THE SPIN PRECESSION TUNE SPREAD IN THE STORAGE RING

I.A. Koop, Ju.M. Shatunov

Institute of Nuclear Physics, 630090 Novosibirsk, USSR

Introduction

Recently the experiments were performed [1] and the new ones were proposed [2] wherein the adiabatic rotation was made of the axial beam polarization into the plane that is normal to the guide magnetic field of the storage ring, after which the spins undergo free precession around the magnetic field direction. The beam depolarization time τ_d in this case will be shown below to be determined by the radiation damping time τ and by the r.m.s. precession tune spread $\langle \delta\Omega^2 \rangle$ in the beam:

$$\tau_d^{-1} = \langle \delta\Omega^2 \rangle \tau \quad . \qquad (1)$$

The precession tune spread is due to the Ω dependence on the amplitudes of betatron and synchrotron oscillations. This dependence is determined by the lattice parameters in particular by the chromaticity. Thus the value of the precession tune spread can be controlled by varying the betatron oscillation chromaticity.

Dependence of the Spin Precession Tune on the Amplitudes of Radial Betatron Oscillations and Synchrotron Oscillations

For a particle moving in the median plane of a storage ring the rotation axis direction is strictly axial, while the angular frequencies of rotation for the velocity vector $\omega \equiv d\Phi_v/dt$ and spin vector $\Omega \equiv d\Phi/dt$ are [3]:

$$\omega = -\frac{q_0}{\gamma} H, \quad \Omega = -\left(\frac{q_0}{\gamma} + q_a\right) H, \qquad (2)$$

where q_0 and q_a are the normal and anomalous parts of the particle gyromagnetic ratio, $H(\theta, x)$ is the magnetic field, γ is the Lorentz factor, Φ_v and Φ are the rotation phases of the velocity and spin vectors. The electric field is considered to be zero.

It is convenient to transform these equations taking the generalized azimuth θ as an independent variable instead of t:

$$\nu_v \equiv \frac{d\Phi_v}{d\theta} = -\frac{q_0}{\gamma} H \frac{dt}{d\theta}, \quad \nu \equiv \frac{d\Phi}{d\theta} - \frac{d\Phi_v}{d\theta} = -q_a H \frac{dt}{d\theta}. \qquad (3)$$

Apparently the value of ν_v averaged over θ and over the radial betatron oscillation phase Φ_x is equal to unity, therefore:

$$\langle H \frac{dt}{d\theta} \rangle_{\theta,\Phi_x} = -\frac{\gamma}{q_0}, \quad \langle \nu \rangle_{\theta,\Phi_x} = \frac{q_a}{q_0}\gamma. \qquad (4)$$

After averaging eq. (4) over the synchrotron oscillation phase also we conclude that the mean differential precession tune $\bar{\nu}$ is uniquely determined by the average particle energy $\bar{\gamma}$, about which the synchrotron oscillations proceed, however $\bar{\gamma}$ is not identical to the equilibrium energy γ_s in the general case. Introducing $\delta = (\gamma - \gamma_s)/\gamma_s$ we put down the betatron and the synchrotron oscillations in the form

$$\delta = \bar{\delta} + a_b \cos \Phi_b, \quad \tilde{X} = a_x f \cos \Phi_x + \tilde{X}_1, \qquad (5)$$

where a_b and a_x are the amplitudes of the first harmonics in the synchrotron and betatron oscillations, f is the absolute value of the Floquet function, $\tilde{X}_1$ is the second order in a_x and periodic in θ correction to the solution of the equations of motion.

The total excursion from the design orbit X will be sought as a superposition

$$X = \Psi\delta + \Psi_1\delta^2 + \tilde{X}. \qquad (6)$$

In the natural coordinate system we have with the account of the second order terms [4]

$$\frac{dt}{d\theta} = \frac{1}{\omega_0}\left[1 + KX + \frac{x'^2}{2}\right], \qquad (7)$$

$$X'' + [K^2 + h' - (2K^2 + h')\delta]X =$$
$$= K(\delta - \delta^2) - h''\frac{X^2}{2} - (K^2 + 2h')KX^2 + K\frac{X'^2}{2} + K'XX', \qquad (8)$$

where ω_0 is the revolution frequency of the equilibrium particle, K is the dimensionless orbit curvature, i. e. $\langle K \rangle_\theta = 1$, and the other values are also dimensionless: X is the excursion divided by the storage ring gross radius, $h' = \frac{1}{\langle H \rangle_\theta}\frac{\partial H}{\partial X}$, $h'' = \frac{1}{\langle H \rangle_\theta}\frac{\partial^2 H}{\partial X^2}$, and the derivatives of X and K are taken with respect to θ. The dispersion functions Ψ, Ψ_1 and the Floquet function absolute value f are the periodic solutions of the known equations:

$$\Psi'' + g_x\Psi = K, \quad f'' + g_x f = \frac{1}{f^3}, \qquad (9)$$

$$\Psi_1'' + g_x\Psi_1 = -K + (g_x + K^2)\Psi -$$
$$-\left[\frac{h''}{2} + K(2g_x - K^2)\right]\Psi^2 + K\frac{\Psi'^2}{2} + K'\Psi\Psi', \qquad (10)$$

where $g_x = K^2 + h'$. The betatron excursion $\tilde{X}$ satisfies an equation

$$\tilde{X}'' + \{g_x + [h''\Psi - g_x - K^2 - K'\Psi' + 2K(2g_x - K^2)\Psi]\delta\}\tilde{X} =$$
$$= K\frac{\tilde{X}'^2}{2} - \left[\frac{h''}{2} + K(2g_x - K^2)\right]\tilde{X}^2 + K'\tilde{X}\tilde{X}' + (K\Psi)'\tilde{X}'\delta. \qquad (11)$$

The periodic correction solution of the eq. (11) $\tilde{X}_1$, that is second order in a_x, satisfies an equation:

$$\tilde{X}_1'' + g_x\tilde{X}_1 = \frac{a_x^2}{2}\left\{\frac{1}{2}K\left(f'^2 + \frac{1}{f^2}\right) - \left[\frac{h''}{2} + K(2g_x - K^2)\right]f^2 + K'ff'\right\}. \qquad (12)$$

The auto-phasing mechanism provides for the isochronism in all particles' motion on the average, i. e. averaging over θ and over the phases of betatron and synchrotron motion $\langle dt/d\theta \rangle = 1/\omega_0$, hence and in combination with eq. (7) we conclude that

$$\langle KX \rangle = -\frac{\langle X'^2 \rangle}{2}. \qquad (13)$$

Thus the trajectory lengthening due to the particle moving at the crossing angle with the design orbit must be cancelled out by the trajectory contraction to the less radius and therefore by the change of the average energy. Substitution of (5) and (6) into (13) yields:

$$\langle KX \rangle = \langle K\Psi \rangle\bar{\delta} + \langle K\Psi_1 \rangle\frac{a_b^2}{2} + \langle KX_1 \rangle,$$
$$\langle X'^2 \rangle = \langle f'^2 + \frac{1}{f^2} \rangle\frac{a_x^2}{2} + \langle \Psi'^2 \rangle\frac{a_b^2}{2}. \qquad (14)$$

Taking account of $\langle K\Psi \rangle = \alpha$, where α is the orbit compaction factor, we find for the energy displacement $\bar{\delta}$ on the average:

$$\bar{\delta} = \alpha^{-1}\left[-\langle KX_1 \rangle - \frac{a_x^2}{4}\langle f'^2 + \frac{1}{f^2} \rangle - \frac{a_b^2}{2}\langle K\Psi_1 + \frac{\Psi'^2}{2} \rangle\right]. \qquad (15)$$

The values $\langle K\Psi_1 \rangle$ and $\langle KX_1 \rangle$ averaged over θ are readily expressed through K, Ψ, f, Ψ' f' by means of one-turn integration of various linear combinations of eqs. (9), (10), (12):

$$\langle K\Psi_1 \rangle = \langle \Psi'^2 - K^2\Psi^2 + \left(K^3 - \frac{h''}{2}\right)\Psi^3 - \frac{7}{2}K\Psi\Psi'^2 - K'\Psi^2\Psi' \rangle,$$

$$\langle KX_1 \rangle = \frac{a_x^2}{2}\langle \left(K^3 - \frac{h''}{2}\right)\Psi f^2 -$$
$$-\frac{3}{2}K\Psi\left(f'^2 + \frac{1}{f^2}\right) - 2K\Psi'ff' - K'\Psi ff' \rangle. \qquad (16)$$

Thus

$$\bar{\delta}=\alpha^{-1}\Big\{-\frac{a_x^2}{2}\langle\frac{1}{2}\left(f'^2+\frac{1}{f^2}\right)+\left(K^3-\frac{h''}{2}\right)\Psi f^2-$$
$$-\frac{3}{2}\left(f'^2+\frac{1}{f^2}\right)K\Psi-2ff'K\Psi'-ff'K'\Psi\rangle-$$
$$-\frac{a_0^2}{2}\langle\frac{3}{2}\Psi'^2-K^2\Psi^2+\left(K^3-\frac{h''}{2}\right)\Psi^3-\frac{7}{2}K\Psi\Psi'^2-K'\Psi^3\Psi'\rangle\Big\}. \quad (17)$$

The betatron contribution into $\bar{\delta}$ can be conveniently expressed via the radial oscillation chromaticity:

$$\gamma\frac{\partial\nu_x}{\partial\gamma}=\frac{1}{2}\langle f^2[h''\Psi-g_x-K^2-K'\Psi'+2K(2g_x-K^2)\Psi]\rangle, \quad (18)$$

where similarly to eq. (17) we can eliminate g_x terms:

$$\gamma\frac{\partial\nu_x}{\partial\gamma}=\langle-\frac{1}{2}\left(f'^2+\frac{1}{f^2}\right)+\left(\frac{h''}{2}-K^3\right)\Psi f^2-$$
$$-\frac{1}{2}(K^2+K'\Psi')f^2+2\left(f'^2+\frac{1}{f^2}\right)K\Psi+2ff'K'\Psi+2ff'K\Psi'\rangle. \quad (19)$$

Substituting (19) into (17) we express the terms of eq. (17) that do not involve K through $\gamma\frac{\partial\nu_x}{\partial\gamma}$:

$$\bar{\delta}=\alpha^{-1}\frac{a_x^2}{2}\Big[\gamma\frac{\partial\nu_x}{\partial\gamma}-\langle K'\Psi ff'+$$
$$+\frac{1}{2}K\Psi\left(f'^2+\frac{1}{f^2}\right)-\frac{1}{2}f^2(K^2+K'\Psi')\rangle\Big]+$$
$$+\alpha^{-1}\frac{a_0^2}{2}\langle\left(\frac{h''}{2}-K^3\right)\Psi^3+K^2\Psi^2-\frac{3}{2}\Psi'^2+$$
$$+\frac{7}{2}K\Psi\Psi'^2+K'\Psi'\Psi^2\rangle. \quad (20)$$

In the storage rings with strong focusing lattices the contribution of the curvature K terms is very small, so is usually the contribution of the synchrotron oscillations, therefore the minimum of the average energy spread and, consequently the minimum of the spin tune spread $\delta\nu=\nu\cdot\bar{\delta}$ occures at the chromaticity close to zero.

Diffusion of the Spin Precession Phase

Consider a given dependence of the spin precession frequency Ω on the squared amplitude a^2 of a specified oscillation. To the first order in a^2 we have

$$\Omega=\Omega_0+\frac{\partial\Omega}{\partial a^2}a^2. \quad (21)$$

We assume the Gaussian distribution over amplitudes with The r.m.s. deviation of σ_a. Then the value of Ω averaged over the particles ensemble is equal to

$$\bar{\Omega}=\Omega_0+\frac{\partial\Omega}{\partial a^2}\sigma_a^2. \quad (22)$$

Let the initial phases of precession coinside for all particles. We find the r.m.s. deviation of the phase Φ for any particle from the average phase $\bar{\Phi}$ within the time t:

$$\overline{(\Phi-\bar{\Phi})^2}=\left(\frac{\partial\Omega}{\partial a^2}\right)^2\cdot\overline{\int_0^t[a^2(t')-\sigma_a^2]\,dt'\cdot\int_0^t[a^2(t'')-\sigma_a^2]\,dt''}. \quad (23)$$

One can show (see Appendix) that the correlation $\overline{[a^2(t')-\sigma_a^2][a^2(t'')-\sigma_a^2]}$ averaged over the initial amplitude distribution is equal to $2\sigma_a^4\exp\left(-\frac{2|t'-t''|}{\tau}\right)$ where τ stands for the damping time of the specified oscillation. After the integration we obtain:

$$\overline{(\Phi-\bar{\Phi})^2}=2\sigma_a^4\left(\frac{\partial\Omega}{\partial a^2}\right)^2\Big[\tau t-\frac{\tau^2}{2}(1-e^{-2t/\tau})\Big]. \quad (24)$$

The factor out of the brackets is actually the mean square of the precession tune spread $\overline{(\Omega-\bar{\Omega}_0)^2}$. Eq. (24) has an evident physical meaning: within the times t much less than the amplitude randomization time τ the phase deviation grows proportionally with time, then at $t\gg\tau$ we observe the establishment of the square root dependence characteristic of a diffusion process.

Acknowledgements. Authors wishes to thank E.A. Perevedentsev and A.A. Polunin for helpful discussions.

Appendix

To calculate the correlations of the form $\overline{a^2(t')a^2(t'')}$ we have to know the form of the non-equilibrium distribution function $f(t, a)$ over the particles' amplitudes. It obeys the kinetic equation

$$\frac{\partial f}{\partial t}-\frac{f}{\tau}-\frac{a}{\tau}\frac{\partial f}{\partial a}-\frac{\sigma_a^2}{\tau}\frac{\partial^2 f}{\partial a^2}=0. \quad (A1)$$

The equilibrium distribution function has the form

$$f_0=\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_a}e^{-a^2/2\sigma_a^2}. \quad (A2)$$

Let all the particles have the amplitude a_0 at the initial time. If there is no diffusion the oscillation amplitude be damped following the law $\tilde{a}=a_0e^{-t/\tau}$. One can prove by substitution into eq. (A1) that the distribution function

$$f=\frac{1}{\sqrt{2\pi}\,\sigma(t)}\Big[\exp\left(-\frac{(a-\tilde{a})^2}{2\sigma^2(t)}\right)+\exp\left(-\frac{(a+\tilde{a})^2}{2\sigma^2(t)}\right)\Big], \quad (A3)$$

where $\sigma^2(t)=\sigma_a^2(1-e^{-2t/\tau})$ satisfies the kinetic equation and has the δ-function limit at the initial time. The second exponential is added for symmetry of f to provide for $\partial f/\partial a=0$ at the origin. Making use of this distribution function we find the correlation

$$\overline{a^2(t)\,a^2(0)}=\sigma_a^2a_0^2(1-e^{-2t/\tau})+a_0^4e^{-2t/\tau}. \quad (A4)$$

Taking the averaging over the initial amplitude a_0, that is assumed to be distributed by the function in eq. (A2), we find the disired correlation

$$\overline{\overline{a^2(t)\,a^2(0)}}=\sigma_a^4(1+2e^{-2t/\tau}). \quad (A5)$$

References

1. I.B. Vasserman et al. Phys. Lett., 1987, v.B198, p.302. Preprint INP 87-117, Novosibirsk, 1987.
2. P.I. Baturin et al. «The dedicated Storage Ring with Longitudinaly Polarized Electrons for Internal Target Experiments». — A Talk at the I-st European Accelerator Conference. Rome, 1988.
3. Ja.S. Derbenev, A.M. Kondratenko, A.N. Skrinšky. Sov. Phys. JETP, 1971, v.60, p.1216.
4. A.A. Kolomensky, A.N. Lebedev. «The Theory of Accelerators». — Moscow, 1962.