Instanton–anti-Instanton Pair Induced Contributions to τ–Lepton Hadronic Width

S.V. Faleev, P.G. Silvestrov

Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

Abstract

The instanton–anti-instanton pair induced asymptotics of perturbation theory expansion for the hadronic width of τ-lepton was found. For $N_f = N_c$ the nonperturbative instanton contribution is finite and may be calculated without phenomenological input. The instanton induced peturbative asymptotics was shown to be enhanced as $(n+10)!$ and in the intermediate region $n < 15$ may exceed the renormalon contribution. Unfortunately, the analysis of $\sim 1/n$ corrections shows that for $n \sim 10$ the obtained asymptotic expressions are at best only the order of magnitude estimate. The instanton–anti-instanton pair nonperturbative contribution to $R_{\tau\to hadrons}$ blows up. On the one hand, this means that instantons could not be considered *ab–initio* at such energies. On the other hand, this result casts a strong doubt upon the possibility to determine the α_s from the τ–lepton width.

1. The possibility to extract the value of the strong coupling $\alpha_s(m_\tau)$ from the hadronic width of the τ-lepton is now actively discussed. The usual belief is [2]-[6] that one may find the value of the strong coupling at the m_τ pole within 10% accuracy.

In this talk we would like to consider the contribution of the instanton–anti-instanton pair to the τ decay width. Previously the attempts to estimate theoretically the instanton contribution to $R_{\tau\to hadrons}$ have been done in refs. [7, 8]. However, it may be shown that authors of both papers [7, 8] have underestimated the instanton effects[1]. As we will see bellow, the instanton–anti-instanton pair induced correction to $R_{\tau\to hadrons}$ turns out to be much larger than the most reasonable semi-phenomenological result of the paper [8].

The semiclassical technique for high order perturbation theory estimates was suggested by L.N.Lipatov [1] almost 20 years ago. Nevertheless up to now the only attempt to find the instanton induced asymptotics of perturbation theory in realistic QCD was done in the paper of I.I.Balitsky [18]. However, unfortunately he could not find correctly either asymptotics of perturbation theory, or nonperturbative contribution of the instanton–anti-instanton pair in the most actual case $N_f = N_c$.

Following G.'t Hooft [10], one can rewrite the integral over the instanton–anti-instanton pair in the Borel form by considering the action as a collective variable. Within this approach the ambiguous, strongly interacting instanton and anti-instanton contribute to the smooth part of the Borel function and, thus, do not effect the asymptotics of perturbation theory. On the other hand, the singular part of the Borel function is saturated by the almost non-interacting pseudoparticles. That is why one can obtain the reliable prediction for the asymptotics starting from such ill defined object as the instanton–anti-instanton pair. The field configuration relevant for the large orders of perturbation theory is a small instanton inside of a very large anti-instanton (or vice versa). The size of small instanton is regulated by the internal momentum in correlator $q\rho_I \sim 1$. The size of anti-instanton (as well as the distance between the centers of pseudoparticles $R \sim \rho_A \gg \rho_I$) determines how close we are to the singularity on the Borel plane.

Thus, unlike the authors of the papers [7, 8], we are able to find the asymptotics of the perturbation theory. As for the nonperturbative contribution, as we will see, in the case $N_f = N_c$ the integral over ρ_A with the logarithmic accuracy comes from the whole region $1/q \ll \rho_A \ll 1/\Lambda_{QCD}$. This means that we have found explicitly the most probable long-wave background for the small instanton.

[1] More concretely, in [7] only the single-instanton contribution proportional to the product of light quark masses $m_u m_d m_s$ was found. The more reasonable is the approach of [8]. However, their result also contains a strong model dependent cancellation of vector and axial contributions.

2. The ratio of hadronic τ decay width to its leptonic width $R_{\tau\to hadrons}$ may be found by the analytical continuation of the correlator of the weak currents from the euclidean q^2 region (see e.g. [7, 8]):

$$R_{\tau\to hadrons} = -6i\pi \oint_{|s|=1} ds(1-s)^2[(1+2s)\Pi^T(-sm_\tau^2) + \Pi^L(-sm_\tau^2)] \ , \tag{1}$$

$$\Pi_{\mu\nu}(q^2) = \Pi^T(q^2)(q_\mu q_\nu - q^2\delta_{\mu\nu}) + \Pi^L(q^2)q_\mu q_\nu = \int dx e^{iqx}\langle j_\mu^+(x) j_\nu(0)\rangle \ ,$$

where

$$j_\mu = V_{ud}u^+\gamma_\mu(1+\gamma^5)d + V_{us}u^+\gamma_\mu(1+\gamma^5)s \ . \tag{2}$$

Here the cut goes along the positive real axis of the complex s-plane.

As we have said before, the most interesting configuration for us turns out to be the small instanton inside of the very large anti-instanton. The classical action for such configuration was found for example in [18] (see also [21])

$$S_{IA} = \frac{4\pi}{\alpha_s}\{1-\xi h\} \ , \ \xi = \frac{\rho_I^2\rho_A^2}{(R^2+\rho_A^2)^2} \ , \ h = 2|TrO|^2 - TrOO^+ \ . \tag{3}$$

Here O is the upper left 2×2 corner of the matrix $U = U_A^+U_I$ describing relative orientations of instanton and anti-instanton.

The features of light fermions are mostly sensitive to the presence of instantons. The Dirac operator $\hat{D}$ for each flavor of massless fermions has two almost zero modes $\Psi_\pm$ with eigenvalues $\lambda_\pm$. Explicit expressions for zero modes in the background of instanton and anti-instanton at $|x-x_I| \sim \rho_I$ are:

$$\Psi_I = \frac{1}{\pi}\frac{\rho_I}{[x^2+\rho_I^2]^{3/2}}\frac{x_\mu\tau_\mu^-}{|x|}\begin{pmatrix}\phi\\ \phi\end{pmatrix} \ , \ \Psi_A = \frac{1}{\pi}\sqrt{\frac{(x-x_I)^2}{(x-x_I)^2+\rho_I^2}}\frac{\rho_A}{[R^2+\rho_A^2]^{3/2}}U\begin{pmatrix}\phi\\ -\phi\end{pmatrix} \ , \tag{4}$$

where $\phi^{\alpha m} = \varepsilon_{\alpha m}/\sqrt{2}$ for $\alpha = 1,2$ and $\phi^{\alpha m} = 0$ for $\alpha > 2$, α is color index, $m = 1,2$ is spinor index, $\varepsilon_{\alpha m}$ is an antisymmetric tensor, and $\tau^\pm = (\mp i, \vec{\tau})$.

Because the instantons interact very slightly, the nonzero modes contribution to the fermionic determinant factorizes. The Green function in at $|x-x_I| \sim \rho_I$ also has rather simple form:

$$S(x,y) = S_\lambda + G_I + O(\xi) \ . \tag{5}$$

Here G_I is the Green function in the background of separate instanton and S_λ is the zero mode contribution. As was shown in [21]:

$$\lambda_\pm = \pm\frac{2\rho_I\rho_A}{(\rho_A^2+R^2)^{3/2}}|TrO| \ , \ S_\lambda(x,y) = \frac{1}{|\lambda|}\left\{\frac{TrO^+}{|TrO|}\Psi_A\Psi_I^+ + \Psi_I\Psi_A^+\frac{TrO}{|TrO|}\right\} \ . \tag{6}$$

The Green function $S(x,y)$ may contain only the terms which convert the right fermions to the left and vice versa. Therefore, S_λ contains only the interference terms $\sim \Psi_I\Psi_A^+$ and the contributions of the zero modes and of "quantum" Green function G_I to the correlation functions are of the same order of magnitude. The Green function in the instanton field has the form [14] ($x_I = 0$):

$$\begin{aligned}2\pi^2 G_I(x,y) \ &= \ \frac{i\gamma_\mu}{\sqrt{T_xT_y}}\frac{(x\tau^-)}{|x|}\left[z_\mu\frac{\rho_I^2+(\tau^+x)(\tau^-y)}{z^4} + \tau_\mu^+\frac{(z\tau^-)\rho_I^2}{2z^2T_x}\right]\frac{(\tau^+y)}{|y|}\left(\frac{1+\gamma_5}{2}\right)\\ &\quad +(\text{c.c.}, x\leftrightarrow y) \ ,\end{aligned} \tag{7}$$

where $T_x = \rho_I^2 + x^2$ and $z = x - y$. We use the Hermitean matrices $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$.

3. Now we can write down the expression for the correlation function:

$$\begin{aligned}\Pi_{\mu\nu} = \ &-4\int e^{iqx}\exp\left\{\frac{4\pi}{\alpha_s}\xi h\right\}T_{\mu\nu}(x,0)\times\\ &\left[Tr\{\gamma_\mu S(x,y)\gamma_\nu S(y,x)\} - Tr\{\gamma_\mu G_0(x,y)\gamma_\nu G_0(y,x)\}\right]\\ &[4\xi^{3/2}|TrO|^2]^{N_f}\frac{d(\rho_I)}{\rho_I^5}\frac{d(\rho_A)}{\rho_A^5}dx\,dx_I\,dx_A\,d\rho_I\,d\rho_A\,dU \ .\end{aligned} \tag{8}$$

Here, $G_0(x-y)$ is the bare Green function and (1) $(|V_{ud}|^2+|V_{us}|^2)=1$. The factor $\sim \xi^{3N_f/2}|TrO|^{2N_f}$ in the square brackets accounts for the contribution of almost zero modes (6) to the fermion determinant.

The instanton density reads [15, 16, 17]:

$$d(\rho)=\frac{c_1 e^{-N_c c_2+N_f c_3}}{(N_c-1)!(N_c-2)!}\left(\frac{2\pi}{\alpha_s(\rho)}\right)^{2N_c}\exp\left(-\frac{2\pi}{\alpha_s(\rho)}\right) . \tag{9}$$

for $\overline{MS}$ scheme $c_1=2e^{5/6}/\pi^2$, $c_2=1.511$, $c_3=0.292$, $c_2-c_3=2\ln 2-1/6$ [15, 17].

We will also use the well known two–loop formula:

$$\frac{4\pi}{\alpha_s(p)}=b\ln\left(\frac{p^2}{\Lambda^2}\right)+\frac{b'}{b}\ln\left(\ln\left(\frac{p^2}{\Lambda^2}\right)\right)+\ldots , \tag{10}$$

where for $N_f=N_c=3$ one has $b=9$ and $b'=64$.

In order to reduce the integral (8) to the Borel one, we have to integrate over all collective variables $x_I, x_A, \rho_I, \rho_A, U$, and, also, over x at fixed action (3), that is to say, at fixed value of the combination ξh. This problem may be divided into two parts. The integration over x and x_I is rather tedious algebraical problem due to the complicated form of the correlation function of quark currents in the instanton field. But from physical point of view, the main problem is the integration over size ρ_A and position $R=x_A-x_I$ of the large anti-instanton. There are two competing effects here. First, the factor $d(\rho_A)\sim\rho_A^b$ tends to make the integral over ρ_A divergent. Second, the almost zero fermion modes $\lambda^{2N_f}\sim\rho_I^{2N_f}/\rho_A^{4N_f}$ (6) tend to suppress the contribution of large anti-instantons. As a result, the value of the integral (8) depends strongly on the number of light quarks N_f. The simple dimensional analysis (let us note, that the integrations over $d\rho_A$ and d^4x_A are not independent and $\rho_A\sim x_A-x_I$ owing to the constraint $\xi h=const$ (3)) shows that the critical value is $N_f=N_c$. If $N_f<N_c$ the integral (8) diverges at large ρ_A. Nevertheless, just in this case the well defined instanton induced asymptotics of perturbation theory may be extracted from (8). For calculation of the integral (8) beyond the perturbation theory at $N_f<N_c$ the new physical income is necessary. The most favorable case is $N_f=N_c$. In this case, the integral over ρ_A in (8) diverges only logarithmically. As a result, we are able not only to obtain the asymptotics of perturbation theory, but also to calculate the finite nonperturbative instanton–anti-instanton pair contribution to $R_{\tau\to hadrons}$.

If $N_f>N_c$, the attraction of pseudoparticles which appears owing to fermionic zero modes prevails. As a result, the integral is saturated by $\rho_A\sim R\sim\rho_I$ and the approximation of almost noninteracting pseudoparticles does not work.

With the use of (9), (10) let us extract the ρ_A dependent part from (8):

$$d(\rho_A)=\phi(\rho_I^2/\rho_A^2)\left(\frac{\rho_I}{\rho_A}\right)^b d(\rho) \quad , \quad \phi(x)=\left[1+b\frac{\alpha_s}{4\pi}\ln(x)\right]^{2N_c-\frac{b'}{2b}} \tag{11}$$

Here and below $\alpha_s\equiv\alpha_s(q)\simeq\alpha_s(\rho_I)$.

The integral over ρ_A and $R=x_A-x_I$ for a fixed value of ξ gives:

$$\int\phi(\rho^2/\rho_A^2)\rho_A^{b-5}\delta\left(\frac{\rho^2\rho_A^2}{(R^2+\rho_A^2)^2}-\xi\right)d\rho_A d^4R=\frac{\pi^2}{2(b-2)(b-1)}\frac{\rho^b}{\xi^{b/2+1}}\phi(\xi) . \tag{12}$$

In fact here we have not integrated over ρ_A, but only have replaced this integral by that over $d\xi$.

The integration over the rest collective variables is straightforward but tedious (see [21]). Finally formula for $\Pi(q^2)$ may be written in the form of Borel integral over the new variable $t=1-h\xi$:

$$\Pi(q^2)=-\frac{e^{8/3}[7!6!]^2}{\pi^2 16!}\left(\frac{4\pi}{\alpha_s}\right)^{12}\left(0.510\int_0^1 dt\frac{e^{-\frac{4\pi}{\alpha_s}t}}{1-t}\phi+0.054\int_1^\infty dt\frac{e^{-\frac{4\pi}{\alpha_s}t}}{t-1}\phi\right) , \tag{13}$$

Here the averaging over $SU(3)$ group was performed numerically.

Both integrals in (13) diverge at $t=1$. In the configuration space these divergences are related to the integration over almost noninteracting instanton and anti-instanton. Since the divergence is only logarithmic one can try out the physical intuition in order to restrict the range of integration in (13). The natural cut-off for ρ_A is $\rho_A \ll 1/\Lambda_{QCD}$, or, in terms of t

$$|t-1|_{min} \sim \left(\frac{\rho_I}{\rho_{Amax}}\right)^2 < \frac{\Lambda^2}{q^2} \ . \tag{14}$$

The use of this cutoff allows to find the finite result

$$\Pi(q^2) = -\frac{e^{8/3}[7!6!]^2}{3\pi^2 16!}\left(\frac{4\pi}{\alpha_s}\right)^{12}\left(0.510\,P\int_0^\infty dt\frac{e^{-\frac{4\pi}{\alpha_s}t}}{1-t}\phi(|1-t|)+\frac{48}{2635}\frac{4\pi}{\alpha_s}e^{-\frac{4\pi}{\alpha_s}}\right) \ . \tag{15}$$

Here P means the principal value integral. The factor $\alpha_s^{-1} \sim \ln(q/\Lambda)$ in the last, nonperturbative, term appears after integration over ρ_A over the wide range $q^{-1} \ll \rho_A \ll \Lambda^{-1}$.

4. Now, integrating over s (1), one can find $R_{\tau\to hadrons}$

$$R_\tau = \frac{33e^{8/3}}{40}\frac{[7!6!]^2}{15!}\left(\frac{4\pi}{\alpha_s}\right)^{11}\left(0.510P\int_0^\infty dt\frac{e^{-\frac{4\pi}{\alpha_s}t}}{1-t}\psi(|1-t|)+\frac{636}{28985}\frac{4\pi}{\alpha_s}e^{-\frac{4\pi}{\alpha_s}}\right) \ , \tag{16}$$

where $\psi(x) = (1+9\alpha_s\ln(x)/8\pi)(1+9\alpha_s\ln(x)/4\pi)^{13/9}$. The expression (16) leads to the following asymptotics of the perturbation theory :

$$R_{\tau\to hadrons} = \sum R_{n\tau}\left(\frac{\alpha_s}{4\pi}\right)^n \ , \quad R_{n\tau} = 61(n+10)!\left(9^4 n\right)^{-\frac{35}{2n}} \ . \tag{17}$$

The last factor $(9^4n)^{-35/2n}$ here accounts for the sum of some $\sim 1/n$ corrections to the asymptotics. The more detailed discussion of the $\sim 1/n$ effects may be found in [21]. In (17) we have taken into account only the corrections which are enhanced like $\ln(n)N_c^2/n$ and $\ln(3N_c)N_c^2/n$. Unfortunately, the unknown corrections also may be huge $\sim N_c^2/n$. Therefore, our final conclusion concerning the asymptotics of perturbation theory is rather contradictory. On one hand, the instanton induced asymptotics may even exceed at $n < 15$ the renormalons [10, 11, 13]. On the other hand, it may be calculated only at $n \gg N_c^2 \sim 10$. The only hope may be that at $n \sim 10$ expression (17) is a reliable order of magnitude estimate.

In order to compare our result with the experiment, let us consider in more detail the pure nonperturbative $R_\tau^{np} \sim e^{-\frac{4\pi}{\alpha}}$ term in (16) (note that this correction in terms of Λ_{QCD} (10) behaves like $(\Lambda/m_\tau)^{18}$). The quantities R_τ^{np} for popular values of $\alpha_s(m_\tau)$ are shown in the first column of the Table. One has to compare these values with the experimental value $R_{\tau\to hadrons} - N_c = 0.56 \pm 0.03$ [4] (here we have subtracted the trivial part $R_\tau \approx N_c$). As we can see, the nonperturbative correction turns out to be dramatically large.

There exist, however, the procedure (used also in the work [8]) which allows to reduce the huge discrepancy with the experiment. The regular way to improve the nonperturbative correction (16) is to calculate the $\sim \alpha_s$ corrections to it. Since, as will be shown below, the result is going to be deceased by $30-50$ times, one has to sum an infinite series of the corrections. Undoubtedly, the exact calculation even of the first correction $\sim \alpha_s$ to (16) is beyond our abilities. All we can do is to use the dependence of the coupling constant on the instanton size $\alpha_s(\rho)$, which is known from the renormalization group-invariance principle. Rather weak justification for taking into account just these particular corrections is the fact that they are enhanced as $\ln(3N_c)$ compared to the other ones. The explicit expression fr the improved nonperturbative correction which has been found in [21] has the form:

$$\begin{aligned} R^{*np}_{\tau\to hadrons} &= \lim_{\varepsilon\to 0}\frac{9e^{8/3}}{\pi\cdot 70\cdot 31}\left(\frac{4\pi}{\alpha_s(m_\tau)}\right)^{13}\exp\left(-\frac{4\pi}{\alpha_s(m_\tau)}\right) \\ &\times \left(1+\frac{9\alpha_s}{4\pi}\frac{\partial}{\partial\varepsilon}\right)^{\frac{53}{9}}\sin(\varepsilon\pi)\frac{(12-\varepsilon)\Gamma(9-\varepsilon)\Gamma(8-\varepsilon)^2\Gamma(5-\varepsilon)}{2^{2\varepsilon}\Gamma(18-2\varepsilon)} \ . \end{aligned} \tag{18}$$

Note, that the leading term here, corresponding to $(\partial/\partial\varepsilon)^0$, is vanishing. This leads to loss of one factor $1/\alpha_s(m_\tau)$ in $R_{\tau\to hadrons}$ compared to the correlation function. Coefficients of the expansion of (18) in powers of α_s were found numerically. The resulting quantities $R^{*np}_{\tau\to hadrons}$ are shown in the last column of the Table. As we can see, taking into account of the two-loop dependence of $\alpha_s(\rho)$ allows to reduce the nonperturbative correction almost by two orders of magnitude. Moreover, passing from $\alpha_s = 0.28$ to $\alpha_s = 0.29$ our correction changes the sign. Such rich behaviour provides us one more evidence that effectively we work at very low energies and listed in the table result is, at best, only the estimate on the order of magnitude.

Table

$\alpha_s(m_\tau)$	$R^{np}_{\tau\to hadrons}$	$R^{*np}_{\tau\to hadrons}$
0.28	5.65	-0.0229
0.29	17.44	0.0507
0.30	49.23	0.476
0.32	311.1	6.70
0.34	1514.1	44.79
0.36	5943.4	190.1

The full nonperturbative contribution of the instanton–anti-instanton pair to the hadronic τ decay width $R^{*np}_{\tau\to hadrons}$ and the contribution accounting for only the leading over α_s term in the sum (18) $R^{np}_{\tau\to hadrons}$ at various values of $\alpha_s(m_\tau)$.

One should remember that the obtained results should be compared with the experimental value $(R_{\tau\to hadrons} - 3)_{exp} = 0.56 \pm 0.03$. We see that, in spite of all our effort, even for $\alpha_s = 0.29$ the instanton contribution to $R_{\tau\to hadrons}$ still remains large. In this situation the only way out may be to ignore completely the instanton contribution to $R_{\tau\to hadrons}$. For example, one may say that the series of the power corrections $\sim 1/m_\tau^n$ is also asymptotic, and it is natural to cut off it somewhere at $n \sim 4-8$ (the instanton contribution behaves like $\sim 1/m_\tau^{18}$). Nevertheless, we do not know to what extent this point of view is justified and we consider our result as indication of the *impossibility* to extract the reliable value of α_s from $R_{\tau\to hadrons}$.

Asknowledgements Authors are thankful to V. L. Chernyak, M. E. Pospelov, A. I. Vainshtein and A. S. Yelkhovsky for valuable discussions. This work was supported by the Russian Foundation for Fundamental Research under Grant 95-02-04607a. The work of S.F. has been supported by the INTAS Grant 93-2492 within the program of ICFPM of support for young scientists.

References

[1] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, (1977) 411.
[2] E. Braaten, S. Narison and A. Pich, Nucl. Phys. **B373** (1992) 581.
[3] F. Le Diberder and A. Pich, Phys. Lett. **B286** (1992) 147.
[4] A. Pich, FTUV-94-71 (1994) [hep-ph/9412273]; S. Narison, CERN-TH-7506-94 [hep-ph/9412295].
[5] ALEPH Collaboration, Phys. Lett. **B307** (1993) 209.
[6] S. Narison, Montpellier preprint PM95/07 [hep-ph/9504334].
[7] P. Nason and M. Porrati, Nucl. Phys. **B421** (1994) 518.
[8] I.I. Balitsky, M. Beneke and V.M. Braun,
[9] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. **B163** (1980) 46.
[10] G.'t Hooft, in: *The why's of subnuclear physics*, (Erice, 1977), ed. A. Zichichi, (Plenum New York 1977).
[11] F. David, Nucl. Phys. **B234** (1984) 237; G.Parisi, Nucl. Phys. **B150** (1979) 163; A.H. Mueller, Nucl. Phys. **B250** (1985) 327.
[12] V.I. Zakharov, Nucl. Phys., **B385** (1992) 452.
[13] A.I. Vainshtein and V.I. Zakharov, Phys. Rev. Lett. **73** (1994) 1207; Report No. TPIMINN-94/9-T, UM-TH-94-09, April 1994.
[14] L.S. Brown,R.D. Carlitz,D.B. Creamer and C. Lee, Phys. Rev. **D17** (1978) 1583.
[15] G.t'Hooft, Phys. Rev. **D14** (1976) 3433.

[16] C. Bernard, Phys. Rev. **D19** (1979) 308.
[17] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B 193** (1981) 210.
[18] I.I. Balitsky, Phys. Lett., **B273** (1991) 282.
[19] N. Andrei and D. J. Gross, Phys. Rev. **D18** (1978) 468.
[20] L. Baiuleu, J. Ellis, M. K. Gailard and W. Zakrzewski, Phys. Lett., **B77** (1978) 290.
[21] S.V. Faleev and P.G. Silvestrov, preprint BudkerINP 95-73 [hep-ph/9509231]