}{4}\right)} \cdot \qquad (2.29)$$

$$\cdot\sqrt{I_0\left(\frac{A_z^2}{4}\right) I_2\left(\frac{A_z^2}{4}\right)} = \frac{\sqrt{2}\, m e^{-A_z^2/4}}{\sqrt{m(m^2-1)}}\sqrt{I_1\left(\frac{A_z^2}{4}\right)\left[I_{\frac{m}{2}-1}\left(\frac{A_z^2}{4}\right) + I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right) + \frac{2m(m+1)}{A_z^2} I_{\frac{m}{2}}\left(\frac{A_z^2}{4}\right)\right]}.$$

From equations (2.21) and (2.27), it can be seen that the quantity ΔA_z^{em} is independent of the aspect ratio $\varkappa$ and linear tune shifts ξ_x, ξ_z, whereas the oscillation frequency Ω_p is proportional to ξ_z and is independent of $\varkappa$ and ξ_x. It should be noted that the independence of ΔA_z^{em} on the current in the opposite beam (that is, on the quantities ξ_x, ξ_z for a fixed ratio ξ_x/ξ_z) and the analogous property of Ω_p is a general property of "nearly linear" oscillations (see /I/). On the other hand the independence of both ΔA_z^{em} and Ω_p on ξ_x/ξ_z is a consequence of the "one-dimensional nonlinearity" assumption (2.19) and the condition $A_z \ll \frac{|m|}{|e|} \frac{\xi_z}{\xi_x} \sqrt{\varkappa}\, A_x$, which is valid only in a finite amplitude region.

For large amplitudes, $A_x \gg \frac{|\ell|}{2}$, $A_z \gg \frac{|m|}{2}$, making use of the asymptotics of the Bessel functions, we get

$$\Omega_p = \frac{2mm_0}{\pi} \frac{\xi_z}{A_x A_z^2} \Delta A_z^{em}. \tag{2.30}$$

It is now possible to find a condition defining those regions of amplitude space in which the damping time is large compared to a libration period. In these regions, synchrotron damping cannot suppres the effect of the resonance and its influence on the equilibrium distribution function is significant. The required condition can be derived from (2.30) and the observation that a resonance can only appreciably perturb the distribution function if it has a large enough width $\Delta A_z^{em} \gtrsim 1$. For a resonance to have a significant effect, we must have

$$A_z^2 A_x \gg 4m \xi_z T_D \tag{2.31}$$

where T_D is the damping time expressed in units of the collision period.

The estimate (2.31) shows that, for most operating machines, the resonances aren't destroyed by damping over a rather large region of amplitude space (A_x, A_z). With VEPP-4 parameters, for example ($T_D \approx 3000$, $\xi_z \approx 0.05$), we get for the resonance $4\nu_x - 2\nu_z = 15$ (i.e. $m = 2$) the condition $A_x^2 A_z \ll 1000$.

The applicability limits of the expressions (2.21)-(2.23) were thoroughly analysed in /9/. It was shown that for typical e^+e^- storage ring parameters, there exists a wide range of amplitudes (5-10) in which these expressions are accurate to within 20-30%.

The most important feature of the above case ($\varkappa \gg 1$) is the nearly vertical nature of the resonant oscillation for $A_x, A_z \sim 1$. This can lead to an enlargement of the transverse beam size in the Z direction, leaving the size in the X direction essentially unperturbed. At times, however, it may be necessary to consider a situation where the resonant oscillation vector have a substantial inclination in the A_x, A_z plane for $A_x, A_z \sim 1$. In this case, the vertical and horizontal oscillations will be strongly coupled and one can expect a substantial decrease of the vertical "blowup". This situation is fundamentally two-dimensional, in contrast to the previous case. The condition for "one-dimensional nonlinearity" $q \ll 1$ is no longer valid and one must take into account the previously neglected terms in the expression (2.14) for the nonlinearity α.

It was shown in /9/ that, for the general case, the resonance width ΔA_z^{em} for $A_x, A_z \sim 1$ (and arbitrary ξ_x, ξ_z, ℓ, m) can be estimated by

$$\Delta A_z^{em} \approx \frac{\Delta A_{z0}^{em}}{\sqrt{1 + C_2 q}}, \tag{2.32}$$

where ΔA_{z0}^{em} is the "zero approximation" resonance width calculated from the "one-dimensional nonlinearity" approximation (2.21-2.23). The two quantities ΔA_z^{em} and ΔA_{z0}^{em} are related by the "dimensionality parameter" q (2.20) and C_2 is of the order of unity (depending only on A_x, A_z). Thus, the expression (2.32) shows that there is a possibility of suppressing the effect of a resonance by enlarging the quantity q. It should be noted that to obtain a large value of q, it is preferable to have $\xi_z < \xi_x$ rather than the more typical situation $\xi_z \gtrsim \xi_x$.

3. The influence of nonlinear resonances on the luminosity of electron-positron rings.

In beam-beam systems, there is always the possibility that stochasticity is present, due to the overlap of two or more resonances. A detailed analysis of phase trajectories has estiblished, however, that for the parameters of VEPP-2M and VEPP-4 in the absence of synchrotron modulations, there should be no strong stochasticity even for the tune shifts as high as $\xi \approx 0.15$. This is illustrated in Fig. 4 where, for the parameters of VEPP-2M, all resonances up to 12th order are shown in the amplitude space (note that the linear tune shifts in VEPP-2M can have very high values; $\xi_x \approx .07$, $\xi_z \approx .14$). This plot was obtained numerically in a fashion similar to that of Fig. 1.

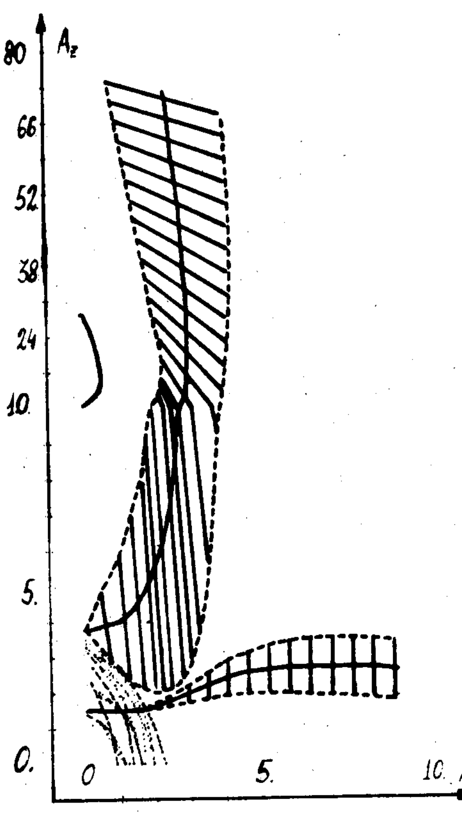

Fig. 4. Nonlinear resonances in A_x, A_z plane for the parameters of VEPP-2M: $\nu_{x0} = 3.06$, $\nu_{z0} = 3.08$; $\xi_x = 0.07$, $\xi_z = 0.14$, $\varkappa = \sigma_x/\sigma_z = 20$. All the resonances of the order $|m| + |\ell| \leq 12$ are shown.

It is clear that all of the resonances, except the resonance (2,-2), are very small so that large scale stochasticity cannot form. Direct observations of trajectories in a simulation have confirmed this. Since there is no stochasticity in this case, those beam-beam effects that do appear must result from the influence of isolated resonances (rather than from several overlapping resonances).

To determine which isolated resonances are most likely to influence the luminosity (or average beam size), we can use the expressions for the resonance characteristics given in section 2. To get a quantitative estimate, we assume that a resonance cannot substantially increase the vertical beam size if the magnitude of its separatrix oscillations are less than $\Delta A_z^{em} < .5$ for small amplitudes, $A_x^2 + A_z^2 < R^2 = 4$. Using the data shown in Figs. 2 and 3, we see that only resonances of order $K = |m| + |\ell| \leq 6$ can seriouslv affect the average vertical size of the beam. It should be noted that an increase of R^2 to 8 doesn't change this result. This condition determining the maximum order of the influential resonances agrees fairly well with the results of numerical simulation /4,5,12/.

Thus, in the absence of synchrotron modulation and machine "imperfections", only isolated resonances of the order $K \leq 6$ can effect the luminosity. In order to see how an isolated resonance can affect luminosity, we consider, as an example, the resonance $4\nu_x - 2\nu_z = 15$ which is close to the operating tune of VEPP-4. Since in VEPP-4 there is only one interaction point, errors in phase advance are absent while dispersion and spurious beam separations are very small. These effects therefore have little influence on the luminosity, in contrast to the cases considered in /4,5/.

A simulation of the beam-beam system was run to observe the effect of the $4\nu_x - 2\nu_z = 15$ resonance on particle motion. The dependencies of the specific luminosity and average vertical beam size on horizontal tune in the vicinity of the resonance $(\ell, m) = (4, 2)$ are shown in Fig. 5.

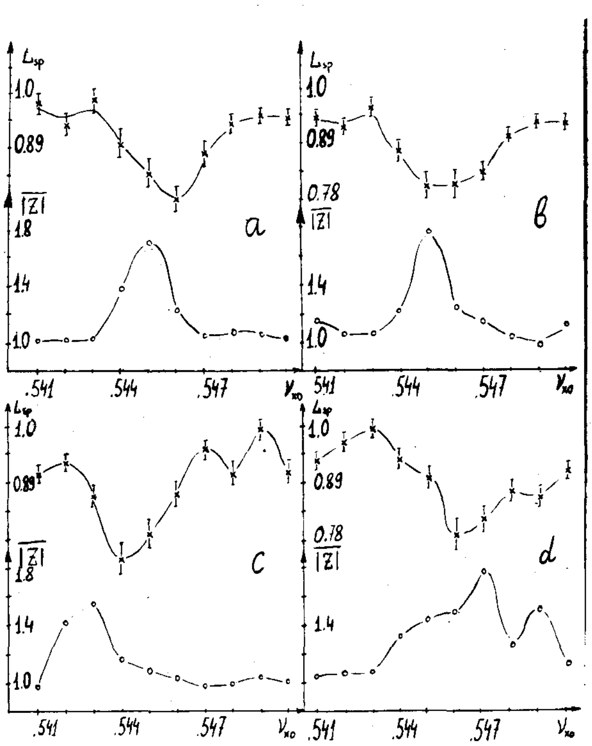

Fig. 5. Specific luminosity L_{sp} (upper curve) and averaged beam size $|\overline{Z}|$ (lower curve) for variable horizontal tune ν_{xo}. The parameters correspond to VEPP-4: $\xi_x = 0.01$, $\xi_z = 0.03$, $æ = 30$, $\nu_{zo} = 9.592$, damping time $N = 3 \cdot 10^3$, number of interaction points $m_0 = 1$. The case (a) refers to a zero machine nonlinearity in the absence of synchrotron modulations, the case (b) to the presence of a synchrotron modulation of the tune ν_{zo} with an amplitude $\delta\nu_z = 0.015$ and of the linear tune shift ξ_z with an amplitude $\delta\xi_z/\xi_z = 0.03$. The modulation frequency is $\nu_s = 0.02$. Machine nonlinearity is absent. (c) - negative machine nonlinearity in the absence of modulations. (d) positive machine nonlinearity in the absence of modulations.

Figure 5a refers to case where both synchrotron modulation and machine nonlinearities are absent. In Fig. 5b, synchrotron modulation of the vertical tune and tune shift (characteristic of VEPP-4) have been added, but apparently fail to significantly affect the luminosity. It should be noted that increasing the damping time in the modulation (with a corresponding reduction in quantum fluctuations to conserve beam size), failed to affect significantly the curves of Figs. 5a,b. Thus, these curves represent the infinite damping-time limit, which is the basis for the theory /13/, and which agrees well with one dimensional estimates of /14/. In other works /5,15/, a strong dependence of the luminosity on damping time was found. In our opinion, this was related to the existence of stochastic regions resulting from the large number of machine imperfections present in the cited works.

Figures 5c, and 5d show the luminosity dependence without modulation but with negative and positive octupole nonlinearity respectively. These different machine nonlinearities cause small displacements (in opposite directions) of the minimum luminosity peak, but don't significantly change the height of the peak.

To understand the mechanism via which this resonance reduces the luminosity, the trajectories of individual particles were "tracked" for the specific cases shown in Figs. 5. By taking two-dimensional cross-sections in the four-dimensional phase space (see /17/), the resonance location and its maximal separatrix width were plotted. The results are shown in Figs. 6-8. We would like to stress that this representation of the separatrix oscillations is more relevant than the "envelopes" used in the work /18/, which give only indirect information about the separatrix. The importance of the phase space regions "inside" the separatrix is shown in the work /12/ where it is seen that, for the one-dimensional case, the equilibrium distribution function is most strongly perturbed in this interior region.

From a comparison of Figs. 5 and 6-8 we see that the greatest influence of the resonance on the luminosity and average beam size occurs when the resonance passes through the region of moderate amplitudes ($A_x, A_z \sim$ ~1.5). The decrease in the resonances influence which appears at smaller amplitudes, where the particle density is larger, can be explained by the rapid decrease of the resonance width ΔA_z with decreasing amplitude (see Figs. 2-3).

4. The influence of nonlinear resonances on the lifetime of electron-positron beams

Measurements were recently made (see /19.20/ of VEPP-4's luminosity and lifetime over a wide range of working tunes. It was found that the behavior of the beam was affected by certain nonlinear resonances, though the resonances that affect luminosity are clearly different from those that affect lifetime. Thus, the measurements of lifetime showed the influence of a large number of isolated resonances of rather high order ($\sim$ 10, 12), which don't manifest themselves in the luminosity measurements. In addition, lifetime measurements turned out to be very sensitive to readjustements of the machine optics, which are assumed to have affected the machine octupole nonlinearity of the magnet lattice.

To understand these results, a number of computer simulations were run in the vicinity of the resonance $10\nu_z = 96$. The particles were given initial conditions

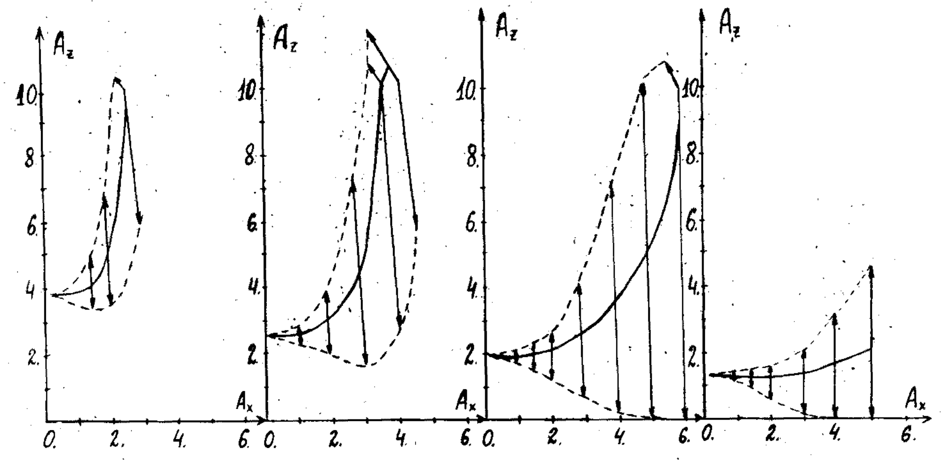

Fig. 6. The location of the resonance $4\nu_x - 2\nu_z = 15$ in the amplitude plane for the parameters of Fig. 5a. The cases a,b,c,d correspond to a successive increase of ν_{x0} ar Fig. 5a: ν_{x0}= 8.544, 8.545, 8.546, 8.547.

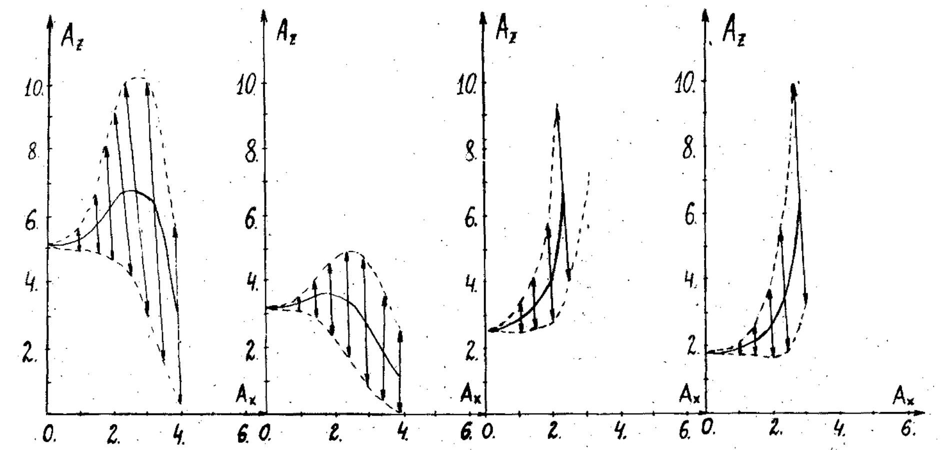

Fig. 7. The same as Fig. 6, but for the parameters of Fig. 5c. The cases a,b,c,d refer to ν_{x0} = 8.543, 8.544, 8.545, 8.546.

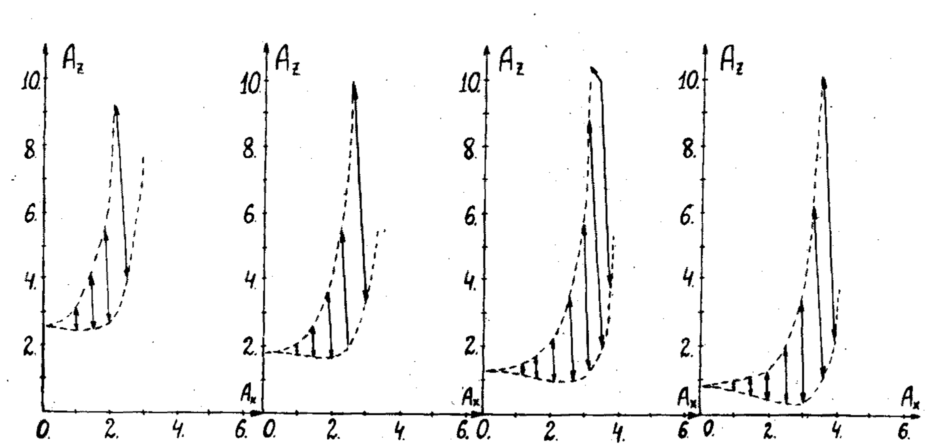

Fig. 8. The same as Fig. 6, but for the parameters of Fig. 5d. The cases a,b,c,d refer to ν_{x0} = 8.545, 8.546, 8.547, 8.548

with $A_z = 5$, and their trajectories were "tracked", until A_z dropped to $A_z = 3$. It was clearly observed that for the case of positive machine nonlinearity (see below) many particles ascended to large vertical amplitudes along the resonance (0,10). Examples of such trajectories are shown Fig. 9b (together with the location of the resonance (0,10) - see below). It should be noted, that in the absence of synchrotron modulations the number of such particles drastically decreased.

To observe the location, width and oscillation direction of the resonance, surface-of-section plots were made, analogous to those of Figs. 6-8. The results of these "tracking" studies are shown in Figs. 9a,b,c.

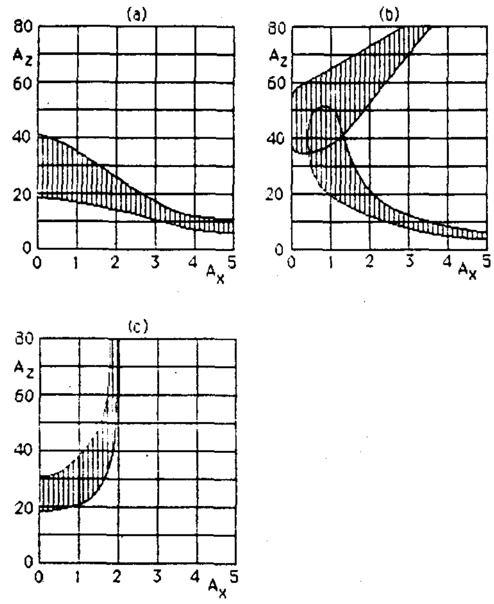

Fig. 9. The resonance $10\,\nu_z = 96$ in the plane A_x, A_z for the cases: a - zero machine nonlinearity, b - positive nonlinearity, c - negative nonlinearity. Betatron tunes are $\nu_{x0} = 8.555$, $\nu_{z0} = 9.599$, tune shifts are $\xi_x = 0.01$, $\xi_z = 0.03$. Machine tune shifts are the same in Fig. 5.

The three cases correspond to zero, positive, and negative nonlinearity, respectively. All three were calculated at $\nu_z = 9.599$, $\nu_x = 8.555$, $\xi_z = 0.03$, $\xi_x = 0.01$. The magnitude of the octupole-induced tune shifts were $\Delta\nu_x = \pm 2.5\cdot 10^{-5}$ at $A_z = 0$, $A_x = 1$ and $\Delta\nu_z = \pm 2.5\cdot 10^{-7}$ at $A_z = 1$, $A_x = 0$.

In the case of zero nonlinearity (no octupole lens) the resonance spans the distance from about $A_z = 20$ to $A_z = 40$ at $A_x = 0$. The period of resonant oscillation is about 20% of the damping time. The direction of resonant oscillation is vertical, so that vertical damping is eliminated inside the resonance. It is therefore expected that at small A_x amplitudes, the density of particles between $A_z = 20$ and $A_z = 40$ is approximately constant. However, the aperture in our case is at about 80, and without additional resonances, it is very difficult for a particle to overcome the very strong damping between 40 and 80. The situation changes dramatically (as shown in Ref. 19) when positive nonlinearity is added (Fig. 9b). On a certain line in amplitude space, the beam-beam induced nonlinearity is cancelled by the octupole induced nonlinearity. Where the resonance crosses this line, it folds back on itself and a region of very wide oscillation appears. Since a sufficiently small vertical damping is completely cancelled inside the resonance, the particle distribution here is dependent only on A_x. At $A_x = 1.5$, the resonance spans the entire distance from $A_z = 15$ to $A_z = 70$. The lower edge of the resonance can be considered a nonlinear dynamic aperture. Thus, the coupling of the beam-beam interaction to a thin octupole lens can seriously affect the rate of particle loss when the lens nonlinearity is positive.

The case of negative nonlinearity is shown in Fig. 9c. Here there is no line of zero z-nonlinearity, and the effect of the octupole is to simply bend the resonance upwards as A_x increases. Since vertical damping is still cancelled inside the resonance, even this situation should be detrimental to beam lifetime. Particles entering the resonance at its bottom edge should be able to diffuse (from quantum-induced fluctuations of A_x) up the resonance to the aperture. However, since the area covered by the resonance is much smaller here than in the previous case, the reduction in lifetime is also expected to be smaller. It is interesting to note that the resonance (4,-2), shown at Figs. 6-8, doesn't show up in the lifetime measurements. This implies that the influence of the resonance on beam lifetime depends not only on it's width (and, possibly, not so much on it's width), but on the location of the resonance line as well. Thus, the negative inclination of the resonance line, generic (for zero machine nonlinearity) for sum resonances (and present as well at Fig. 9) makes possible a "streaming" of particles along the resonance (see /21/), which can lead to a decrease of the beam lifetime.

Finally, we would like to express our sincere gratitude to A.A.Zholents, A.B.Temnykh, and G.M.Tumaikin for fruitfull discussions and helpfull comments.

References

1. F.M.Izrailev, "Nearly linear mappings and their applications" INP Preprint 80-149, Novosibirsk, 1980; Physica 1D (1980) 243.
2. A.Chao, "The Beam-Beam Instability" Lecture at the 3rd Summer School on High Energy Particle Accelerators, New York (1983); SLAC-PUB-3179 (1983).
3. L.R.Evans, "The Beam-Beam Interaction" Lecture at the CERN Accelerator School (1983); CERN SPS/83-38 (DI-MST).
4. A.Piwinski, IEEE Trans. NS-24 (1977) 1408; DESY-77/18 (1977); IEEE Trans. NS-28 (1981) 2440.
5. S.Myers, Nucl. Instr. and Meth. 211(1983) 263-282.
6. A.L.Gerasimov, F.M.Izrailev, I.B.Vasserman, J.L.Tennyson. "Beam-Beam effects for elliptic beam with a large aspect ratio" Report at the XII International High Energy Accelerator Conference, Batavia, USA, 1983; Preprint 84-16 INP, Novosibirsk, 1984.
7. V.V.Vecheslavov, "Approximate representation of the beam potential for elliptic beams", Preprint 80-72 INP, Novosibirsk, 1980- in Russian.
8. B.V.Chirikov, Physics Reports, 52 (1979) 263.
9. A.L.Gerasimov, F.M.Izrailev, J.L.Tennyson, "Nonlinear resonances description for colliding beams with a large aspect ratio", Preprint INP 86-98, Novosibirsk, 1986.
10. Handbook of Mathematical Functions, ed. M.Abramovits, I.Stegun, Washington, D.C., 1968.
11. S.Peggs "Some aspects of Machine Physics at CESR" Ph. D. thesis, Cornell University (1981).
12. S.Peggs, R.Talman Proc. 11th Intern. Conf. on High En. Accel., July 1980, p.754.
13. J.Shonfeld "Statistical Mechanics of Colliding Beams" Fermilab-Pub-83/93-THY.
14. G.Guignard, Beam-beam limit as a function of the damping time in electron rings, IEEE Trans. on Nuclear Science, vol.NS-28, No 3, June 1981.
15. S.Myers. Lecture given in Joint CERN/US School on "Nonlinear dynamics" held in Sardinia, Italy, Feb. 1985.

16. A.B.Temnykh "The influence of a zero harmonic of a cubic nonlinearity of a leading magnetic field of a storage ring on the beam-beam effects. INP Preprint 84-143, Novosibirsk 1984 - in Russian.
17. A.Lichtenberg, M.Lieberman "Regular and Stochastic motion", Springer.
18. P.Bambade "Beam-beam driven coupling" LAL/RT/84-02 (1984.
19. A.B.Temnykh "Observations of betatron resonances at VEPP-4 storage ring, INP Preprint 84-131, Novosibirsk, 1984 - in Russian.
20. J.L.Tennyson, A.L.Gerasimov, F.M.Izrailev, Lecture given at Joint CERN/US School on "Nonlinear dynamics" held in Sardinia, Italy, Feb. 1985; INP Preprint 86-100.
21. J.L.Tennyson Physica, 1982, v.5D, p.123.