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Bosonic Chiral Anomalies in Gravitational Field. Macroscopic Implications

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Abstract

The notion of chirality for electromagnetic field is defined. The chirality is classically conserved in gravitational interaction.. The corresponding chiral current is however anomalous in external gravitational field. This anomaly is analogous to the well-known fermionic triangle anomaly. The result obtained permits to calculate radiative corrections to the fermionic chiral anomaly in gravitational field. The relation between the number of zero modes of antisymmetric tensor gauge field (with zero spin) and the anomaly for vector field is considered. The possible observational manifestations of the chiral anomaly in the gravitational field of the Kerr black hole are discussed.

1 Introduction

It is well known that the axial current of massless fermions, which is formally conserved because of classical equations of motions, possesses the anomaly connected with the triangle diagram [1,2]

$$D_\mu a^\mu = \frac{Q^2\alpha}{2\pi} F_{\mu\nu}\widetilde{F}^{\mu\nu} - \frac{1}{192\pi^2} R_{\mu\nu\kappa\lambda}\widetilde{R}^{\mu\nu\kappa\lambda} \tag{1}$$

where $a^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. ψ is the massless Dirac field with electric charge Q, D_μ is the covariant derivative in external gravitational field,

$$D_\mu a^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,a^\mu\right),$$

$F_{\mu\nu}$ is the electromagnetic field strength tensor, $R_{\mu\nu\kappa\lambda}$ is the Riemann tensor, and

$$\widetilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \qquad \widetilde{R}^{\mu\nu\kappa\lambda} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\kappa\lambda}.$$

We will consider here effects connected with the external gravitational field (the second term in eq. (1)). The existence of analogous equation for bosons can be understood by the following heuristic arguments. Let $R\widetilde{R} \neq \emptyset$ in some space region. Then because of eq. (1) there must exist the flow of leptonic charge from that region. Of course, gravitational interaction "knows" nothing about leptonic charge, but for massless Weyl fermions there exists one-to-one correspondence between chirality and leptonic charge. So leptonic charge nonconservation (1) is natural to interpret as chirality nonconservation connected with the interaction of the particle spin with the gravitational field.

Gravitational interaction is known to be universal. Hence the anomaly of the same form as (1) must exist for any spinning particle, for boson as well as for fermion. Indeed it can be shown that the analog of eq. (1) for photon is of the form [3]:

$$\langle D_\mu K^\mu \rangle = -\frac{1}{96\pi^2} R_{\mu\nu\kappa\lambda} \widetilde{R}^{\mu\nu\kappa\lambda} \quad (2)$$

where $K^\mu = -\frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} A_\mu \partial_\kappa A_\lambda$ and A_ν is the electromagnetic vector-potential. Using the operator identity $D_\mu K^\mu = -\frac{1}{2} F_{\mu\nu} \widetilde{F}^{\mu\nu}$ we can rewrite eq. (2) in the equivalent form:

$$\left\langle F_{\mu\nu} \widetilde{F}^{\mu\nu} \right\rangle = \frac{1}{48\pi^2} R_{\mu\nu\kappa\lambda} \widetilde{R}^{\mu\nu\kappa\lambda} \quad (3)$$

In contrast to fermionic case, current K^μ is not conserved even on classical level. Anomalous property of eq. (2) and (3) is that the average $\langle F\widetilde{F} \rangle$ in curved spacetime, which proved to be nonzero, naively vanishes because of the formal photon chirality conservation in a gravitational field. This conservation is broken by the triangle diagram.

The notion of the photon chirality deserves a more detailed discussion. It is well known that the Maxwell equations in a gravitational field are invariant under duality transformation:

$$F'_{\mu\nu} = F_{\mu\nu} \cos\alpha + \widetilde{F}_{\mu\nu} \sin\alpha. \quad (4)$$

This invariance however does not permit to construct the corresponding Noether current since transformation (4) is not expressible in terms of vector potential A_μ.

The corresponding conserved charge has been constructed in the first-order formalism where it turns out non-local [4].

However, following our paper [5], we use here another approach, the light-cone one, when photons are described by complex field A and the action is bilinear in A and A^*. In terms of these variables the photon chirality can be defined in the same way as the electron chirality. In this framework the conserved chiral current of photons can be determined, but only by the price of noncovariance of the light-cone formalism. Current K^μ defined above generates chiral transformations and is explicitly covariant, but nonconserved.

Eqs. (2) and (3) present one-loop anomaly. We will show that using these equations one can easily calculate two-loop corrections (of order α) to the fermion chiral anomaly in gravitational field [5].

The chiral anomaly of gauge vector field leads to vanishing of the chiral anomaly of antisymmetric tensor gauge field $\varphi_{\mu\nu}$. Nonzero result of ref. [5] was obtained without unknown at the time anomaly of the vector ghosts which exactly compensates for the contribution of the φ-loop. This relation permits to find the connection between the number of zero modes of $\varphi_{\mu\nu}$ and the anomaly of vector field which has no zero modes.

Gravitational fields with $R\widetilde{R} \neq \emptyset$ exist around rotating massive bodies. It is natural to expect that the nonconservation of the chiral current leads to the particle production (both bosons and fermions) in sufficiently strong external gravitational fields [8]. It resembles charged fermion production by the field of dyon [9]. The problem is more complicated however because in contrast to the dyon case the topological charge of the rotating body vanishes,

$$\int d\vec{r}\, R\widetilde{R} = 0. \quad (5)$$

Anomalous fermion nonconservation in the cosmological background with $R\widetilde{R} \neq \emptyset$ was considered in ref. [10].

The existence of the chiral anomaly for vector fields was confirmed by different from ours methods in ref. [11] and by S.V. Kostyuk. In another form such anomaly was found in ref. [12] in the first-order formalism:

$$\left\langle \widetilde{F}(F - dA) \right\rangle \neq 0.$$

The existence of bosonic anomaly for nonabelian gauge fields was pointed in ref. [13].

2 Photonic chiral current

Let consider first the case of a free electromagnetic field. It is evident that the Maxwell equations

$$\partial_\mu F_{\mu\nu} = 0, \qquad \partial_\mu \widetilde{F}_{\mu\nu} = 0$$

are invariant under duality transformation (4). We define the eigenvectors of this transformation,

$$F^{\pm}_{\mu\nu} = F_{\mu\nu} \mp i\widetilde{F}_{\mu\nu}$$

as the fields of definite chirality.

The attempt to generalize the notion of chirality to quantal case encounters difficulty because the theory is quantized in terms of potentials A_μ whereas transformation (4) is defined in terms of $F_{\mu\nu}$. Note that transformation (4) can be formulated in terms of A_μ only for potentials satisfying the Maxwell equations.

We will introduce the notion of chirality for photons and the corresponding chiral current as consequences of $U(1)$-symmetry of the action for arbitrary potential A_μ. This proves to be possible in the light-cone formalism. We show that the chiral rotation of A_μ results in transformation (4) only for photons on-mass-shell.

We use the spinor notation and the gauge condition

$$A_{2\dot{2}} = \emptyset \tag{6}$$

(for details see ref. [5].

The equation of motion for $A_{1\dot{1}}$ does not contain the time derivatives, so $A_{1\dot{1}}$ is not a dynamical variable in this formalism and can be expressed through two independent quantities $A_{1\dot{2}}$ and $A_{2\dot{1}}$:

$$A_{1\dot{1}} = \partial_{2\dot{2}}^{-1}\left(\partial_{2\dot{1}} A_{1\dot{2}} + \partial_{1\dot{2}} A_{2\dot{1}}\right). \tag{7}$$

The latter satisfy the equations:

$$\Box A_{1\dot{2}} = 0, \qquad \Box A_{2\dot{1}} = 0, \qquad \Box = \partial_\mu \partial_\mu = \frac{1}{2}\partial_{\beta\dot{\beta}}\partial^{\beta\dot{\beta}}. \tag{8}$$

Action S can be written in terms of the field variables $A = A_{2\dot{1}}/\sqrt{2}$ and $\bar{A} = A_{1\dot{2}}/\sqrt{2}$ as follows

$$S = \int d^4x\, \bar{A}\Box A. \tag{9}$$

This expression is explicitly invariant with respect to the global $U(1)$-transformations:

$$A \to A e^{i\varphi}, \qquad \bar{A} \to \bar{A} e^{i\varphi}. \tag{10}$$

Taking into account gauge condition (6) one can express field strength tensor $F_{\mu\nu}$ through A and $\bar{A}$ as follows:

$$\begin{aligned}
F_{\alpha\dot{\alpha}\beta\dot{\beta}} &= (\sigma^\mu)_{\alpha\dot{\alpha}} (\sigma^\nu)_{\beta\dot{\beta}} F_{\mu\nu}, \\
F^{+}_{\alpha\dot{\alpha}\beta\dot{\beta}} &= \varepsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}, \qquad F^{-}_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}}, \\
f_{\alpha\beta} &= \frac{1}{2}\left(\partial_\alpha{}^{\dot{\alpha}} A_{\beta\dot{\alpha}} + \partial_\beta{}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}\right), \\
f_{11} &= \frac{1}{2}\partial_\xi^{-1}\partial^2 A - \partial_\xi^{-1}\Box\bar{A}, \\
f_{22} &= \frac{1}{2}\partial_\xi A, \qquad f_{12} = f_{21} = \frac{1}{2}\bar{\partial} A,
\end{aligned} \tag{11}$$

where

$$\partial = \sqrt{2}\,\partial_{2\dot{1}}, \quad \bar{\partial} = \sqrt{2}\,\partial_{1\dot{2}}, \quad \partial_\xi = \sqrt{2}\,\partial_{2\dot{2}}, \quad \partial_i = \sqrt{2}\,\partial_{1\dot{1}},$$

$$\Box = \partial_\mu \partial_\mu = \frac{1}{2}\left(\partial_\xi \partial_i - \partial\bar{\partial}\right).$$

Quantities $\bar{f}_{\dot{\alpha}\dot{\beta}}$ are the complex conjugates of $f_{\alpha\beta}$. Note that the field of the wrong chirality $\bar{A}$ enters into the expression for $f_{\alpha\beta}$ in the form $\Box A$ i.e. it vanishes on-mass-shell.

Expressing $F_{\mu\nu}$ through A in gauge (6) one can check that transformation (10) over the potentials leads to the chiral rotation (4) for the field strength only if $A_{\alpha\dot{\alpha}}$ satisfy the equation of motion.

This trivial exercise for the free electromagnetic field permits to consider the case of the photons interacting with gravity along the same lines. The action is of the well-known form:

$$S = \frac{1}{4}\int d^4x \sqrt{-g}\, g^{\mu\nu} g^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda}. \tag{12}$$

It is convenient to represent real symmetric matrix g in the form

$$g = e^H$$

where H is also real symmetric matrix. Since $\sqrt{-g} = \exp(-\mathrm{Tr}\, H/2)$, action (12) depends only on the traceless part of H, i.e. on $h = H - I(\mathrm{Tr}\, H)/4$:

$$S = \frac{1}{4}\int d^4x\, \mathrm{Tr}\left(e^h F e^h F\right) \tag{13}$$

This is a consequence of the conformal invariance of the theory. In the last expression the contraction is made with the flat space metric tensor $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. In what follows we use the perturbation theory expanding the action as

$$S = \frac{1}{4}\int d^4x\, \mathrm{Tr}\left(F^2 + 2hFF + h^2F^2 + hFhF + \ldots\right) \tag{14}$$

Using the spinor notations for the field strength of definite chirality we get the following expressions for the terms of the zeroth, first and second order in h respectively:

$$\begin{aligned}
S^{(0)} &= -\frac{1}{8}\int d^4x \left(f_{\alpha\beta} f^{\alpha\beta} + \bar{f}_{\dot{\alpha}\dot{\beta}} \bar{f}^{\dot{\alpha}\dot{\beta}}\right), \\
S^{(1)} &= \frac{1}{8}\int d^4x\, h^{\alpha\dot{\alpha}\beta\dot{\beta}} f_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}} \\
S^{(2)} &= -\int d^4x \left[\frac{1}{48}\left(\mathrm{Tr}\, h^2\right)\left(f_{\alpha\beta} f^{\alpha\beta} + \bar{f}_{\dot{\alpha}\dot{\beta}} \bar{f}^{\dot{\alpha}\dot{\beta}}\right) + \right. \\
&\quad + \frac{1}{64} h^{\alpha\dot{\alpha}\beta\dot{\beta}} h^{\gamma}{}_{\dot{\beta}}{}^{\delta}{}_{\dot{\alpha}} f_{\{\beta\gamma} f_{\delta\alpha\}} + \\
&\quad \left. + \frac{1}{64} h^{\alpha\dot{\alpha}\beta\dot{\beta}} h_{\beta}{}^{\dot{\gamma}}{}_{\alpha}{}^{\dot{\delta}} \bar{f}_{\{\dot{\beta}\dot{\gamma}} \bar{f}_{\dot{\delta}\dot{\alpha}\}}\right].
\end{aligned} \tag{15}$$

The zero order term at first sight violates chirality conservation. But $S^{(0)}$ vanishes on-mass-shell and off-mass-shell chirality is conserved as it follows from eq. (9).

The first order term conserves chirality because in this order one can neglect the wrong chirality contribution to $f_{\alpha\beta}$ which is proportional to $\Box\bar{A}$ (see eq. (11)) and so $f \sim A$ and $\bar{f} \sim \bar{A}$.

It is explicitly seen that $S^{(2)}$ breaks chirality conservation, but in the second order in h there are terms proportional to $\Box\bar{A}$ in f which also violate chirality conservation and their contribution exactly cancels chirality violation due to $S^{(2)}$ (see ref. [5]). This cancellation is closely connected with the transversality of the gravitational amplitudes which permits to reconstruct the whole amplitude by the pole terms [14]. Pole terms are absent for chirality violating amplitudes and correspondingly the whole amplitude vanishes. The conservation of chirality in photon interaction with gravity was also noted in ref. [15].

In the light-cone formalism one can define the operator of chiral charge

$$Q = \frac{i}{4}\int d^3x\, \bar{A}(x)\, \overleftrightarrow{\partial}_\xi\, A(x) \tag{16}$$

where $d^3x = d^2x_\perp d\xi$, $\overleftrightarrow{\partial}=\overrightarrow{\partial} - \overleftarrow{\partial}$. Charge Q is the integral of the ξ-component of the conserved current

$$j_\mu = \frac{i}{4}\bar{A}(x)\, \overleftrightarrow{\partial}_\mu\, A(x). \tag{17}$$

This current is not of course a Lorentz-vector because the functions A and $\bar{A}$ are not scalars. Lorentz-covariant current with ξ-component coinciding with that of j_μ is K_μ (eq. (2)). The last current however is not conserved. Nevertheless its matrix elements in a gravitational field would vanish if there were no anomaly. Indeed the operator

$$F_{\mu\nu}\widetilde{F}^{\mu\nu} = \frac{i}{2}\left(f^{\alpha\beta} f_{\alpha\beta} - \bar{f}^{\dot\alpha\dot\beta}\bar{f}_{\dot\alpha\dot\beta}\right)$$

changes chirality by ± 2 and gravitational interaction conserves chirality. Thus the nonvanishing expectation value of $D_\mu K^\mu$ in a gravitational field can be called chiral anomaly.

Evidently current K^μ is not gauge invariant, but the charge

$$Q = \int d\vec{r}\, K^0$$

does not depend upon the gauge. Moreover, matrix elements of K^μ in a gravitational field are also gauge independent.

Let us mention in conclusion the analogy between the current K^μ and the Pauli-Lubański vector [16]

$$\Gamma^\mu = \varepsilon^{\mu\nu\kappa\lambda} P_\nu M_{\kappa\lambda}$$

where P_ν and $M_{\kappa\lambda}$ are the generators of the Lorentz translations and rotations respectively.

3 The triangle diagram

The calculation of anomaly (2) connected with the triangle diagram can be done by the dispersion relation method of ref. [17]. We shall consider bosonic and known fermionic cases in parallel. The amplitudes of production of two photons and two gravitons by the fermionic axial current and the amplitude of production of two gravitons by bosonic current K^μ are determined by a single form-factor and can be written as follows

$$\langle 2\gamma | a_\mu | 0\rangle = f_1(q^2) q_\mu F_{\kappa\lambda}\widetilde{F}^{\kappa\lambda}, \tag{17.a}$$

$$\langle 2g | a_\mu | 0\rangle = f_2(q^2) q_\mu R_{\kappa\lambda\rho\sigma}\widetilde{R}^{\kappa\lambda\rho\sigma}, \tag{17.b}$$

$$\langle 2g | K_\mu | 0\rangle = f_3(q^2) q_\mu R_{\kappa\lambda\rho\sigma}\widetilde{R}^{\kappa\lambda\rho\sigma}. \tag{17.c}$$

Such a form of the amplitudes is dictated by the gauge invariance with respect to external fields. It is assumed that the external photons and gravitons are on-mass-shell.

The imaginary parts of these amplitudes are determined by the unitarity condition and at first sight are to vanish because of chirality conservation. (Note that one has to take into account, where necessary, the contact diagrams ensuring transversality.) The anomalies arise as a result of the infrared regularization of the amplitudes which violates chiral symmetry and leads to nonvanishing $\mathrm{Im}\, f$:

$$\mathrm{Im}\, f_1(q^2) = \lim_{m\to 0}\left(-\frac{\alpha}{4q^2}\right)(1-v^2)\ln\frac{1+v}{1-v} = -\frac{\alpha}{2}\,\delta(q^2), \tag{18.a}$$

$$\mathrm{Im}\, f_2(q^2) = \lim_{m\to 0}\frac{(1-v^2)^2}{128\pi q^2}\ln\frac{1+v}{1-v} = \frac{1}{192\pi}\,\delta(q^2), \tag{18.b}$$

$$\mathrm{Im}\, f_3(q^2) = \lim_{m\to 0}\frac{v^2(1-v^2)}{128\pi q^2}\ln\frac{1+v}{1-v} = \frac{1}{96\pi}\,\delta(q^2), \tag{18.c}$$

where $v = (1-4m^2/q^2)^{1/2}$ is the c.m. velocity of the particles in the intermediate states and m is their mass ($m \to 0$) introduced for infrared regularization.

Prescribing a nonvanishing mass to photon gives rise, in contrast to fermions, to extra degrees of freedom of the vector field. They however do not contribute to amplitude f_3. Indeed, if in addition to the mass term

$$S_m = \frac{m^2}{2}\int d^4x\,\sqrt{-g}\, g^{\mu\nu} A_\mu A_\nu \tag{19}$$

one takes into account the gauge fixing term

$$\Delta S = -\frac{1}{2}\xi\int d^4x\,\sqrt{-g}\,(D_\mu A^\mu)^2, \tag{20}$$

one can see that the result for f_3 does not depend upon gauge parameter ξ. The gauge invariance of f_3 means that the mass term does not introduce here any undesirable degree of freedom.

4 Electromagnetic corrections to the fermionic chiral anomaly in gravitational field

Taking expectation value of eq. (1) in external gravitational field and using for $\langle F\widetilde{F}\rangle$ expression (3) we obtain the radiative correction to the second term in anomaly (1):

$$D_\mu a^\mu = -\frac{1}{192\pi^2}\left(1-\frac{2\alpha Q^2}{\pi}\right) R_{\mu\nu\kappa\lambda}\widetilde{R}^{\mu\nu\kappa\lambda}. \tag{21}$$

This result corresponds to the following specific regularization of the two-loop diagrams: it is assumed that the infrared mass of the photon is much larger than that of the fermion,

$$m_\gamma \gg m_f.$$

In this case only the two-photon intermediate state contributes into two-loop correction for Im f_2 [5]. Note the relation of this result to the Adler-Bardeen theorem [18].

5 Vanishing of the chiral anomaly for antisymmetric tensor gauge field

The chiral current for antisymmetric tensor field $\varphi_{\mu\nu}$ was introduced in ref. [6] in the Feynman gauge and looks as follows

$$j_\mu = -D_\lambda\widetilde{\varphi}^{\lambda\nu}\varphi_{\mu\nu} - \widetilde{\varphi}_{\mu\nu}D_\lambda\varphi^{\lambda\nu} \tag{22}$$

where $\widetilde{\varphi}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}\varphi_{\alpha\beta}$.

This current is conserved because of the equations of motion, but as it was stated in ref. [6] there existed the anomaly:

$$\langle D_\mu j^\mu\rangle = -\frac{1}{48\pi^2}R_{\mu\nu\alpha\beta}\widetilde{R}^{\mu\nu\alpha\beta}. \tag{23}$$

The discovery of this anomaly put a very interesting problem about the equivalence of the description of massless field with zero spin either by scalar field, or by antisymmetric tensor gauge field. These two ways of description are known to be equivalent at the classical level [19]. One naturally would expect therefore that the anomaly found through the calculation of the imaginary part of the amplitude, as it has been done in the preceeding section. should vanish. The

contradiction is resolved by taking into account the anomaly for vector fields (eq. 2). Indeed the chirality of $\varphi_{\mu\nu}$ could only be connected with the nonphysical degrees of freedom which appear at the quantization as a result of gauge fixing. As is well known, several ghost fields are to be introduced in this case. Among them in particular there are two complex vector fields η and $\tilde{\eta}$. These vector fields give contribution to the chiral current and their anomaly exactly cancels out anomaly (23) [20].

The problem is a little bit more complicated however because current j_μ (22) is not gauge invariant and the classical equations of motion are not self-consistent when the interaction with this current is taken into account. Besides, prescribing nonzero mass to the fields, which is necessary for the infrared regularization, changes the number of the degrees of freedom and it is not evident that the extra degrees of freedom do not survive in the limit $m \to 0$. These problems are considered in ref. [20].

Let $\Phi_{\mu\nu}$ be a massive antisymmetric tensor field with the Lagrangian

$$L = -\frac{1}{2}\partial_\lambda\widetilde{\Phi}_{\lambda\nu}\partial_\kappa\widetilde{\Phi}_{\kappa\nu} - \frac{1}{4}m^2\Phi^2_{\mu\nu} + h_\mu K_\mu(\Phi). \tag{24}$$

where h_μ is an external field interacting with chiral current $K_\mu(\Phi)$:

$$K_\mu(\Phi) = -\partial_\lambda\widetilde{\Phi}_{\lambda\nu}\Phi_{\mu\nu}. \tag{25}$$

It is known that field $\Phi_{\mu\nu}$ is equivalent to massive vector field b_μ described by the Lagrangian:

$$L(b) = -\frac{1}{4}b^2_{\mu\nu} + \frac{m^2}{2}b^2_\mu + h_\mu K_\mu(b). \tag{26}$$

The correspondence between the fields are given by the relations

$$b_\mu = \frac{1}{m}\partial_\lambda\widetilde{\Phi}_{\lambda\mu}, \qquad b_{\mu\nu} \equiv (\partial_\mu b_\nu - \partial_\nu b_\mu) = -m\widetilde{\Phi}_{\mu\nu}. \tag{27}$$

In terms of b_μ chiral current (25) can be written as

$$K_\mu = -b_\nu\tilde{b}_{\mu\nu} = -\varepsilon_{\mu\nu\kappa\lambda}b_\nu\partial_\kappa b_\lambda. \tag{28}$$

At the classical level its divergence is:

$$\partial_\mu K_\mu = \frac{1}{2}b_{\mu\nu}\tilde{b}_{\mu\nu} = \frac{m^2}{2}\Phi_{\mu\nu}\widetilde{\Phi}_{\mu\nu}. \tag{29}$$

R.h.s. of eq. (29) is proportional to m^2, but the propagator of $\Phi_{\mu\nu}$ is singular in m^2, so the vanishing of $\partial_\mu K_\mu$ in the limit $m^2 \to 0$ is up to now only formal.

To reveal this singularities it is convenient to make the following substitutions

$$\Phi_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m}a_{\mu\nu}, \qquad a_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{30}$$

The enlarging the number of the fields leads to the gauge freedom:

$$\delta\varphi_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad \delta a_\mu = -m\xi_\mu. \tag{31}$$

To fix the gauge we introduce the following term

$$L_{gauge} = \frac{1}{2}\left[\partial_\lambda \varphi_{\lambda\nu} - m a_\nu - h_\lambda\left(\tilde{\varphi}_{\lambda\nu} - \frac{1}{m}\tilde{a}_{\lambda\nu}\right)\right]_+^2 + O(h_\mu^2). \tag{32}$$

It is necessary also to add the ghost Lagrangian

$$L_{ghost} = -\frac{1}{2}\bar{\eta}_{\mu\nu}\eta_{\mu\nu} + m^2 \bar{\eta}_\mu \eta_\mu - 2h_\mu \bar{\eta}_\nu \tilde{\bar{\eta}}_{\mu\nu} \tag{33}$$

where η_μ and $\bar{\eta}_\mu$ are vector Grassman fields, $\eta_{\mu\nu} = \partial_\mu \eta_\nu - \partial_\nu \eta_\mu$, and $\tilde{\bar{\eta}}_{\mu\nu} = \frac{1}{2}$.

After straightforward calculations we obtain in the first order in h:

$$\begin{aligned} K_\mu(b) = & -\partial_\lambda \tilde{\varphi}_{\lambda\nu}\varphi_{\mu\nu} - \partial_\lambda \varphi_{\lambda\nu}\tilde{\varphi}_{\mu\nu} - a_\nu \tilde{a}_{\mu\nu} \\ & - 2\eta_\nu \tilde{\bar{\eta}}_{\mu\nu} + \frac{1}{m}\Big(\partial_\lambda \varphi_{\lambda\nu}\tilde{a}_{\mu\nu} - \partial_\lambda \tilde{\varphi}_{\lambda\nu} a_{\mu\nu}\Big) + m a_\nu \tilde{\varphi}_{\mu\nu}. \end{aligned} \tag{34}$$

It is evident that $\langle \varphi_{\lambda\nu} a_\mu \rangle = 0$, so

$$\langle K_\mu(b) \rangle = \langle\, j_\mu(\varphi) + K_\mu(a) + K_\mu(\eta, \bar{\eta})\,\rangle, \tag{35}$$

and since $\langle K_\mu(b)\rangle = \langle K_\mu(a)\rangle$, we get finally

$$\langle\, j_\mu(\varphi) + K_\mu(\eta, \bar{\eta})\,\rangle = 0. \tag{36}$$

So first, the total chiral current of the field φ and the ghosts $\eta, \bar{\eta}$ in gravitational background vanishes in accordance with physical expectations, and so much the more its divergence. Second, since $\langle K_\mu(\eta,\bar{\eta})\rangle = -2\langle K_\mu(A)\rangle$, the obvious relation arises between $\langle \partial_\mu K_\mu(A)\rangle$ and the number of zero modes of the field φ that controls $\langle \partial_\mu j_\mu(\varphi)\rangle$.

6 Macroscopic manifestation of the chiral anomalies in the field of the rotating black hole

In the Kerr metric the anomalous term is of the following form

$$\frac{1}{2}\frac{\varepsilon^{\kappa\lambda\rho\sigma}}{\sqrt{-g}} R_{\mu\nu\kappa\lambda} R^{\mu\nu}{}_{\rho\sigma} = 12 r_g^2 (\bar{a} r)\frac{(3r^2 - a^2\cos^2\theta)(r^2 - 3a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^6} \tag{37}$$

where r_g is the gravitational radius of the massive rotating body, $\bar{a} = \bar{M}/m$, $\bar{M}$ and m are respectively its angular momentum and mass. It is noteworthy that the space average of $R\tilde{R}$ vanishes (see eq. (5)).

It is known that in the electromagnetic field of a dyon where $F\tilde{F} \neq 0$ anomaly (1) leads to the charged fermion production. This makes one to suspect that the analogous process of massless spinning particle production by the gravitational field of a rotating black hole also takes place. We have presented arguments in favor of this phenomenon in ref. [8], but we do not have a strict proof of it because in contrast to a dyon with nonvanishing topological charge the topological charge of a black hole is zero. An extra argument supporting the idea of the anomalous particle production by a rotating black hole present the numerical calculations of ref. [21]. It is shown there that even in the case of the limiting value $a = r_g/2$ when the Hawking temperature is zero, the particle emission and the corresponding loss of the angular momentum by the black hole does not vanish. The production of bosons could be explained by the phenomenon of the superradiance, but this is known to be absent for neutrino. So the production of the latter can only be connected with the chiral anomaly.

Except for the particle production, anomalies (1) and (2) in the field of a rotating massive body lead to the formation of condensate $\langle F\tilde{F}\rangle$ even if the body is electrically neutral. There are also to exist dipole vacuum currents falling off at large distance as r^{-3}. These effects though interesting from the qualitative point of view are far from the possibility of observation.

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PART III

MATHEMATICAL ASPECTS OF QUANTUM FIELD THEORIES