The semiclassical Green's function and Delbrück scattering in a screened Coulomb field

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We obtain a simple integral representation for the classical Green's function of the Dirac equation in an arbitrary centrally symmetric decreasing field. The approach uses semiclassical radial wave functions and the contribution of large orbital angular momenta. The derived Green's function is used to calculate the amplitude of Delbrück scattering in a screened Coulomb field. © *1995 American Institute of Physics.*

1. INTRODUCTION

A convenient way to allow for the effect of an external electromagnetic field on QED processes is to use the Furry representation. This requires knowing the Green's function $G(\mathbf{r},\mathbf{r}'|\varepsilon)$ of the Dirac equation in this field. Unfortunately, explicit expressions for the Green's function are known for only a very limited number of potentials. Hence, one is forced to use numerical methods to find the Green's function. In this connection obtaining new analytic representations for Green's functions is a problem of unquestionable interest. In many high-energy processes the orbital angular momenta providing the main contribution to cross sections are large. Hence we can use the semiclassical approximation. In this paper we find an explicit expression for the semiclassical Green's function for the Dirac equation in an arbitrary centrally symmetric decreasing potential.

The semiclassical Green's function for a Coulomb field was found in Refs. 1 and 2 by summing the integral representation of the exact Green's function of the Dirac equation over the orbital angular momenta.[3] We demonstrate that knowing the exact Green's function is not obligatory for finding the semiclassical Green's function and that it is sufficient to use semiclassical wave functions of the radial equation for large orbital angular momenta. The semiclassical method was used by Olsen *et al.*[4] to obtain wave functions in the Sommerfeld–Maue approximation[5] in investigating bremsstrahlung and pair production in a screened Coulomb potential at high energies.

The integral representation of the Green's function derived here proves useful in conducting analytic calculations of amplitudes of various QED processes in an arbitrary centrally symmetric field at high energies. We demonstrate this by calculating the amplitude of Delbrück scattering[6] (coherent photon scattering via virtual electron–positron pairs) in a screened Coulomb potential. Delbrück scattering is one of the few nonlinear QED processes observed in experiments (see the recent review article by Milstein and Schumacher[7]). Thus far, the Delbrück scattering amplitude has been studied in detail for a Coulomb field exactly in the parameter $Z\alpha$ at high photon energies $\omega \gg m$, where m is the electron mass, $Z|e|$ is the charge of the nucleus, $\alpha = e^2 = 1/137$ is the fine-structure constant, e is the electron charge, and $\hbar = c = 1$. The approaches used depend strongly on the value of momentum transfer $\Delta = |\mathbf{k}_2 - \mathbf{k}_1|$, where $\mathbf{k}_1$ is the initial photon momentum, and $\mathbf{k}_2$ is the final photon momentum. For $\Delta \ll \omega$ the amplitudes were found in Refs. 8–10 by summing, in a certain approximation, the perturbation theory diagrams in the interaction with the Coulomb field, and in Refs. 1 and 2 via the semiclassical Green's function. For $m \ll \Delta \sim \omega$ the amplitudes were found in Refs. 11–13 via the exact Green's function of an electron in a Coulomb field[3] in the limit $m = 0$. Calculations for arbitrary photon energies but in the lowest perturbation-theory order in the parameter $Z\alpha$ have also been done (a review of the numerous investigations done in this approximation can be found in Ref. 14). It was found, however, that for $\omega \gg m$ the result that is exact in $Z\alpha$ differs considerably from the one obtained in the lowest perturbation order.

The effect of screening on the Delbrück scattering amplitudes is considerable for low momentum transfer, $\Delta \sim 1/r_c \ll m$, where r_c is the screening radius. This is the momentum transfer range considered in this paper.

2. THE GREEN'S FUNCTION

We examine the Green's function of the Dirac equation in an external centrally symmetric potential $V(r)$:

$$G(\mathbf{r},\mathbf{r}'|\varepsilon) = \frac{1}{\gamma^0[\varepsilon - V(r)] - \gamma\mathbf{p} - m + i0}\,\delta(\mathbf{r}-\mathbf{r}'), \quad (1)$$

where the γ^μ are the Dirac matrices, and $\mathbf{p} = -i\nabla$. We are interested in the Green's function at $|\varepsilon| \gg m$. We write the function G in the form

$$G(\mathbf{r},\mathbf{r}'|\varepsilon) = \{\gamma^0[\varepsilon - V(r)] - \gamma\mathbf{p} + m\} D(\mathbf{r},\mathbf{r}'|\varepsilon), \quad (2)$$

where

$$D(\mathbf{r},\mathbf{r}'|\varepsilon) = \frac{1}{[\varepsilon - V(r)]^2 - \mathbf{p}^2 - [\boldsymbol{\alpha}\mathbf{p}, V(r)] - m^2 + i0} \times \delta(\mathbf{r}-\mathbf{r}'), \quad (3)$$

with $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$. As is known (see Ref. 5), at high energies $\varepsilon \gg m$ we can ignore the term $V^2(r)$ in Eq. (3) and allow only for the first term of the expansion of D in the commutator $[\boldsymbol{\alpha}\mathbf{p}, V(r)]$. Performing the expansion and representing the commutator as

$$[\boldsymbol{\alpha}\mathbf{p},V(r)]=\frac{1}{2\varepsilon}[\boldsymbol{\alpha}\mathbf{p},H],\quad H=\mathbf{p}^2+2\varepsilon V(r),\tag{4}$$

we arrive at the following representation for D:

$$D(\mathbf{r},\mathbf{r}'|\varepsilon)=\left[1-\frac{i}{2\varepsilon}(\boldsymbol{\alpha}\nabla+\nabla')\right]D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon),\tag{5}$$

where

$$D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon)=\frac{1}{\kappa^2-H+i0}\,\delta(\mathbf{r}-\mathbf{r}'),\tag{6}$$

with $\kappa^2=\varepsilon^2-m^2$. Thus, the problem is reduced to calculating the semiclassical Green's function $D^{(0)}$ of the Schrödinger equation with Hamiltonian H.

Now we introduce the impact parameter by $\rho=|\mathbf{r}\mathbf{r}'|/|\mathbf{r}-\mathbf{r}'|$. For high-energy processes the important distances are of order $|\mathbf{r}-\mathbf{r}'|\sim\kappa/m^{-2}\gg 1/m$ and $\rho\gg 1/m$. This implies that the characteristic value of the angular momentum $l\sim\kappa\rho\gg 1$ and that the semiclassical approximation can be employed. Another interesting case is the one in which $\rho\ll|\mathbf{r}-\mathbf{r}'|$. Here the angle between $\mathbf{r}$ and $-\mathbf{r}'$ or the angle between $\mathbf{r}$ and $\mathbf{r}'$ is small.

We consider the eigenfunctions of the Hamiltonian H and use the completeness relation for these functions in (6) by replacing the delta function with a sum of products of the eigenfunctions. Naturally, the main contribution to $D^{(0)}$ is provided by the functions of the continuous spectrum with large angular momenta. We use the system of functions of the continuous spectrum that are represented far from the target by a plane wave and an outgoing spherical wave. The system of equations represented far from the target by incoming spherical waves leads to the same result for the Green's function. The eigenfunction of the Hamiltonian H with eigenvalue q^2 that is represented by a plane wave with momentum $\mathbf{q}$ and an outgoing spherical wave far from the target is

$$\psi_{\mathbf{q}}(\mathbf{r})=\frac{1}{qr}\sum_{l=0}^{\infty}i^l e^{i\delta_l}(2l+1)u_l(r)P_l(\cos\vartheta).\tag{7}$$

Here $P_l(x)$ is a Legendre polynomial, and ϑ is the angle between the vectors $\mathbf{q}$ and $\mathbf{r}$. In the semiclassical approximation the functions $u_l(r)$ and $\delta_l=\delta(l/q)$ are specified as follows:[4]

$$u_l(r)=\sin\left[qr-\frac{l\pi}{2}+\frac{l^2}{2qr}+\lambda\delta\left(\frac{l}{q}\right)+\lambda\Phi(r)\right],$$
$$\Phi(r)=\int_r^\infty V(\zeta)\,d\zeta,\quad \delta(\rho)=-\int_0^\infty V(\sqrt{\zeta^2+\rho^2})\,d\zeta,\tag{8}$$

with $\lambda=\varepsilon/q$. Taking into account the completeness relation, we arrive at the following expression for $D^{(0)}$:

$$D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon)=\int\frac{\psi_{\mathbf{q}}(\mathbf{r})\,\psi^*_{\mathbf{q}}(\mathbf{r}')}{\kappa^2-q^2+i0}\frac{d\mathbf{q}}{(2\pi)^3}.\tag{9}$$

Substituting (7) into (9) and evaluating the integral over the angles of $\mathbf{q}$ via the well-known relationship for Legendre polynomials,

$$\int P_l(\mathbf{n}_1\mathbf{n}_2)P_{l'}(\mathbf{n}_1\mathbf{n}_3)\,d\Omega_1=\frac{4\pi}{2l+1}P_l(\mathbf{n}_2\mathbf{n}_3)\delta_{ll'},$$

where $\mathbf{n}_i$ are the unit vectors, we find that

$$D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon)=\frac{1}{2\pi^2rr'}\int_0^\infty\frac{dq}{\kappa^2-q^2+i0}\sum_{l=0}^{\infty}(2l+1)\times u_l(r)u_l(r')P_l(\mathbf{n}\mathbf{n}'),\tag{10}$$

where $\mathbf{n}=\mathbf{r}/r$ and $\mathbf{n}'=\mathbf{r}'/r'$. Using (8), we write the product $u_l(r)u_l(r')$ as follows:

$$u_l(r)u_l(r')=\frac{1}{2}\cos\left\{q(r-r')+\frac{l^2(r'-r)}{2qrr'}+\lambda[\Phi(r)-\Phi(r')]\right\}-\frac{1}{2}(-1)^l\cos\left\{q(r+r')+\frac{l^2(r+r')}{2qrr'}+2\lambda\delta\left(\frac{l}{q}\right)+\lambda[\Phi(r)+\Phi(r')]\right\}.\tag{11}$$

If the angle θ between $\mathbf{n}$ and $-\mathbf{n}'$ is small, $P_l(\mathbf{n}\mathbf{n}')$ can be replaced by $(-1)^lJ_0(l\theta)$, where $J_\nu(x)$ is a Bessel function. Here the main contribution to the sum over l is provided by the second term in (11). For this term summation over l can be replaced by integration. In Eq. (10) we carry out an exponential parametrization of the energy denominator:

$$\frac{1}{\kappa^2-q^2+i0}=-i\int_0^\infty\exp[is(\kappa^2-q^2)]\,ds.$$

The integrals with respect to q and then with respect to s are evaluated by the method of stationary phase, which is applicable under the assumptions. After performing simple calculations we find for the case $\theta\ll 1$ considered here that

$$D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon)=\frac{i\exp[i\kappa(r+r')]}{4\pi\kappa rr'}\int_0^\infty dl\,lJ_0(l\theta)\times\exp\left\{i\left[\frac{l^2(r+r')}{2\kappa rr'}+2\lambda\delta\left(\frac{l}{\kappa}\right)+\lambda(\Phi(r)+\Phi(r'))\right]\right\}.\tag{12}$$

Here $\lambda=\varepsilon/\kappa$. In the relativistic case considered here, $\lambda=+1$ for $\varepsilon>0$ and $\lambda=-1$ for $\varepsilon<0$.

If the angle $\theta_1=\pi-\theta$ between $\mathbf{n}$ and $\mathbf{n}'$ is small, $J_0(l\theta_1)$ can be substituted for $P_l(\mathbf{n}\mathbf{n}')$. In this case in the summation over l the main contribution is provided by the first term in (11). Since this term does not contain $\delta(l)$, after performing transformations similar to those done in deriving Eq. (12) we can integrate with respect to l. For $\pi-\theta\ll 1$ we obtain

$$D^{(0)}(\mathbf{r},\mathbf{r}'|\varepsilon)=-\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\exp\{i\kappa|\mathbf{r}-\mathbf{r}'|+i\lambda\,\mathrm{sign}(r-r')[\Phi(r)-\Phi(r')]\}.\tag{13}$$

Substituting (12) and (13) into Eq. (5), we arrive at the following expression for D when $\theta \ll 1$:

$$D(\mathbf{r},\mathbf{r}'|\varepsilon)=\frac{i\exp[i\kappa(r+r')]}{4\pi\kappa rr'}\int_0^\infty dl\, l\left[J_0(l\theta)-i\frac{\alpha(\mathbf{n}+\mathbf{n}')}{\kappa\theta}\delta'\left(\frac{l}{\kappa}\right)J_1(l\theta)\right]\times\exp\left\{i\left[\frac{l^2(r+r')}{2\kappa rr'}+2\lambda\delta\left(\frac{l}{k}\right)+\lambda(\Phi(r)+\Phi(r'))\right]\right\}. \quad (14)$$

Here $\delta'(\rho)=\partial\delta(\rho)/\partial\rho$.

When $\pi-\theta\ll 1$,

$$D(\mathbf{r},\mathbf{r}'|\varepsilon)=-\left\{1-\mathrm{sign}(r-r')\frac{1}{4\kappa}\times[V(r)-V(r')]\,\alpha(\mathbf{n}+\mathbf{n}')\right\}\times\frac{\exp\{i\kappa|\mathbf{r}-\mathbf{r}'|+i\lambda\,\mathrm{sign}(r-r')[\Phi(r)-\Phi(r')]\}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \quad (15)$$

Substituting (14) and (15) into Eq. (2), we arrive at the final expression for the semiclassical Green's function in a centrally symmetric field with $\theta\ll 1$:

$$G(\mathbf{r},\mathbf{r}'|\varepsilon)=\frac{i\exp[i\kappa(r+r')]}{4\pi krr'}\int_0^\infty dl\, l\exp\left\{i\left[\frac{l^2(r+r')}{2\kappa rr'}+2\lambda\delta\left(\frac{l}{\kappa}\right)+\lambda(\Phi(r)+\Phi(r'))\right]\right\}\left\{\left[\gamma^0\varepsilon+m-\frac{1}{2}(\gamma,\mathbf{n}-\mathbf{n}')\left(\kappa+\frac{l^2}{2krr'}\right)\right]J_0(l\theta)+i\left[\frac{l^2(r-r')}{2rr'}(\gamma,\mathbf{n}+\mathbf{n}')+l\delta'\left(\frac{l}{\kappa}\right)\gamma^0\left(1-(\gamma,\mathbf{n})(\gamma,\mathbf{n}')-\frac{(\gamma,\mathbf{n}+\mathbf{n}')m}{\kappa}\right)\right]\frac{J_1(l\theta)}{l\theta}\right\}, \quad (16)$$

while with $\pi-\theta\ll 1$ we have

$$G(\mathbf{r},\mathbf{r}'|\varepsilon)=-\frac{1}{4\pi R}\left[\gamma^0\varepsilon+m-\frac{1}{R}\left(\kappa+\frac{i}{R}\right)(\gamma,\mathbf{R})\right]\times\exp\{i\kappa R+i\lambda\,\mathrm{sign}(r-r')[\Phi(r)-\Phi(r')]\},\quad \mathbf{R}=\mathbf{r}-\mathbf{r}'. \quad (17)$$

For the Coulomb field $V(r)=-Z\alpha/r$ we have

$$2\delta(\rho)+\Phi(r)+\Phi(r')=Z\alpha\ln\left(\frac{4rr'}{\rho^2}\right),\qquad \delta'(\rho)=-Z\alpha/\rho. \quad (18)$$

Substituting (18) into (16), we arrive at an expression for the semiclassical Green's function in a Coulomb field that agrees with the one obtained in Refs. 1 and 2.

3. DELBRÜCK SCATTERING

We now use the derived expressions for the Green's function to calculate the amplitude of Delbrück scattering in a screened Coulomb field. In the Thomas–Fermi model the screening radius $r_c\sim(m\alpha)^{-1}Z^{-1/3}$. The characteristic impact parameter $\rho\sim 1/\Delta$. If $R\ll 1/\Delta\ll r_c$, where R is the radius of the nucleus, screening is unimportant and the amplitude coincides with that of scattering in a Coulomb field. If $\Delta/1\sim r_c\gg 1/m$, screening must be taken into account. For such momentum transfer the values of ρ contributing to the amplitude range from $1/m$ to r_c. The corresponding orbital angular momenta $l\sim\omega\rho\gg 1$, with the result that the semiclassical approximation holds.

Suppose that a photon with momentum $\mathbf{k}_1$ creates at point $\mathbf{r}_1$ a pair of virtual particles that at point $\mathbf{r}_2$ transform into a photon with momentum $\mathbf{k}_2$. The uncertainty relation implies that the virtual electron–positron pair has a lifetime $\tau\sim|\mathbf{r}_2-\mathbf{r}_1|\sim\omega/(m^2+\Delta^2)$. Hence for $\omega/m^2\gg r_c$ the angles between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, and $\mathbf{r}_2$ and $-\mathbf{r}_1$ are small. It is in this range of photon energies that we operate. According to Feynman's rules, in the Furry representation the Delbrück scattering amplitude is

$$M=2i\alpha\int d\mathbf{r}_1\,d\mathbf{r}_2\exp[i(\mathbf{k}_1\mathbf{r}_1-\mathbf{k}_2\mathbf{r}_2)]\times\int d\varepsilon\,\mathrm{Sp}\,\hat{e}_2^*G(\mathbf{r}_2,\mathbf{r}_1|\omega-\varepsilon)\hat{e}_1G(\mathbf{r}_1,\mathbf{r}_2|-\varepsilon), \quad (19)$$

where e_1^μ and e_2^μ are the initial and final photon polarization vectors, and $\hat{e}=e^\mu\gamma_\mu$. In (19) one must subtract from the integrand its value in zero field. We assume that such a subtraction has been done, but we perform it explicitly in the final result. The main contribution to the amplitude M emerges from integration with respect to ε from m to $\omega-m$. Thus, $\lambda=+1$ in the first Green's function in (19) and $\lambda=-1$ in the second. The representation (2) provides a convenient means of writing Eq. (19) in the form

$$M=i\alpha\int d\mathbf{r}_1\,d\mathbf{r}_2\exp[i(\mathbf{k}_1\mathbf{r}_1-\mathbf{k}_2\mathbf{r}_2)]\times\int d\varepsilon\,\mathrm{Sp}[(2\mathbf{e}_2^*\mathbf{p}_2-\hat{e}_2^*\hat{k}_2)D(\mathbf{r}_2,\mathbf{r}_1|\omega-\varepsilon)]\times[(2\mathbf{e}_1\mathbf{p}_1+\hat{e}_1\hat{k}_1)D(\mathbf{r}_1,\mathbf{r}_2|-\varepsilon)]+2i\alpha\mathbf{e}_2^*\mathbf{e}_1\int d\mathbf{r}\exp[i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}]\int d\varepsilon\,\mathrm{Sp}\,D(\mathbf{r},\mathbf{r}|\varepsilon). \quad (20)$$

Here $\mathbf{p}_{1,2}=-i\nabla_{1,2}$. After normalization the last term on the right-hand side contributes nothing to the amplitude at high energies, since this term is independent of ω and depends solely on the momentum transfer Δ. On the other hand, for $\omega\gg\Delta$ the amplitude is proportional to ω (see, e.g., Ref. 7). Further calculations amount to doing the following. We substitute (14) into (20), find the derivative of the result, and take the trace in the γ-matrices. It is convenient to direct the axis of the spherical coordinate system along the vector $\mathbf{k}_1+\mathbf{k}_2$. Allowing for the smallness of the angles, we can write $d\Omega_{1,2}\approx\theta_{1,2}\,d\theta_{1,2}\,d\phi_{1,2}=d\boldsymbol{\theta}_{1,2}$, with $(\boldsymbol{\theta}_{1,2}\mathbf{k}_1+\mathbf{k}_2)=0$. The Bessel functions depend on the two vectors $\boldsymbol{\theta}_{1,2}$ only as a function of the combination $\theta=|\boldsymbol{\theta}_1+\boldsymbol{\theta}_2|$. We transform to variables $\boldsymbol{\theta}=\boldsymbol{\theta}_1+\boldsymbol{\theta}_2$ and $\boldsymbol{\xi}=r_1\boldsymbol{\theta}_1-r_2\boldsymbol{\theta}_2$. After this we can easily evaluate the integral with respect to $d\boldsymbol{\xi}$. Next, to simplify the derivation, we consider only calculations for zero momentum transfer ($\mathbf{k}_2=\mathbf{k}_1=\mathbf{k}$), and then give the result of similar calculations for the case where $\Delta\sim 1/r_c$.

3.1. Zero momentum transfer

We assume that $\mathbf{k}_1=\mathbf{k}_2$ and $\mathbf{e}_1=\mathbf{e}_2$ and evaluate the integral with respect to $d\boldsymbol{\theta}$ using the relationship[15]

$$\int_0^\infty dx\, xe^{icx^2}J_\nu(ax)J_\nu(bx)=\frac{ie^{i\pi\nu/2}}{2c}J_\nu\left(\frac{ab}{2c}\right)\exp\left(\frac{-i(a^2+b^2)}{4c}\right)$$

and the relationships obtained by differentiating the above with respect to a parameter. We change variables in the integral representation of the Green's function:

$$l_1=\kappa_1\rho_1,\quad l_2=\kappa_2\rho_2,$$

where

$$\kappa_1=\sqrt{\varepsilon^2-m^2},\quad \kappa_2=\sqrt{(\omega-\varepsilon)^2-m^2}.$$

We then go from the variables r_1 and r_2 to the variables s and x:

$$r_1=\frac{\kappa_1\kappa_2}{m^2\omega sx},\quad r_2=\frac{\kappa_1\kappa_2}{m^2\omega s(1-x)}.$$

As a result, integration with respect to ε becomes elementary and we arrive at the following expression:

$$M=\frac{2i\alpha\omega m^2}{3}\int_0^1\frac{dx}{x(1-x)}\left[1+\frac{1}{x(1-x)}\right]\times\int_0^\infty\int_0^\infty\rho_1\rho_2\,d\rho_1\,d\rho_2\int_0^\infty\frac{ds}{s}\exp\left\{\frac{i}{2}\left[m^2(\rho_1^2+\rho_2^2)s-\frac{1}{sx(1-x)}\right]\right\}\sin^2[\delta(\rho_2)-\delta(\rho_1)]J_0(m^2s\rho_1\rho_2).\quad(21)$$

Here we have subtracted from the integrand its value at zero field. We change variables in the following manner: $\rho_1=\rho e^{-\tau/2}$ and $\rho_2=\rho e^{\tau/2}$. We rotate the contour of integration with respect to the variable s in such a way that it goes from zero to $i\infty$ and evaluate the integral with respect to s via the relationship[15]

$$\int_0^\infty\exp\left[-\frac{x(a^2+b^2)}{2}-\frac{1}{2x}\right]I_\nu(abx)\frac{dx}{x}=2I_\nu(a)K_\nu(b),$$

where $a<b$, and $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and third kinds, respectively. As a result we obtain

$$M=\frac{8i\alpha\omega m^2}{3}\int_0^1\frac{dx}{x(1-x)}\left[1+\frac{1}{x(1-x)}\right]\int_0^\infty\rho^3\,d\rho\times\int_0^\infty d\tau\,\sin^2[\delta(\rho e^{\tau/2})-\delta(\rho e^{-\tau/2})]I_0(y_1)K_0(y_2),\quad(22)$$

where $y_{1,2}=m\rho\exp[\mp\tau/2]\sqrt{x(1-x)}$.

We partition the integration with respect to τ into two domains: one from 0 to τ_0 and the other from τ_0 to ∞, where $1\gg\tau_0\gg 1/(mr_c)$. We start our calculations with the second domain. There the main contribution is provided by impact parameters $\rho<r_c$, and the field can be assumed to be of the Coulomb form. Evaluating the integrals with respect to x and ρ, and then with respect to τ, we obtain

$$M_2=i\frac{28\alpha\omega}{9m^2}\int_{\tau_0}^\infty d\tau\frac{\cosh\tau\sin^2(Z\alpha\tau)}{\sinh^3\tau}=-i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\mathrm{Re}\,\psi(1-iZ\alpha)+C+\ln 2\tau_0-\frac{3}{2}\right].\quad(23)$$

Here $\psi(x)=d\ln\Gamma(x)/dx$, and $C=0.577...$ is Euler's constant.

In the first domain the difference $\delta(\rho e^{\tau/2})-\delta(\rho e^{-\tau/2})$ is small, and we can expand in this difference. Hence this domain contributes only in the lowest Born approximation in the interaction with the external field. We partition the integration with respect to ρ into two domains: from zero to ρ_0 and from ρ_0 to ∞, where $r_c\gg\rho_0\gg 1/(m\tau_0)$. In the integral from zero to ρ_0 the field can be assumed to be of the Coulomb form, with the result that the integrals can easily be evaluated. The corresponding contribution is

$$M_{11}=i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln(m\tau_0\rho_0)+C-\frac{11}{21}\right].\quad(24)$$

In the integral from ρ_0 to ∞ we can use the asymptotic behavior of the Bessel functions $I_0(x)$ and $K_0(x)$ for large values of the argument and continue integration with respect to τ to infinity. As a result we find that

$$M_{12}=i\frac{28\alpha\omega}{9m^2}\int_{\rho_0}^\infty\rho\left(\frac{\partial\delta}{\partial\rho}\right)^2d\rho.\quad(25)$$

Adding Eqs. (23), (24), and (25), we get

$$M=i\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln\left(\frac{m\rho_0}{2}\right)+\frac{1}{(Z\alpha)^2}\int_{\rho_0}^\infty\rho\left(\frac{\partial\delta}{\partial\rho}\right)^2d\rho-\mathrm{Re}\,\psi(1-iZ\alpha)+\frac{41}{42}\right].\quad(26)$$

For $\rho\ll r_c$ the integral in this expression is equal to $\ln(r_c/\rho_0)+A$, where A is a constant of order unity. Hence M in Eq. (26) is independent of ρ_0, and we can select, say, $\rho_0=2/m$. Thus, we have arrived at an expression for the Delbrück forward scattering amplitude in an arbitrary screened potential. The value of the constant depends on the shape of the potential. We consider the Moliere potential,[16] which approximates the potential in the Thomas–Fermi model:

$$V(r)=-\frac{Z\alpha}{r}\sum_{i=1}^{3}\alpha_i e^{-\beta_i r}, \qquad (27)$$

where $\alpha_1=0.1$, $\alpha_2=0.55$, $\alpha_3=0.35$, $\beta_i=mZ^{1/3}b_i/121$, $b_1=6$, $b_2=1.2$, $b_3=0.3$. For this potential the scattering phase is

$$\delta(\rho)=Z\alpha\sum_{i=1}^{3}\alpha_i K_0(\beta_i\rho). \qquad (28)$$

Substituting this into (26), we arrive at the final expression for the Delbrück forward scattering amplitude in the Moliere potential:

$$M=i\,\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[\ln(183Z^{-1/3})-C-\mathrm{Re}\psi(1-iZ\alpha)-\frac{1}{42}\right]. \qquad (29)$$

As is known, the imaginary part of the photon's forward scattering amplitude is related to the total cross section σ of electron–positron pair production by the photon in the field through the formula $\sigma=\mathrm{Im}M/\omega$. Hence Eq. (29) agrees with the result obtained by Davies *et al.*[17] for the total cross section of pair production in a screened potential. Note that the real part of the Delbrück forward scattering amplitude in a screened potential is zero, in contrast to the case of an unscreened Coulomb potential.[2,9]

3.2. Nonzero momentum transfer

For finite momentum transfer it is convenient to proceed with our discussion using the language of helical amplitudes. We select the polarization vectors in the form

$$\mathbf{e}_{1,2}^{\pm}=([\boldsymbol{\lambda}\boldsymbol{\nu}_{1,2}]\pm i\boldsymbol{\lambda})\sqrt{2}, \quad \boldsymbol{\lambda}=[\boldsymbol{\nu}_1\boldsymbol{\nu}_2]/|[\boldsymbol{\nu}_1\boldsymbol{\nu}_2]|, \qquad (30)$$

where $\boldsymbol{\nu}_{1,2}=\mathbf{k}_{1,2}/\omega$. There are two independent variables: $M^{++}=M^{--}$ and $M^{+-}=M^{-+}$. In terms of linear polarization, due to parity conservation, the amplitude is finite only when the polarization vectors of the initial and final photons both lie in the scattering plane ($M^{\|}$) or are perpendicular to it ($M^{\perp}$). Here

$$M^{\|}=M^{++}+M^{+-}, \quad M^{\perp}=M^{++}-M^{+-}.$$

In the event of zero momentum transfer the amplitude M^{+-} is zero in view of conservation of the projection of angular momentum on to the direction of motion of the initial photon, and the amplitude M^{++} coincides with (29). As in the case with zero momentum transfer, we partition the integration with respect to the parameter τ into two domains: from 0 to τ_0, and from τ_0 to ∞, where $1\gg\tau_0\gg 1/mr_c$. The angle θ_0 between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ is equal to $\Delta/\omega \ll m/\omega$. In the domain of integration from τ_0 to ∞, we can assume a Coulomb field and ignore the angle θ_0. As a result, the contribution of this domain to M^{++} coincides with M_2 of Eq. (29), while the contribution to M^{+-} is zero. In the integration with respect to τ from 0 to τ_0 we again partition the integration with respect to the variable ρ into two domains: from zero to ρ_0, and from ρ_0 to ∞, where $r_c\gg\rho_0\gg 1/m\tau_0$. In the integral from zero to ρ_0 we can again assume a Coulomb field and ignore θ_0. The contribution of this domain to M^{++} coincides with M_{11} of Eq. (24), while the contribution to M^{+-} is zero. Screening affects the magnitude of the integral only in the second domain, from ρ_0 to ∞. Here the main contribution in integrating over the angles is provided by the values $\theta\sim\rho/r\sim\rho m^2/\omega\gg m/\omega\gg\theta_0$. The argument of the Bessel functions in the Green's functions is $l\theta\sim\omega\rho\theta\sim(m\rho)^2\gg 1$, and we can employ the asymptotic behavior of the Bessel functions for large values of the argument. We must retain the first two terms in their asymptotic expansions, since the first term is compensated for in the pre-exponential factor. After this is done, the integrals with respect to θ other variables can easily be evaluated, and we arrive at the following expression for the contribution of this domain to M^{++}:

$$M_{12}^{++}=i\,\frac{28\alpha\omega}{9m^2}\int_{\rho_0}^{\infty}\rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 J_0(\rho\Delta)\,d\rho. \qquad (31)$$

Adding Eqs. (31), (23), and (24), we get

$$M^{++}=i\,\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left[(Z\alpha)^{-2}\int_{2/m}^{\infty}\rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 J_0(\rho\Delta)\,d\rho-\mathrm{Re}\psi(1-iZ\alpha)+\frac{41}{42}\right]. \qquad (32)$$

We substitute (28) into (32) and evaluate the integral with respect to ρ via Eq. (6.578(10)) of Ref. 15. As a result we arrive at the final expression for the scattering amplitude M^{++} in the Moliere potential:

$$M^{++}=i\,\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left\{-\mathrm{Re}\psi(1-iZ\alpha)-C+\frac{41}{42}-\frac{1}{2}\sum_{i,j}\alpha_i\alpha_j\left[\ln\left(\frac{\beta_i\beta_j}{m^2}\right)+\frac{u}{\sqrt{u^2-1}}\times\ln(u+\sqrt{u^2-1})\right]\right\}, \qquad (33)$$

where $u=(\Delta^2+\beta_i^2+\beta_j^2)/2\beta_i\beta_j$. When $\Delta\ll 1/r_c$, Eq. (33) transforms into (29). When $m\gg\Delta\gg 1/r_c$, Eq. (33) becomes

$$M^{++}=i\,\frac{28\alpha\omega(Z\alpha)^2}{9m^2}\left\{\ln\frac{m}{\Delta}-\mathrm{Re}\psi(1-iZ\alpha)-C+\frac{41}{42}\right\}, \qquad (34)$$

which coincides with the result of Refs. 2, 8, and 9. Reasoning as we did in deriving (31), we obtain the following expression for the amplitude M^{+-}:

$$M^{+-}=i\,\frac{4\alpha\omega}{9m^2}\int_0^{\infty}\rho\left(\frac{\partial\delta}{\partial\rho}\right)^2 J_2(\rho\Delta)\,d\rho. \qquad (35)$$

Here integration with respect to ρ from ρ_0 to ∞ was replaced by integration from zero to ∞ because the domain from zero to ρ_0 contributes nothing to the integral. For the Moliere potential we have

$$M^{+-}=i\,\frac{2\alpha\omega(Z\alpha)^2}{9m^2}\left\{1+\frac{1}{\Delta^2}\sum_{i,j}\alpha_i\alpha_j\left[(\beta_i^2-\beta_j^2)\ln\frac{\beta_i}{\beta_j}\right.\right.$$

$$\left.\left.-\frac{u(\beta_i^2+\beta_j^2)-2\beta_i\beta_j}{\sqrt{u^2-1}}\ln(u+\sqrt{u^2-1}\,)\right]\right\}. \quad (36)$$

As $\Delta\to 0$, the amplitude M^{+-} specified by Eq. (36) tends to zero. When $m \gg \Delta \gg 1/r_c$, Eq. (36) becomes

$$M^{+-}=i\,\frac{2\alpha\omega(Z\alpha)^2}{9m^2}, \quad (37)$$

which coincides with the result of Refs. 2, 8, and 9. Thus, using the calculation of the Delbrück scattering amplitudes in a screened Coulomb potential, we have found that the semiclassical Green's function obtained for the case of an arbitrary decreasing centrally symmetric potential can be effectively used in studies of QED processes at high energies.

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