DYNAMIC APERTURE LIMITATION IN STORAGE RINGS DUE TO SOLENOIDS

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Abstract: Focusing solenoids incorporated in the storage ring lattice may be especially useful in colliding beam machines at intermediate energies (≤ 0.5 GeV). An example to this can be the Novosibirsk ϕ-factory project which envisages for a new generation collider with ultra-high luminosity at the ϕ-meson resonance energy [1]. Here strong superconducting solenoids perform a combined service to provide for round colliding beams: *i)* focusing to obtain equal β-functions of very low value at the collision point ($\beta_x = \beta_z \simeq 1$ *cm*); *ii)* equalizing transverse emittances of the colliding bunches due to coupling of the betatron modes which are alternately excited by radiative diffusion in the two arcs of ϕ-factory.

However, these advantages of solenoids are paid for with major nonlinear perturbations in transverse motion due to their end-fields. This results in tune shifts (and spreads), excitation of nonlinear resonances and limitation of dynamic aperture.

The paper presents analytic estimates of the end-field nonlinearity effect and simulation results to determine the dynamic aperture in one of the ϕ-factory lattice versions.

Equations of motion

Using the cylindric coordinates r, θ, s and the kinetic momenta p_r, p_θ, p_s, we put down the equations of motion for a particle of momentum p in the axially symmetric magnetic field H_r, H_s with the vector potential A_θ [2]:

$$\dot{p}_r - \dot{\theta}\, p_\theta = \frac{e}{c}\, r\, \dot{\theta}\, H_s \; ; \qquad (1)$$

$$p_\theta + \frac{e}{c}\, r\, A_\theta = const = r_0^2\, \dot{\theta}_0 \; ; \qquad (2)$$

while

$$r\, A_\theta \equiv \frac{\Phi}{2\pi} = \int_0^r H_s\, r' dr' \; , \qquad (2')$$

where Φ stands for the magnetic flux;

$$\dot{p}_s = - \frac{e}{c}\, r\, \dot{\theta}\, H_r \; . \qquad (3)$$

Taking for the ultra-relativistic particle $p \cong \gamma mc$ and substituting (2′) to (2), we obtain in place of (1) and (2):

$$\ddot{r} - r\, \dot{\theta}\, (\, \dot{\theta} + \frac{e}{p} H_s) = 0 \; , \qquad (4)$$

$$\dot{\theta} = \dot{\theta}_0 r_0^2 / \, r^2 - \frac{e}{2p} \frac{\Phi}{\pi r^2} \; . \qquad (5)$$

Introducing the paraxial field description by its on-axis longitudinal component $H(s)$ and by the relevant derivatives $H' = dH(s)/ds$:

$$H_s = H - \frac{1}{4} H'' r^2 + \ldots \qquad (6)$$

we obtain the radial motion equation to the first approximation:

$$\ddot{r} + \left[\left(\frac{eH}{2p}\right)^2 - \left(\dot{\theta}_0 r_0^2 / r^2\right)^2 \right] r = \frac{1}{8}\left(\frac{e}{p}\right)^2 HH'' \cdot r^3 \qquad (7)$$

for any trajectory, including non-meridianal ones: $\dot{\theta}_0 \neq 0$.

In the cartesian coordinates x, z, s the nonlinear "centrifugal" force in the left-hand side of eq. (7) will naturally disappear. Then changing to the normalized variables $X = x/\sqrt{\beta}$, $Z = z/\sqrt{\beta}$, ψ, ($\beta_x = \beta_z$, $\psi_x = \psi_z$ for the round beams) we obtain instead of eq. (7):

$$\ddot{X} + X = \frac{1}{8}\left(\frac{e}{pc}\right)^2 HH'' \beta^3 \cdot R^2 X \; , \qquad (8)$$

$$\ddot{Z} + Z = \frac{1}{8}\left(\frac{e}{pc}\right)^2 HH'' \beta^3 \cdot R^2 Z \; , \qquad (8')$$

where $R^2 = X^2 + Z^2$. These equations are uncoupled in the frame, rotating at the Larmor frequency taken on the axis, while the nonlinear contribution to θ in (5) will be negligible for sufficiently short ranges of nonlinear field ($r \cong const$). Thus the problem is reduced to the 1-dimensional motion, and the main nonlinear term from the solenoid end-field is just a cubic nonlinearity, acting radially.

End-field nonlinearity

The micro-β condition at the collision point apparently gives large β-values in the focusing solenoids ($\beta \simeq 15$ *m* at the nearest solenoid entrance in the ϕ-factory lattice). If the betatron phase advance over the end region is small enough: $\Delta\psi \simeq$(coil radius)$/\beta \simeq 2\cdot 10^{-3}$, we can account for this lumped perturbation as a thin octupole lens:

$$\ddot{X} + X = -\alpha\, R^2 X\, \delta(\psi), \qquad (9)$$

$$\ddot{Z} + Z = -\alpha\, R^2 Z\, \delta(\psi), \qquad (9')$$

with the integral strength α :

$$\alpha = - \frac{1}{8}\left(\frac{e}{pc}\right)^2 \int_{-\infty}^{+\infty} H\, H''\, \beta^2\, ds \qquad (10)$$

If the β-function varies slowly over the end range, we can give a very simple estimate:

$$\alpha \simeq \left(\frac{e}{pc}\right)^2 \frac{\beta^2}{8} \int_{-\infty}^{+\infty} (H')^2\, ds \; , \qquad (11)$$

which comes to $\simeq 100$ cm^{-1} for the worst of our ends. Here the contribution of the 5th-order terms, omitted in (8), is within 1 per cent for our maximal amplitudes.

From eqs. (10), (11) one can see an important general property of the axi-symmetric field: *its contribution to the cubic nonlinearity is positively definite for sufficiently short end range.* Hence we are not able to compensate for this perturbation by means of an axi-symmetric corrector. To minimize this effect one can only reduce β-values at the ends, or shape the solenoid coil so as the end range be longer.

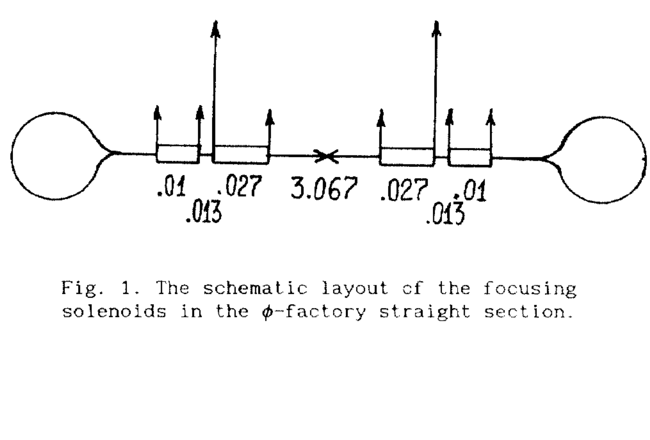

Fig. 1. The schematic layout of the focusing solenoids in the ϕ-factory straight section.

Fig. 1 shows the relative strength of the solenoid end perturbations in the ϕ-factory lattice and the phase relations between them. The special feature of our lattice is that the phase advance between the strongest two perturbations is π, hence in an analytical approach below we can merge all the end-field perturbations in one octupole lens with the total strength. Besides, the arcs have the transport matrices Tx= Tz, that enables a one-dimensional treatment of motion all round the turn, because the meridianal trajectories are thus preserved.

Analytic estimates

Let us describe the motion in the ϕ-factory lattice in terms of the phase space map $\mathcal{M}$ after each passage of 1/2 revolution: first the particles pass through the thin octupole lens, and from (9): $\Delta R' = -\alpha \cdot R^3$; then they pass through the linear optics of the arc and gain the betatron phase advance $p = 2\pi\{\nu/2\}$, here ν is the tune. Hence the transport matrix in the normalized variables will correspond to a simple rotation:

$$T = \begin{pmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{pmatrix} .$$

The fixed points of the map $\mathcal{M}^n$ are the periodic trajectories closed over n turns, they are easy to find analytically. We will use their positions for a reasonable estimation of the dynamic aperture limits. Keeping in mind the positive cubic nonlinearity, we will see that the study of the fixed points only for $2 \le n \le 6$ will be sufficient.

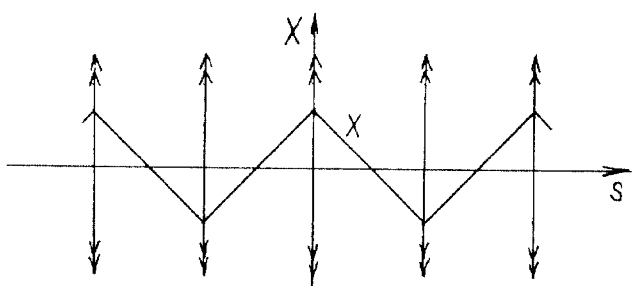

Fig. 2. The 2nd-order periodic trajectories.

Let us consider the simplest case n=2 in detail. For both periodic trajectories closed over two turns, the displacement X and the angle X' are readily found:

$$X = a \sin(p/2-\varphi), \qquad X' = - a \cos(p/2-\varphi),$$

φ=0 at the position of the octupole, whose action gives:

$$\Delta X' = - 2 a \cos p/2 = - \alpha X^3 .$$

Hence, we find for the stationary amplitude a_{2r}:

$$a_{2r} = \frac{2 \cos p/2}{\alpha \sin^3 p/2} \qquad (12)$$

Linearizing the motion in the vicinity of this resonance: $X^3 - X_r^3 = 3 X_r^2 \Delta X$, we obtain for the optical strength of the emerging linear lens: $1/F = 3 \alpha X_r^2$, and the stability of the fixed point is easily judged from the matrix M_2 of the relevant symplectic map:

$$M_2 = \begin{pmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} ,$$

$$\mathrm{Sp}\, M_2 = -(1 + 4\cos^2 p/2) < - 1 ,$$

thus the 2nd order fixed points are always unstable.

Now consider the fixed points for n=4: here we have two couples of points, the "bad" one is unstable. It can be obtained from the previous study if we place in Fig. 2 an additional octupole lens in the nodes of the resonance trajectory or, in other words, if we replace $p/2$ by p (see curve 1 in Fig. 3a). The 2nd fixed point corresponds to the trajectory 2 in Fig. 3a. It can be found from the set of equations:

$$X_1 = a \sin p/2 = b \cos p/2 = X_2 ,$$

$$X_1' - X_2' = -\alpha X_1^3 ,$$

or: $$a \cos p/2 - b \sin p/2 = \alpha a^3 \sin^3 p/2 ,$$

hence $$a_{4r} = \frac{\cos p}{\alpha \sin^3 p/2 \cos p/2} . \qquad (13)$$

Now the linearization yields $\mathrm{Sp} M_4 = -2\cos p$, and we can see that this "good" fixed point can only exist and be stable for $\pi/3 < p < \pi/2$.

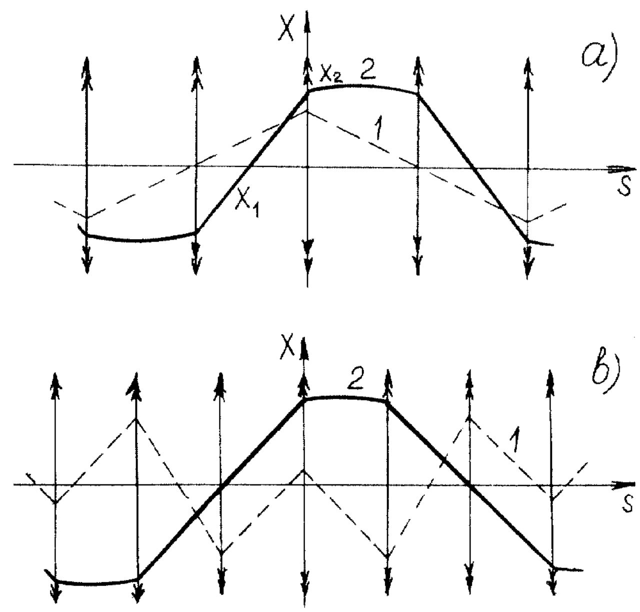

Fig. 3. The 4th- and the 6th-order periodic trajectories.

The case n = 6 is studied similarly and yields for the triplet of "good" fixed points, whose trajectories resemble curve 2 in Fig. 3b :

$$a_{6r} = \frac{\cos 3p/2}{\alpha \sin p \cos^3 p/2} ; \quad b_{6r} = \frac{\cos 3p/2}{\alpha \sin^3 p \cos p/2} ; \qquad (14)$$

$$\mathrm{Sp}\, M_6 = 2(2\cos p - 1)^3 - 1 ,$$

with the stability range of $0 < p < \pi/3$ just coinciding with the existance range (mind that the nonlinearity is positive!). The other triplet of fixed point of $\mathcal{M}^6$ is un stable as it should be in general [3].

The fixed points of $\mathcal{M}^3$ and $\mathcal{M}^5$ are obtained in the same manner, but the emerging transcedental equations do not allow for the same simple form of the results as for n = 2,4 and 6. The numerical solution reveals a simple general relation for $2 \le n \le 5$: in the range of its existance $p \in [\, 0,\ 2\pi/n]$ the "good fixed point" is unstable for $p \in [0,\ \approx\pi/(n-1)]$, that is just in the interesting range! And only beginning from the 6th order the resonances acquire the "good fixed point" that is stable throughout the resonance existance range.

This conclusion enables a simple estimation of dynamic aperture based on eq. (14) for the 6th order fixed points (in the case $p \ll 1$ and $\beta' = 0$) :

$$X_6 = \sqrt{\frac{\beta}{\alpha\ p}}\ , \qquad X'_6 = 1\ /\ \sqrt{\alpha\ \beta\ p^3}\ , \tag{15}$$

which restrict the phase space domain, circumscribed by the last resonance yet having the stable fixed point Somewhat more rigorous definition of dynamic aperture might be: the domain which is circumscribed by the largest invariant trajectory which is separated from the resonance by the width of its manifold [3] (*i.e.* its stochastic layer). For our parameters the width appeared to be small as one could see from the simulation results (see Fig. 5). For the ϕ-factory lattice in question eq.(15) gives $\simeq 4$ *cm* or about 15σ for estimated dynamic aperture.

Note that in the case of small deviation from meridianal motion which is practically interesting for us the two-dimensional problem is reduced to radial beating and some additional slow precession of the 1D trajectories analyzed above. These phenomena (see Fig. 7) manifest the energy exchange between the two modes.

Simulation

The numerical simulation was based on the realistic ϕ-factory lattice with 8 ends of the focusing solenoids, actual betatron phase relations and β-values (see Fig.1).To raise the efficiency of the code we used the normalized variables defined above:X, X', Z, Z' for iterations so as the account of the linear optics be trivial, and thin octupole lenses at each solenoid end acted radially, with the integral strength (10) accurately calculated once before tracking with the account of $\beta \neq$ *const* over the end range.

The clues to understanding of the simulation results are given in the previous section, and the agreement was good between the simulation and the analytical approach both in qualitative insight and in quantitative estimates. Typical patterns are shown in Fig.5.

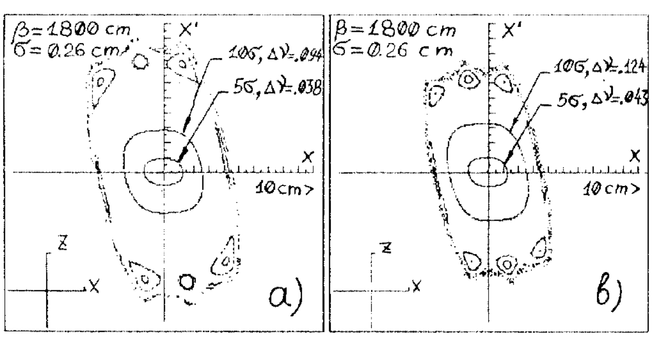

Fig. 5. Paraxial (*a*) and non-paraxial (*b*) map for the ϕ-factory lattice with solenoids.

The simulation included non-paraxial effects due to large angular spread of particle trajectories in the collision straight, resulting from low β^* and high β' This results in modulation of longitudinal velocity for the subsequent passages through solenoids. We preferred to track these passages in terms of time rather than azimuth according to eq. (3). This effect caused the dynamic aperture reduction by $\simeq$15%, however its behaviour resembled the main effect of end-fields when the latter were switched off. The similarity is shown in Figs. 5a,b and also can be seen in Fig. 6 where the tune shifts are plotted vs.the amplitude squared.

The effect of non-meridianal trajectories was simulated for various angular momenta. It caused some insignificant ($\simeq$15%) decrease in dynamic aperture:Fig.7.

Though apparently useless, the "normal octupole" lens was tried in simulation to compensate for the tune growing with amplitudes. It did not work because of precession which averaged out its action. Of course more elaborate schemes using βx- βz differences at the octupoles of opposite signs are not forbidden however they do not seem to be practical in a very tightly-bound lattice like that of the ϕ-factory.

Conclusions

The analytical estimates and the simulation results agree in an optimistic conclusion: although the nonlinear end-fields of the strong superconducting solenoids placed in the ϕ-factory collision straight for micro-β focusing and for round colliding beams formation do severely restrict the dynamic aperture of the machine, we still can hope for about 15σ. This could be enough unless some other effects (say, sextupoles) would not do worse.

Better situation can be striven for by:

i) shaping the coil ends to expand the effective length of the end-field;

ii) keeping the betatron tune closer to the integer;

iii) choosing reasonably low β-values in solenoids;

iv) partial compensation in a lattice with opposite H' and β' signs at the ends ;

v) "normal" octupoles in an elaborate lattice providing for large βx- βz differences in the octupoles of opposite polarity, e.t.c..?

Neither of these means is simple, and they must reach a compromise with the luminosity.

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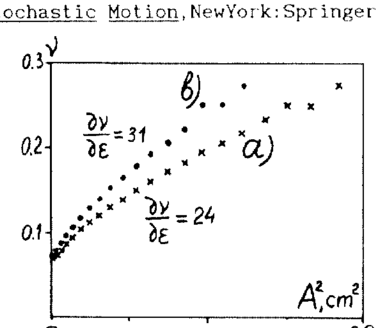

Fig. 6. Tune shifts dependence on amplitudes in the paraxial (*a*) and non-paraxial (*b*) maps.

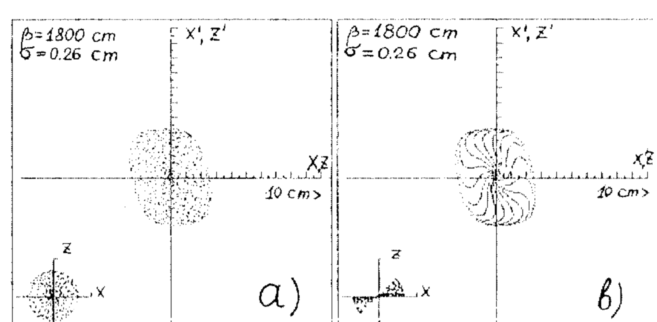

Fig. 7. Non-meridianal trajectories: strong (*a*) and weak (*b*) effect.