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The Status of Renormalon

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Abstract

It is shown that the series of renormalon–type graphs, which consist in the chain of insertions to one soft(hard) gluon(photon) line is in fact ill defined. Each new type of insertions, which appears in the higher orders of perturbation theory, generates the correction to renormalon of the order of ~ 1. However, this series of the corrections to the asymptotics although have no small parameter but hopefully is not the asymptotic one. The consideration based on the use of the renormalization group equation for effective charge is supported by the direct diagrammatic picture.

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1. For many years it was generally accepted that the true asymptotics of the perturbation theory in theories with running coupling constant is determined by the renormalons [1, 2]. Nowadays, the renewed interest is demonstrated in asymptotic estimates of perturbative series [3-11]. It results even in the recent attempts [12] to use the renormalon for direct calculation of experimentally measurable quantities.

However, the accurate determination of renormalon-type asymptotics appears to be not so simple problem. It was recognized by the specialists [6, 7, 8] that the overall normalization factor of the renormalon could not be found without taking into account of all terms of the expansion of, say, the Gell-Mann–Low function. However, the usual proof of this fact do not refer on the direct counting of the Feynman graphs (see also the discussion after our equation (12)). Also recently Vainshtein and Zakharov [10] have found the new source of uncertainty, which makes difficult the quantitative finding of the renormalon asymptotics (at least for the ultraviolet renormalon). The generally considered renormalon chain of graphs is formed by dressing of one gluon(photon) line by various insertions. The authors of ref. [10] have broken this tradition and considered the contribution to the ultraviolet renormalon of diagrams with two, three etc. dressed lines. Their analysis based on the operator product expansion, showed that contributions to the asymptotics from diagrams with any few dressed gluon lines are all of the same order of magnitude and also are parametrically larger than the usual renormalon with only one dressed line.

In the present paper we would like to continue the analysis of the renormalon–type chain of Feynman diagrams. We consider the dressing of single gluon(photon) line as for the traditional renormalon, but try to estimate the role of the arbitrary high order insertions to this "dressed" gluon. It will be shown that each new type of insertions generates the correction to renormalon of the order of ~ 1. Thus in order to find the overall normalization of the asymptotics one has to calculate all the coefficients $b_0, b_1, b_2,$ ... of the expansion of the renormalization group equation for, say, effective charge (see the eq. (2) below). Our approach is equally valid for both infrared and ultraviolet renormalons. However, because there exists another source of problems for the ultraviolet renormalon [10], we will concentrate our attention on the infrared one.

It is a tradition now to consider the renormalon for QED. In this paper we will also discuss only the QED–type diagrams of the perturbation theory, without the self-interaction of gluons. However, while considering the renormalization group coefficients we would like to use the QCD values $b_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$, $b_1 = \left(\frac{1}{4\pi}\right)^2\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_f N_c + N_f/N_c\right)$, ... Thus by hand we avoid the problem of the gauge–independent definition of the renormalon chain in QCD.

2. The contribution of the diagrams with exchange of one soft gluon(photon) to some "physical" quantity (say $R_{e^+e^-\to hadrons}$, or the correlator of two currents) has the generic form

$$R = \int_{k \ll Q} \alpha(k) \frac{k^2 dk^2}{Q^4} \quad . \tag{1}$$

This is the first infrared renormalon, because of the contribution proportional to dk^2/Q^2 vanishes due to the gauge invariance. The Feynman graphs corresponding to this value are shown in fig. 1. More precisely the graphs of fig. 1 correspond to (1) with the running coupling constant $\alpha(k)$ replaced by the fixed value $\alpha_0 = \alpha(Q)$. In (1) we have written down the effective running coupling constant $\alpha(k) = \alpha_{eff}(k)$ instead of α_0. $\alpha_{eff}(k)$ is trivially connected with the transverse part of the gluon propagator $G_{\mu\nu}(k)$. The function $\alpha(k)$ satisfies the renormalization group equation:

$$\begin{aligned} \frac{d\alpha}{dx} &= b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + ... \quad , \\ x &= \ln\left(\frac{Q^2}{k^2}\right) \quad , \end{aligned} \tag{2}$$

where for practical applications we will use $N_f = N_c = 3$, $4\pi b_0 = 9$, $16\pi^2 b_1 = 64$. Starting from b_2 the coefficients of the expansion (2) depend on the renormalization scheme. It is to be noted here that we have fixed the renormalization scheme by considering the effective charge. Thus our coefficients $b_2, b_3, ...$ are neither the free parameters, nor the known, say, for $\overline{MS}$ scheme, $b_2(\overline{MS})$, $b_3(\overline{MS})$.

In order to find the leading contribution to the asymptotics, one may neglect b_1, b_2, etc. in (2). Now instead of (1)

$$\begin{aligned} R &= \frac{2}{\alpha_0}\int_0^\infty \alpha(x) e^{-2x} dx = \int_0^\infty \frac{e^{-2x}}{1 - b_0\alpha_0 x} 2dx = \\ &= \int_0^\infty \sum_{N=0} (b_0\alpha_0 x)^N e^{-2x} 2dx = \sum_{N=0} \left(\frac{b_0\alpha_0}{2}\right)^N N! \quad . \end{aligned} \tag{3}$$

Here the first equality is also the explicit definition of the quantity, which we are going to consider in this paper.

It is seen immediately from (3) that if $b_0 > 0$, our renormalon is ill defined due to the Landau pole. However, in this paper we will consider only the asymptotics of the perturbation theory and will not concern the issue of the nonperturbatve definition of the integral (3).

The way of calculation of the renormalon now seems straightforward. One should find, step by step, the function $\alpha(x)$ from (2), substitute it into (3), and look for the asymptotics. Nevertheless, before passing to this formal way, let us illustrate the role of complicated contributions to the renormalon by the explicit estimate of corresponding Feynman graphs.

The fig. 2 shows the chain of diagrams corresponding to renormalization of the soft gluon line in the leading approximation (3). As we have said above, we show only the QED–type diagrams consisting of fermionic bubbles without gluon self-interaction. In the N-th order of perturbation theory each of the N bubbles from fig. 2 generates the factor

$$b_0\alpha_0 \ln\left(\frac{Q^2}{k^2}\right) \tag{4}$$

in the integrand of (1),(3). Now the difference between theories (QCD,QED) is hidden in the factor b_0, accompanying the single bubble.

Now let us replace two of the simple bubbles by the more complicated two loop diagram, as it is shown in fig. 3. Again we have shown only the QED diagram. Moreover we have shown explicitly only one of the two diagrams (compare with fig. 1) of the second order in α_0. As it is indicated on the figure, the – two loop bubble generates the factor

$$b_1\alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right) \tag{5}$$

in the integrand, which has one power of large logarithm less (or one α_0 more) than the leading order contribution (3). On the other hand a large combinatorical factor $N-1$ appears due to a number of permutations of the second order bubble among the simple bubbles. As a result one has

$$(N-1)b_1\alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right)\left[b_0\alpha_0 \ln\left(\frac{Q^2}{k^2}\right)\right]^{N-2} \to \left(\frac{b_0\alpha_0}{2}\right)^N N!\frac{2b_1}{b_0^2}\ . \tag{6}$$

Thus we can see that taking into account one second order insertion into the soft gluon line leads to the correction of the order of one to the trivial asymptotics (3). Summation over the number of the second order insertions shown in fig. 3 leads to a simple exponentiation of this correction

$$\left(\frac{b_0\alpha_0}{2}\right)^N N!\exp\left(\frac{2b_1}{b_0^2}\right)\ . \tag{7}$$

In last few years it became very popular [9, 11] to consider the renormalon in the limit $N_f \to \infty$. From this point of view the correction (6) is nothing more than the $\sim 1/N_f$ correction, because of $2b_1/b_0^2 \sim 1/N_f \ll 1$. However for practically interesting $N_c = 3$ and $N_f = 3, 4, 5$ one has, respectively, $2b_1/b_0^2 = \frac{128}{81}, \frac{906}{625}, \frac{678}{529}$.

Consider now the more complicated diagram of fig. 4 with dressing of the internal gluon line of the second order bubble. At this point it is natural to write down explicitly the last integration over internal momentum of the two loop diagram

$$b_1\alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right) = b_1\alpha_0^2 \int_{k^2}^{Q^2} \frac{dq^2}{q^2}\ . \tag{8}$$

Now the dressing of the gluon line evidently leads to

$$b_1\alpha_0^2\int_{k^2}^{Q^2}\left[b_0\alpha_0\ln\left(\frac{Q^2}{q^2}\right)\right]^n\frac{dq^2}{q^2}=\frac{1}{n+1}b_1\alpha_0^2\ln\left(\frac{Q^2}{k^2}\right)\left[b_0\alpha_0\ln\left(\frac{Q^2}{k^2}\right)\right]^n\ . \qquad (9)$$

Thus up to the overall factor $\frac{1}{n+1}$ the contribution of diagram of fig. 4 coincides with that of fig. 3. Summation over the number of simple bubbles inserted into the large bubble – n naturally leads to $\ln(N)$. Also taking into account a number of large bubbles of fig. 4 allows to exponentiate the correction

$$\left(\frac{b_0\alpha_0}{2}\right)^N N!\exp\left(\frac{2b_1}{b_0^2}\ln(N)\right)=\left(\frac{b_0\alpha_0}{2}\right)^N N^{\frac{2b_1}{b_0^2}}N!\ . \qquad (10)$$

This is the generally recognized expression for the infrared renormalon. Our arguments up to this stage simply repeat the line of reasoning of the paper [5], though may be in more details.

It is clear, that the argument of the exponent in (10) was found with the $\sim\ 1/\ln(N)$ accuracy and therefore the nontrivial overall factor as well as the function of N, weaker than N^γ, may appear in (10), as we consider in detail in the following section.

Now let us consider the three loop correction to the renormalon, shown in fig. 5. Like it was done before, we show only one example of the third order diagram. All these contributions generate the factor in the integrand of (3)

$$b_2\alpha_0^3\ln\left(\frac{Q^2}{k^2}\right)\ . \qquad (11)$$

Thus here we have two extra α_0, which at first glance could not be compensated by one combinatorical factor N and hence the diagram of fig. 5 seems to generate only the $\sim\ 1/N$ correction to renormalon. In particular such conclusion was drown by Zakharov [5]. However, let us see, what happens if one dresses the internal gluon lines of the three loop diagram (fig. 6). Now summation over the number of trivial corrections inserted into a large diagram n_1, n_2 gives:

$$\begin{aligned} & b_2\alpha_0\ln\left(\frac{Q^2}{k^2}\right)\sum_{n_1,n_2}\alpha_0^2\frac{1}{n_1+n_2+1}(N-n_1-n_2-2) \\ \sim\ & b_2\alpha_0\ln\left(\frac{Q^2}{k^2}\right)\times(\alpha_0 N)^2\sim b_2\alpha_0\ln\left(\frac{Q^2}{k^2}\right)\ . \end{aligned} \qquad (12)$$

Here the factor $(n_1+n_2+1)^{-1}$ in the sum appears after integration over the internal momentum of the large bubble (see (8)), while the factor $(N-n_1-n_2-2)$ accounts for the number of permutations of our large bubble with simple small bubbles on the main gluon line. So we see that after dressing of all gluon lines the three loop ($\sim b_2$)

diagram generates the correction to renormalon of the order of ~ 1. One can easily show that four loop ($\sim b_3$), five loop ($\sim b_4$) etc. diagrams generate the corrections of the same order of magnitude. Previously the analogous proof of the importance of the high loop corrections to the renormalon chain was done by Mueller [13], but this result was not published.

3. In this section we would like to turn back to the direct analysis of the renormalon (1,3) without referring to the Feynman diagrams. It is easy to integrate formally the renormalization group equation (2)

$$-\frac{1}{\alpha}+\frac{1}{\alpha_0}-\frac{b_1}{b_0}\ln\left(\frac{\alpha}{\alpha_0}\right)-c_2(\alpha-\alpha_0)-c_3\frac{\alpha^2-\alpha_0^2}{2}-...=b_0x\ , \tag{13}$$

where, $c_2=b_2/b_0-b_1^2/b_0^2\ ,\ c_3=b_3/b_0-2b_2b_1/b_0^2+b_1^3/b_0^3\ ...$. Also it is convenient to introduce the new set of variables

$$\begin{aligned} t &= 2x\ , \\ \frac{b_0\alpha}{2} &\to \alpha\ ,\ \frac{b_0\alpha_0}{2}=a \\ \beta_1 &= \beta=\frac{2b_1}{b_0^2}\ ,\ \beta_n=\left(\frac{2}{b_0}\right)^n c_n\ ...\ . \end{aligned} \tag{14}$$

Now the renormalon (3) takes the form

$$\begin{aligned} R &= \int_0^\infty \frac{\alpha}{a}e^{-t}dt\ , \\ \frac{\alpha}{a} &= \frac{1}{1-at-\beta_1 a\ln(\alpha/a)-\beta_2(\alpha-a)a-\beta_3(\alpha^2-a^2)a-...}\ . \end{aligned} \tag{15}$$

Although we have shown, that the corrections to renormalon generated by the high order contributions to the renormalization group equation $b_2\alpha^4, b_3\alpha^5$... (2) are not small, the correction induced by the second term $b_1\alpha^3$ still plays an outstanding role due to the additional enhancement by $\ln(N)$ (see (10)). Therefore one should be more careful, while treating these enhanced contributions. Moreover, the iterations of the second order diagram of fig. 3, e.g. as it is shown in fig. 7, may also have some additional enhancement. Thus at the first stage let us omit $\beta_2,\beta_3,\beta_4\ldots$ in (15) and consider the truncated effective charge:

$$\frac{\alpha}{a}=\frac{1}{1-at-\beta a\ln(\alpha/a)}\ . \tag{16}$$

This is the transcendental equation for the function $\alpha=\alpha(a)$, which may be solved iteratively. For estimate of the N-th order of perturbation theory we will often use the formula for N-th term of the expansion of the integral in powers of a

$$\left\{\int\frac{e^{-t}dt}{1-at}\left[\frac{\beta a}{1-at}\right]^k\left[\ln\frac{1}{1-at}\right]^m\right\}_N=a^N N!\frac{\beta^k}{k!}\left[\ln\frac{N}{k}\right]^m\left(1+O\Big(\frac{1}{\ln N}\Big)\right)\ . \tag{17}$$

Here both m and k are supposed to be large $m, k \sim \ln(N)$. Everywhere in this section we suppose that not only the N, but also the $\ln(N)$ is a large parameter and neglect all the corrections of the order of $\sim 1/\ln(N)$. In order to derive the formula (17) one has to use the asymptotics of gamma-function as well as the trivial identity

$$(\ln(p))^n = \lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial \varepsilon} \right)^n p^{\varepsilon} \ . \tag{18}$$

Now it is easy to make the first iteration in (16,15)

$$\begin{aligned} \{R_1\}_N = \int \frac{e^{-t} dt}{1 - at + \beta a \ln(1 - at)} &= \int \frac{e^{-t} dt}{1 - at} \sum_p \left[\frac{\beta a}{1 - at} \ln \frac{1}{1 - at} \right]^p = \\ = a^N N! \sum_p \frac{1}{p!} \left[\beta \ln \left(\frac{N}{p} \right) \right]^p &= \left(\frac{b_0 \alpha_0}{2} \right)^N \left[\frac{N}{\beta \ln N} \right]^{\beta} N! \left(1 + O\Big(\frac{1}{\ln N} \Big) \right) \ . \end{aligned} \tag{19}$$

Here for brevity we "forget" to indicate, that only the N-th term of the expansion of both integrals over a is considered. The asymptotics (19) should be compared with (10). One can see that the consistent treatment of the diagrams of fig. 4, which in fact we have done in (19), results in the nontrivial small factor $(\ln N)^{-\beta}$ as compared to the naive result (10).

Consider now the effect of iteration of two loop correction (fig. 7), which in terms of α means

$$\left(\frac{a}{\alpha} \right)_{second\ iteration} = 1 - at + \beta a \ln(1 - at + \beta a \ln(1 - at)) \ . \tag{20}$$

To this end one has simply to replace in (19)

$$\begin{aligned} \left[\frac{\beta a}{1 - at} \ln \frac{1}{1 - at} \right]^p &\to \left[\frac{-\beta a \ln(1 - at + \beta \ln(1 - at))}{1 - at} \right]^p = \\ = \left[\frac{\beta a}{1 - at} L \right]^p & \left\{ 1 - \frac{1}{L} \ln \left(1 - \frac{\beta a}{1 - at} L \right) \right\}^p \ , \end{aligned} \tag{21}$$

where we introduce $L = \ln\left(\frac{1}{1-at}\right)$. As we have seen from (17,19), from the point of view of perturbation theory $\frac{\beta a}{1-at} \approx (\ln N)^{-1}$, while $L \approx \ln N$. As a result the expression in curly brackets in (21) is of the form

$$1 - \frac{1}{L} \ln \left(1 - \frac{\beta a}{1 - at} L \right) \sim 1 - O \left(\frac{\ln(\ln N))}{\ln N} \right) \tag{22}$$

and may be written as the exponent of a small quantity. Thus instead of (21) one gets

$$\left[\frac{\beta a}{1 - at} L \right]^p \left[1 - \frac{\beta a}{1 - at} L \right]^{-\frac{p}{L}} = \sum_{n=0} \left[\frac{\beta a}{1 - at} L \right]^{p+n} \frac{n^{\frac{p}{L} - 1}}{\Gamma(p/L)} \ . \tag{23}$$

In terms of Feynman graphs the variables p and n are respectively the number of large second order bubbles and number of internal second order bubbles (see fig. 7). Also all wavy lines in fig. 7 are supposed to be renormalized by arbitrary number of simple one loop insertions (like in fig. 2).

Substitution of (23) into (19) gives

$$\{R_2\}_N = a^N N! \sum_{q=0}^{\infty} \sum_{n=0}^{q} \frac{1}{q!} \left[\beta \ln\left(\frac{N}{q}\right)\right]^q \frac{n^{\frac{q-n}{\ln N}-1}}{\Gamma\left(\frac{q-n}{\ln N}\right)} \ . \tag{24}$$

Here we have denoted $p+n$ (23) by q. Due to $[\beta \ln N]^q/q!$ the series in q (24) has a narrow peak. Therefore, up to $\sim 1/\ln N$ corrections

$$\{R_2\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta x^{\beta-x-1} e^{-x\ln(\ln N)} \frac{dx}{\Gamma(\beta - x)} \ , \tag{25}$$

where $x = n/\ln(N)$ (24). The explicit integration here may be performed only if the $\ln(\ln N) \gg 1$

$$\{R_2\}_N \approx \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta\ln(\ln N)}\right]^\beta N! \left(1 + O\Big(\frac{1}{\ln(\ln N)}\Big)\right) \ . \tag{26}$$

Thus again we can see (compare (26) and (19)) that taking into account the diagrams of fig. 7, or in other words, making the second iteration (20) in the transcendental equation (16), leads to parametrically large renormalization of the renormalon.

In the analogous way one may perform the third iteration of (16). The result reads

$$\{R_3\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta dx_1 \int_0^{x_1} dx_2 \frac{(x_1-x_2)^{\beta-x_1-1}}{\Gamma(\beta-x_1)} \frac{x_2^{x_1-x_2-1}}{\Gamma(x_1-x_2)} e^{-x_2\ln(\ln N)} \ . \tag{27}$$

Here the integration may be performed analytically only if the $\ln(\ln(\ln N)) \gg 1$ resulting in the same expression as (26), but with $\ln(\ln N)$ replaced by $\ln(\ln(\ln N))$ both in the main formula and in the estimate of the error.

For the case of arbitrary $k+1$ iterations in (16) the generalization of (25) and (27) leads to

$$\{R_{k+1}\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta dx_1 \int_0^{x_1} dx_2 ... \int_0^{x_{k-1}} dx_k \frac{\beta - x_1}{\Gamma(1+\beta-x_1)} \tag{28}$$
$$\times \frac{(x_1-x_2)^{\beta-x_1}}{\Gamma(1+x_1-x_2)} \frac{(x_2-x_3)^{x_1-x_2}}{\Gamma(1+x_2-x_3)} ... \frac{(x_{k-1}-x_k)^{x_{k-2}-x_{k-1}}}{\Gamma(1+x_{k-1}-x_k)} \frac{x_k^{x_{k-1}-x_k}}{x_k} e^{-x_k\ln(\ln N)} \ .$$

Again one can calculate this integral analytically if the k-th logarithm is large $\ln(\ln(... \ln N) ...) \gg 1$, though the expression (28) itself has the accuracy $\sim 1/\ln N$.

Looking at the formula (28), one may even doubt, whether it has any finite limit for large k. In order to prove this finiteness consider in more details the last integral in (28)

$$\int_0^{x_{k-1}} dx_k \frac{(x_{k-1}-x_k)^{x_{k-2}-x_{k-1}}}{\Gamma(1+x_{k-1}-x_k)} \frac{x_k^{x_{k-1}-x_k}}{x_k} e^{-x_k \ln(\ln N)} = \tag{29}$$
$$x_{k-1}^{x_{k-2}-x_{k-1}-1}\Big(1 + \sum_{m+n>0} c_{mnl} x_{k-2}^m x_{k-1}^n (\ln x_{k-1})^l\Big) \ .$$

Substitution of this result into (28) naturally leads to the following estimate

$$\{R_{k+1}\}_N = \{R_k\}_N \left(1 + O\Big(\frac{const^k}{k!}\Big)\right) \ . \tag{30}$$

By the way we have shown here that the integral in (28) at large k forgets effectively about the value of $\ln(N)$.

Thus we arrive at the surprising result. If one performs only one (a few) iterations while solving the transcendental equation for the effective charge (16), the asymptotics of the perturbation theory differs drastically (19,25–27) from the generally recognized renormalon (10). Nevertheless, after taking into account the infinite number of iterations (28) the renormalon (10) is restored up to some overall constant.

4. In the present paper we have no plan to perform the complete analysis of the contribution to renormalon from the high order terms of the renormalization group equation (2) $b_2\alpha^4, b_3\alpha^5$ ... (in particular because the coefficients b_2, b_3 themselves are not known). It will be enough for us to show that all the corrections to renormalon due to these high order terms are equally important. To this end we will consider the contribution to renormalon from the multi-loop corrections in the equation (2) only in the leading nontrivial approximation.

As we have seen (13–15) in the formal solution of the renormalization group equation the coefficients b_n have been converted into $\beta_n = 2^n b_n / b_0^{n+1} + \ ...$. In the linear in $\beta_2, \beta_3, ...$ approximation the effective charge (15) takes the form

$$\frac{\alpha}{a} = \frac{\alpha_1}{a}\Big(1 + \beta_2\alpha_1^2 + \beta_3\alpha_1^3 + \ ...\Big) \ , \tag{31}$$

where α_1 is the solution of the equation (16). While going from (15) to (31) we have ignored the $\beta_2 a^2$ in comparison with $\beta_2\alpha^2$, $\beta_3 a^3$ in comparison with $\beta_3\alpha^3$ and so on. With the experience of the previous section one can easily show, that the omitted contributions are of the order of $\sim 1/N$.

Now it is easy to repeat all the logics, which led us to the equation (28)

$$\{R_{k+1}\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta dx_1 \tag{32}$$
$$\left(1 + \frac{\beta_2}{2!}(\beta - x_1)^2 + \frac{\beta_3}{3!}(\beta - x_1)^3 + ...\right) \int_0^{x_1} dx_2 ... \ ,$$

where the part of the formula after integration over dx_2 simply repeats the corresponding part of the equation (28). Keeping in mind the finiteness of both (28) and (32) (see (29) and (30)) one can see that all β_2, β_3, ... make contributions of the order of ~ 1 to the renormalon.

Nevertheless the formula (32) allows one to draw not only the pessimistic conclusions. All the high order ($\sim \beta_n$) terms in (32) have appeared in the combination $\beta_n/n!$. Thus one may hope, that at least the series of the corrections to the renormalon is not the asymptotic one.

5. The formulas like (28) still are too complicated for practical use. In this section we would like to develop another method for calculation of the renormalon–type asymptotics, which at least enables to perform the simple numerical computations.

Let us again introduce the new variables instead of (14)

$$f = \frac{\alpha}{a} \ , \ s = at \ , \ \gamma = \frac{2b_1}{b_0^2 t} = \frac{2b_1}{b_0^2 N} \ . \tag{33}$$

Here in the last equality we use, that in all corrections to the trivial renormalon chain (3) the variable t may be replaced by the saddle–point value $t = N$ up to corrections of the order of $\sim [\ln N]^2/N$. Now the truncated formula for the effective charge (16) takes the form

$$f = \frac{1}{1 - s - \gamma s \ln f} \ , \tag{34}$$

where $\gamma \sim 1/N$. The N-th term of the expansion of f in the series over s just gives the asymptotics

$$f = 1 + \sum_1^\infty C_n(\gamma) s^n \ , \tag{35}$$
$$\{R\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N N! C_N(\beta/N) \ .$$

It is easy to show that $f(s)$ satisfies the differential equation

$$f' = \frac{df}{ds} = \frac{f(f-1)}{s} + \gamma s f f' \ , \tag{36}$$

which leads to the recursion relation for the coefficients of the expansion (35)

$$C_{k+1} = \frac{1}{k}\sum_{n=1}^{k} C_n C_{k+1-n} + \gamma C_k + \frac{\gamma}{2}\sum_{n=1}^{k-1} C_n C_{k-n} \ , \tag{37}$$

with the initial condition $C_1 = 1$. For analytical investigation this formula is even more complicated, than the original equations (15), (16). On the other hand, it is easy to solve (37) numerically.

We have performed the numerical simulations with $b_0 = 9,\ b_1 = 64$ and N up to ~ 16000. The result reads

$$R = 0.27 \sum \left(\frac{9}{2}\alpha_0\right)^N N^{\frac{128}{81}} N! \ \ . \tag{38}$$

Let us recall that this result is obtained for $\beta_2 = \beta_3 = .. = 0$ (see (14),(15)) or, in other words, we were interested in the solution of the equation (compare with (2))

$$\frac{d\alpha}{dx} = \frac{b_0\alpha^2}{1 - \frac{b_1}{b_0}\alpha} \ . \tag{39}$$

If one expands the r.h.s. of this equation in series in α, the terms proportional to $b_1^2, b_1^3,$ ... will mimic effectively the $\sim b_2, b_3,$ ... terms of the full equation (2). On the other hand, one may consider the pure truncated renormalization group equation:

$$\frac{d\alpha}{dx} = b_0\alpha^2 + b_1\alpha^3 \ . \tag{40}$$

In this case the formulas (34), (36) are modified

$$f = \frac{1}{1 - s - \gamma s \ln\left(\frac{f}{1+\gamma s f}\right)} \ , \quad f' = \frac{f(f-1)}{s} + \gamma s f^2 (f - 1 - \gamma s f) \ . \tag{41}$$

More precisely this formula for the effective charge corresponds to the approximate solution of (40), but the omitted terms lead only to the $\sim 1/N$ corrections to renormalon. Again it is useful to expand $f(s)$ in the series (35). The corresponding recursion relation takes the more complicated form

$$\begin{aligned} C_{k+1} &= \frac{1}{k}\sum_{n=1}^{k} C_n C_{k+1-n} \\ &+ \frac{\gamma}{k}\Big[C_k + 2\sum C_n C_{k-n} + \sum C_n C_m C_{k-n-m}\Big] \\ &- \frac{\gamma^2}{k}\Big[\delta_{k,1} + 3C_{k-1} + 3\sum C_n C_{k-n-1} + \sum C_n C_m C_{k-n-m-1}\Big] \ , \end{aligned} \tag{42}$$

where again $C_1 = 1$ and all C-s with nonpositive numbers are supposed to be zero. After numerical simulation of about 1000 terms of the series (42) we have found

$$R = 0.13 \sum \left(\frac{9}{2}\alpha_0\right)^N N^{\frac{128}{81}} N! \quad . \tag{43}$$

All the difference between the two definitions of the effective charge (39) and (40) lies in the different choice of higher order corrections to the renormalization group equation b_2, b_3 ... . Thus the two asymptotics (38), (43) show explicitly that these corrections make the contribution of the order of ~ 1 to the overall normalization of renormalon, in complete agreement with the conclusion of the preceding section.

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Figure captions

Fig 1. The renormalon-type graphs with exchange of one soft gluon. The internal gluon line will be dressed in the following figures.

Fig 2. The example of the simplest chain of diagrams corresponding to the renormalization of the soft gluon line. Each bubble generates the factor $b_0\alpha_0 \ln\left(Q^2/k^2\right)$.

Fig 3. The example of the chain of diagrams with one two – loop insertion in soft gluon line. This second order bubble generates the factor $b_1\alpha_0^2 \ln\left(Q^2/k^2\right)$.

Fig 4. The dressing of the internal gluon line of the second order bubble, shown in fig. 5, by n simple bubbles.

Fig 5. Three – loop insertion to soft gluon line. This diagram generate the factor $b_2\alpha_0^3 \ln\left(Q^2/k^2\right)$ and makes only $1/N$ correction to the renormalon.

Fig 6. The same as in fig. 5, but with dressing of two internal gluon lines by the simple chains of bubbles. The summation over n_1 and n_2 allows to compensate the extra α-s of the contribution of fig. 5, thus leading to the correction of the order of ~ 1 to the renormalon.

Fig 7. An example of the graph which corresponds to the second iteration of two – loop correction (see fig. 3). Here p is the number of (large) dressed bubbles which lie along the soft gluon line and n is the total number of internal (small) dressed bubbles.

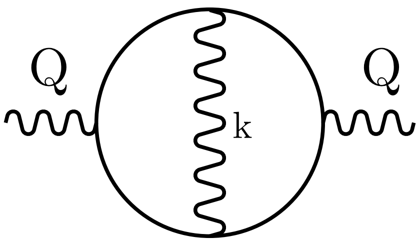

Fig.1

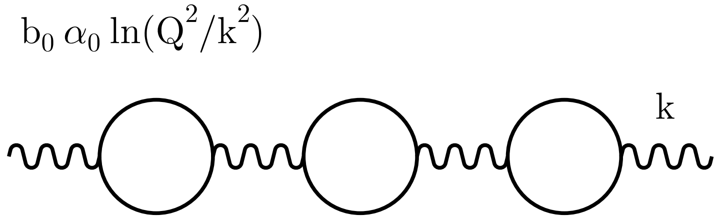

Fig.2

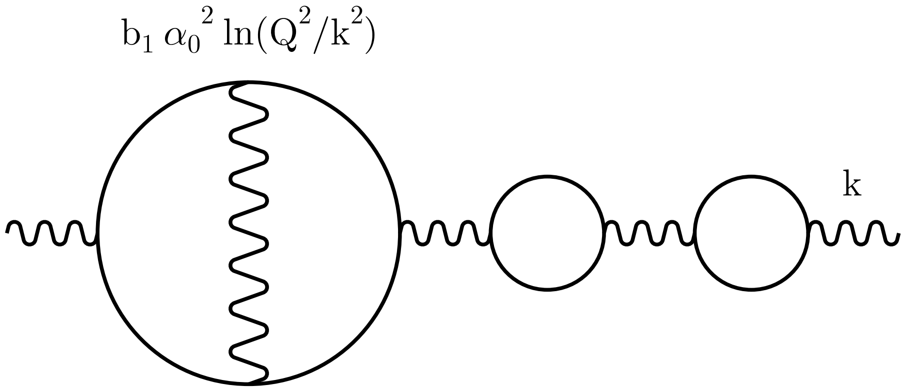

Fig.3

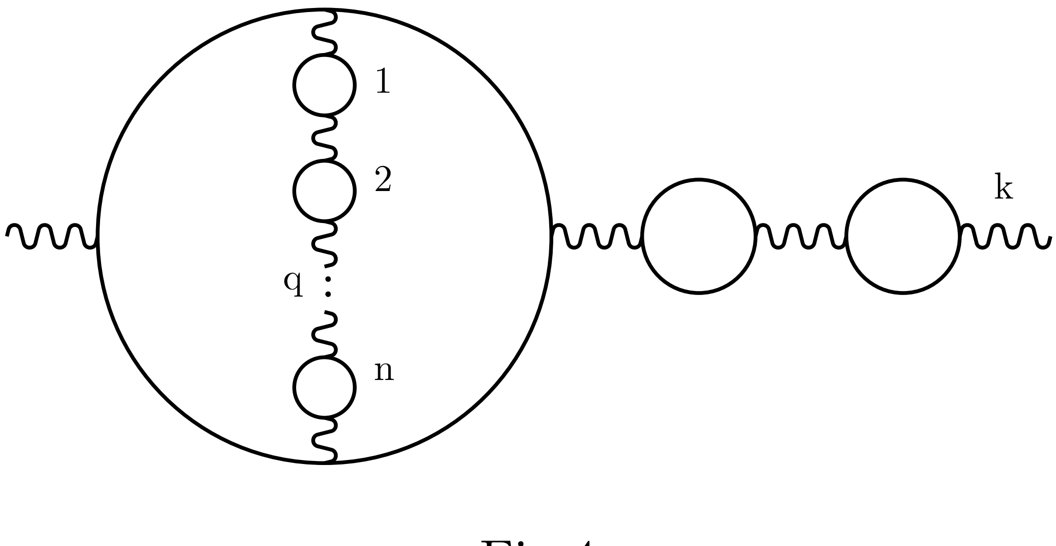

Fig.4

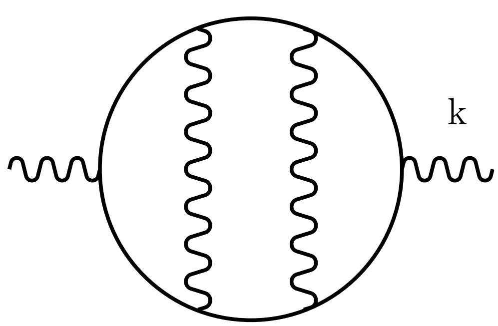

Fig.5

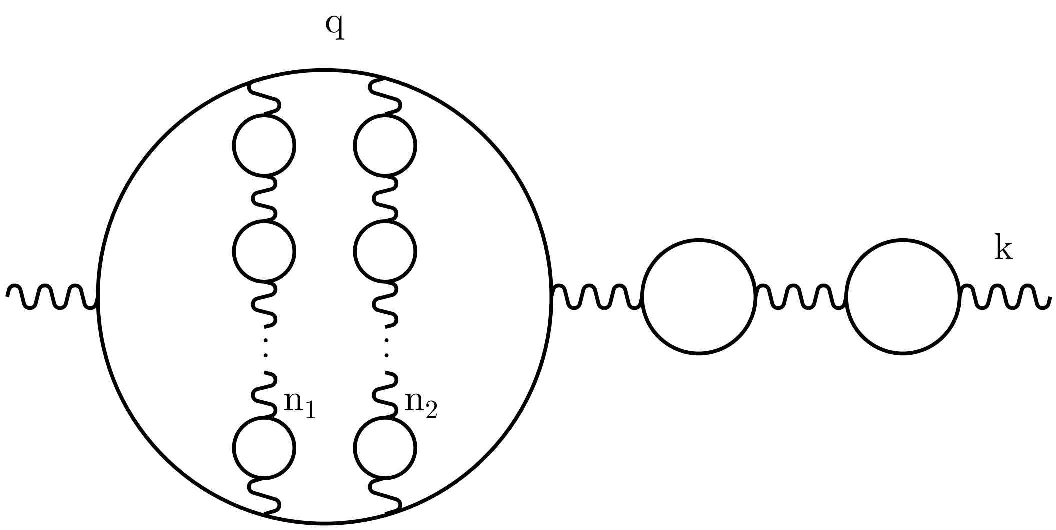

Fig.6

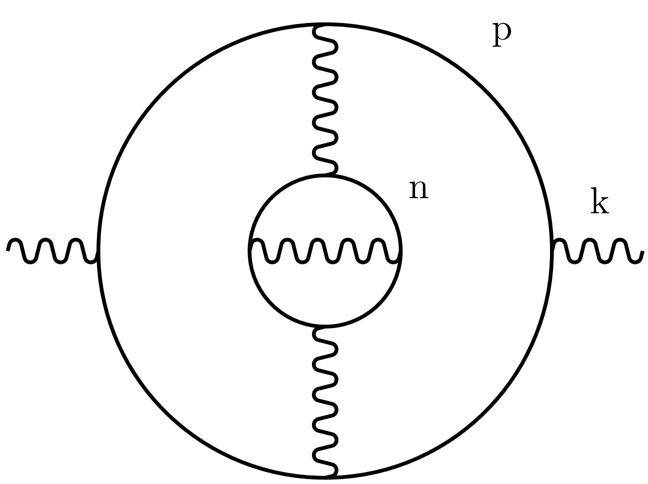

Fig.7