GLUON REGGEIZATION IN QCD IN THE NEXT TO LEADING ORDER *

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Abstract

Gluon Reggeization can significally simplify calculation of subleading terms for parton distributions at small x, where x is the longitudinal momentum fraction carried by a parton. We check the Reggeization in elastic scattering processes of QCD at large energy $\sqrt{s}$ and fixed momentum transfer $\sqrt{-t}$. For amplitudes of these processes with gluon quantum numbers and negative signature in the t channel s-channel discontinuities are calculated in the two-loop approximation. The two-loop correction to the gluon trajectory is expressed in terms of these discontinuities. Remarkable cancellations lead to the independence of the trajectory on the properties of the scattered particles, confirming the gluon Reggeization.

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1. INTRODUCTION

Quantum Cromodynamics (QCD) was accepted as a true quantum field theory of the strong interactions (for a review see [1]) in much extent due to its successes in discribing hard processes (for a review see [2]). An usual tool for a theoretical study of such processes is the perturbation theory improved by the renormalization group and combined with an operator product expansion. The applicability of the perturbative QCD is guaranteed here by the smallness of the running coupling constant $\alpha_s(Q^2)$, where Q is the hard scale (typical virtuality).

For semihard processes [3] the hard scale is small compared to the c.m.s. energy $\sqrt{s}$ of colliding particles, so that the ratio $x = Q^2/s$ becomes an important parameter. The convergence of the perturbative series is spoiled by powers of $\ln(1/x)$, consequently large logarithmic terms of the type $\alpha_s^n(ln(1/x))^m$, with $m \leq n$ (for the scattering channel which is considered in this paper) must be resummed to all orders in α_s.

Investigation of the small x behaviour of parton distributions is one of the most important problems of the perturbative QCD [4]. With the appearance of measurements in the small x region performed at HERA this problem has acquired a particular phenomenological interest [5]. Unfortunately, up to now our understanding of the small x phenomena is far from being complete. The problem of summing up the logarithmically enhanced terms is solved [6] in the leading logarithmic approximation (LLA) only, which means here summing up the terms with $m = n$. This approximation results in a power growth of cross sections with the energy. In terms of parton distributions this means a fast increase of the gluon density $g(x, Q^2)$ with decreasing x:

$$g(x, Q^2) \sim x^{-j_0} \tag{1}$$

where $j_0 = 1 + \omega_0$ is the LLA position of the singularity of the partial wave with the vacuum quantum numbers in the t channel [6],

$$\omega_0 = \frac{4\alpha_s}{\pi} N \ln 2 \ , \tag{2}$$

with $N = 3$ for QCD. Clearly, the behaviour (1) violates the Froissart bound so that it should be modified in the asymptotically small x region. However we will not discuss here the unitarization problem. In the region of parameters accessible in the modern experiments it appears that the observed behaviour of structure functions is consistent with LLA results (1) [7]. Nevertheless, to confirm definitively that this is the case, subleading corrections must be taken into consideration. It is all the more clear so far as the scale dependence of the running coupling constant α_s is beyond of the accuracy of LLA, that diminishes the predictive power of LLA, permitting to change strongly numerical results by changing a scale. Therefore, the problem of calculating radiative corrections to LLA becomes very important now.

For solving this problem the key point can be [8] the gluon Reggeization. It was proved [6, 9] in LLA that gauge bosons in the non-Abelian SU(N) gauge theories are

Reggeized with the trajectory

$$j(t) = 1 + \omega(t) \ , \tag{3}$$

where [6]

$$\omega(t) = \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{(D-1)}} \frac{N}{2} \int \frac{d^{D-2}k_\perp}{k_\perp^2 (q-k)_\perp^2} \ . \tag{4}$$

Here g is the gauge coupling constant ($\alpha_s = \frac{g^2}{4\pi}$), q is the momentum transfer, $t = q^2 \approx q_\perp^2$, and $D = 4 + \varepsilon$ is the space-time dimension, different from four for regularization of Feynman integrals. The integration in Eq.(4) is performed over the $(D-2)$-dimensional momenta orthogonal to the initial particle momentum plane.

The problem of calculating next-to-leading corrections can be reduced to calculating corrections to the kernel of the Bethe-Salpeter type equation for the t-channel partial wave with vacuum quantum numbers [6] (which is known now as BFKL equation). The kernel is expressed in terms of the gluon trajectory and the Reggeon-Reggeon-gluon vertex. Corrections to the vertex were calculated already [10, 11], so the calculation of the contribution $\omega^{(2)}(t)$ to the trajectory in the next (two-loop) approximation and verification of the gluon Reggeization at this step appear to be the most urgent problem now.

The paper is devoted to a solution of this problem. We use for this purpose the elastic scattering processes (quark-quark, gluon-gluon and quark-gluon) in the Regge region ($s \gg |t| \sim m^2$). The correction $\omega^{(2)}(t)$ to the trajectory can be determined through the s-channel discontinuity of the scattering amplitude of any of these processes calculated in the two-loop approximation. The coincidence of the obtained results serves as a check of the Reggeization.

2. METHOD OF CALCULATION

Let us consider the amplitude $\left(\mathcal{A}_8{}^{(-)}\right)_{AB}^{A'B'}$ of the scattering process $A + B \to A' + B'$ with gluon quantum numbers and negative signature in the t channel. Assuming that it is given by the Reggeized gluon contribution, we have

$$\left(\mathcal{A}_8{}^{(-)}\right)_{AB}^{A'B'} = \Gamma^c_{A'A} \frac{s}{t} \left[\left(\frac{s}{-t}\right)^{\omega(t)} + \left(\frac{-s}{-t}\right)^{\omega(t)} \right] \Gamma^c_{B'B} \ , \tag{5}$$

where $\Gamma^c_{A'A}$ are the particle-particle-Reggeon (PPR) vertices. In the helicity basis they can be presented [10, 12, 13] as

$$\Gamma^i_{A'A} = g \langle A'|T^i|A\rangle \left[\delta_{\lambda_{A'},\lambda_A} \left(1 + \Gamma^{(+)}_{AA}(t)\right) + \delta_{\lambda_A, -\lambda_{A'}} \Gamma^{(-)}_{AA}(t) \right] \ , \tag{6}$$

where $\Gamma^{(\pm)}_{AA}(t)$ are the radiative corrections to the helicity conserving LLA vertices [6]. To calculate the two-loop correction to the trajectory $\omega^{(2)}(t)$ we can consider only the part of the scattering amplitude (5) conserving helicities of each of the colliding particles. Let us write the two-loop contribution to the s- channel discontinuity of this part in the following form:

$$\left[\left(\mathcal{A}_8{}^{(-)}\right)_{AB}^{A'B'}(\text{two-loop})\right]_s^{(+)} = g^2\langle A'|T^i|A\rangle\langle B'|T^i|B\rangle\left(-\tfrac{2\pi i s}{t}\right)\Delta_{AB}\ . \tag{7}$$

Calculating this contribution from Eq.(5) with the help of Eq.(6) we find

$$\omega^{(2)}(t) = \Delta_{AB} - \left(\omega^{(1)}(t)\right)^2\ln\left(\frac{s}{-t}\right) - \left[\Gamma_{AA}^{(+)}(t) + \Gamma_{BB}^{(+)}(t)\right]\omega^{(1)}(t)\ . \tag{8}$$

Since one-loop corrections to the PPR vertices were calculated for the quark [13] as well as the gluon [10, 12] case, we can obtain $\omega^{(2)}(t)$ calculating the discontinuity (7) for any of the elementary scattering processes in QCD. Of course, the trajectory cannot depend on scattering particles; therefore, comparing the results of the calculation we can verify the gluon Reggeization beyond LLA.

Before calculating the discontinuity Δ_{AB}, let us explain unambiguosly the meaning of δ-symbols entering Eq.(6), because these symbols can have a literal sense only in the physical case $D = 4$ and only for a suitable choice of relatives phases in spin wave functions. For the quark-quark-Reggeon (QQR) vertex we adopt here the convention of Ref. [13] (see Eqs.(27) and (37) therein), so that the vertex $\Gamma_{QQ}^{(+)}(t)$ that will be used in the following, is given by the sum of contributions presented in Eqs.(39), (63), (64) and (68) in Ref. [13]. For the gluon-gluon-Reggeon vertex we use the definition of Ref. [17] (see Eq.(9) therein). Then the vertices $\Gamma_{GG}^{(\pm)}(t)$ are given by Eqs. (10) - (14) of this Ref.

3. TWO-LOOP s-CHANNEL DISCONTINUITIES

In the two-loop approximation the discontinuity Δ_{AB} is given by the sum of the two- and three-particle intermediate states in the s-channel unitarity condition:

$$\Delta_{AB} = \Delta_{AB}^{(2)} + \Delta_{AB}^{(3)}\ . \tag{9}$$

In the case of the two-particle intermediate state only the helicity conserving part of elastic scattering amplitudes with the octet colour state and negative signature in the corresponding t channels contribute to the unitary conditions. The reason is that the Born amplitudes are real and in the Regge region they contain only the octet colour state with negative signature in the t channel, as well as the real parts of the one loop amplitudes. Consequently, one can use the representation (5) for them. Because of this circumstance, the two-particle contribution can be written in the general form

$$\Delta_{AB}^{(2)} = \frac{g^2Nt}{(2\pi)^{D-1}}\int\frac{d^{(D-2)}q_1}{q_1^2(q_1-q)^2}\left[\omega^{(1)}(q_1^2)\ln\left(\frac{s}{-q_1^2}\right) + \Gamma_{AA}^{(+)}(q_1^2) + \Gamma_{BB}^{(+)}(q_1^2)\right]\ . \tag{10}$$

Here and below all vectors are $D-2$ dimensional and transversal to the (p_A, p_B) plane.

Let us now consider the three-particle contribution. The results of the straightforward calculations for the quark-quark [14] and gluon-gluon [15] scattering can be written in the form

$$\Delta_{AB}^{(3)} = \frac{g^2 t}{2(2\pi)^{D-1}} \int \frac{d^{(D-2)} q_1}{q_1^2 (q_1 - q)^2} \left[f_A(q_1, q) + f_B(q_1, q) - 2 f_A(q_1, q_1) - 2 f_B(q_1, q_1) \right] . \tag{11}$$

Here the functions f_A and f_B are defined by the properties of the particles A and B correspondingly; they coincide in the case of the gluon-gluon scattering whereas one of them can be obtained from the other one simply by the substitution $m_A \leftrightarrow m_B$ in the case of the quark-quark scattering, m_A and m_B being the corresponding quark masses. Eq.(11) is valid because the three-particle discontinuity can be presented as the sum of the contributions of the fragmentation regions of the particles A and B respectively, and the contribution of the fragmentation region of one particle does not depend on the properties of the other one. This leads us to the conclusion that Eq.(11) holds for the quark-gluon scattering as well, therefore it holds for all the elementary scattering processes of QCD.

The gluon can fragment or in two gluons $(2g)$ or in a quark-antiquark pair $(q\bar{q})$. Accordingly, we have

$$f_G(q_1, q) = f_G^{(2g)}(q_1, q) + f_G^{(q\bar{q})}(q) . \tag{12}$$

For the $(2g)$ contribution we find

$$f_G^{(2g)}(q_1, q) = \frac{g^2 N^2 q^2}{2(2\pi)^{D-1}} \int \frac{d^{(D-2)} q_2}{q_2^2 (q_2 - q)^2} \left[\frac{1}{2} \ln \left(\frac{s}{-(q_1 - q_2)^2} \right) \right.$$

$$\left. + \psi(1) - \psi(D-2) + \frac{1}{(D-3)(D-4)} + \frac{1}{(D-1)(D-2)} \right] , \tag{13}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function $\Gamma(x)$. In turn for the $(q\bar{q})$ contribution we get

$$f_G^{(q\bar{q})}(q) = \frac{g^2 N}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(3 - \frac{D}{2}\right)}{2 - D} \sum_f \left\{ \int_0^1 \int_0^1 \frac{dx_1 dx_2 \theta(1 - x_1 - x_2)}{[m_f^2 - q^2 x_1 x_2]^{3 - \frac{D}{2}}} \right.$$

$$\left. \times \left[\frac{2-D}{2} (x_1 + x_2) q^2 + 2(1 - x_1 - x_2) \left((x_1 + x_2)^2 q^2 - 2 m_f^2 \right) \right] + \frac{2}{3} m_f^{D-4} \right\} . \tag{14}$$

We notice that this contribution does not depend on q_1.

In the quark case we can have in the fragmentation region only the same quark and a radiated gluon. Nevertheless, it is convenient to split the function $f_Q(q_1, q)$ into two terms according to their colour coefficients:

$$f_Q(q_1,q)=f_Q^{(\mathrm{a})}(q)+f_Q^{(\mathrm{na})}(q_1,q)\ . \tag{15}$$

The first term has an *abelian* nature, and only this part survives in the abelian case. It does not depend on q_1, as well as $f_G^{(q\bar{q})}$, and reads

$$f_Q^{(\mathrm{a})}(q)=\frac{g^2\Gamma\left(2-\frac{D}{2}\right)}{2(4\pi)^{\frac{D}{2}}(D-3)}\int_0^1 dx\left\{\frac{\left((D-3)(D-4)+4\right)q^2-8m_Q^2}{4\left[m_Q^2-q^2x(1-x)\right]^{3-\frac{D}{2}}}+2m_Q^{D-4}\right\}\ . \tag{16}$$

The second term in Eq.(15) is essentially *non abelian* and is given by

$$f_Q^{(\mathrm{na})}(q_1,q)=\frac{g^2N^2}{4}\int\frac{d^{(D-2)}q_2}{(2\pi)^{D-1}}\int_{\beta_0}^1 d\beta\frac{(1-\beta)^2}{\beta}$$

$$\times\left\{\frac{q^2\left[1+(1-\beta)^2+\frac{D-4}{2}\beta^2\right]-4m_Q^2\frac{\beta^2}{1-\beta}}{(m_Q^2\beta^2-q_2^2)[m_Q^2\beta^2-(q_2-(1-\beta)q)^2]}+\frac{4m_Q^2\beta^2}{(1-\beta)(m_Q^2\beta^2-q_2^2)^2}\right\}\ , \tag{17}$$

where

$$\beta_0=\sqrt{-\frac{(q_1-q_2)^2}{s}}\ . \tag{18}$$

4. TWO-LOOP CONTRIBUTION TO THE GLUON TRAJECTORY

Using the representation (4) for $\omega^{(1)}(t)$ and Eqs.(10) and (11) respectively for the two- and three-particle contribution to the discontinuity $\Delta_{AB}(t)$ (9), from Eq.(8) we arrive at

$$\omega^{(2)}(t)=\frac{g^2t}{2(2\pi)^{D-1}}\int\frac{d^{(D-2)}q_1}{q_1^2(q_1-q)^2}\left[F_A(q_1,q)+F_B(q_1,q)-2F_A(q_1,q_1)-2F_B(q_1,q_1)\right]\ , \tag{19}$$

where

$$F_A(q_1,q)=f_A(q_1,q)-\frac{N}{2}\omega^{(1)}(q^2)\ln\left(\frac{s}{-q^2}\right)-N\Gamma_{AA}^{(+)}(q^2)\ . \tag{20}$$

For the gluon case all the terms in the RHS of Eq.(20) are shown in Eqs.(4), (12)-(14) and in the Eqs.(10) - (13) of Ref. [17]. For the quark case we have the same expression (4) for $\omega^{(1)}(t)$, $f_Q(q_1,q_2)$ is given by Eqs.(15) - (18) and, finally, $\Gamma_{QQ}^{(+)}(t)$ is given by Eqs.(39),(63), (64) and (68) of Ref. [13].

As a result of remarkable cancellations we find that $F_A(q_1,q)$ does not depend on the properties of the scattered particles. Therefore, Eq.(19) can be rewritten as

$$\omega^{(2)}(t)=\frac{g^2t}{(2\pi)^{D-1}}\int\frac{d^{(D-2)}q_1}{q_1^2(q_1-q)^2}\left[F(q_1,q)-2F(q_1,q_1)\right]\ , \tag{21}$$

with

$$F(q_1, q) = \frac{g^2}{4}\left\{\frac{N^2 q^2}{(2\pi)^{D-1}}\int\frac{d^{(D-2)}q_2}{q_2^2(q_2-q)^2}\right.$$

$$\times\left[\ln\left(\frac{q^2}{(q_1-q_2)^2}\right) - 2\psi(D-3) - \psi\left(3-\frac{D}{2}\right) + 2\psi\left(\frac{D}{2}-2\right) + \psi(1)\right.$$

$$\left. + \frac{2}{(D-3)(D-4)} + \frac{D-2}{4(D-1)(D-3)}\right]$$

$$\left. + \frac{8N\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}}\sum_f\int_0^1 dx\frac{x(1-x)}{[m_f^2 - q^2x(1-x)]^{2-\frac{D}{2}}}\right\}. \quad (22)$$

Eqs.(21) and (22) give us a closed expression for the two- loop correction to the gluon trajectory (for the case of massless quarks it was presented in Ref. [16]). Its independence on the properties of the scattered particles, which appears as a result of remarkable cancellations among the various terms in Eq.(20), sets up a stringent test of the gluon Reggeization beyond the leading logarithmic approximation.

The two-loop correction to the trajectory contains both ultravilolet and infrared divergencies. The former ones can be easily removed by the charge renormalization in the total expression for the trajectory. As for the latter ones, we expect they cancel in the total expression for the corrected kernel to the BFKL equation. We hope to demonstrate it in subsequent papers.

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The Status of Renormalon

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Abstract

It is shown that the series of renormalon–type graphs, which consist in the chain of insertions to one soft(hard) gluon(photon) line is in fact ill defined. Each new type of insertions, which appears in the higher orders of perturbation theory, generates the correction to renormalon of the order of ~ 1. However, this series of corrections to the asymptotics although have no small parameter but is not the asymptotic one.

1. The renewed interest in the renormalon asymptotics[1, 2] of perturbative serie have been demonstrated in last few years [3-11]. It results even in attempts [12] t use renormalon for calculation of experimentally measurable quantities.

However, the accurate determination of renormalon-type asymptotics appears t be not so simple problem. It was recognized[6-8,10,15] that the overall normalizatio of the renormalon asymptotics could not be found without taking into account o all terms of the expansion of, say, the Gell-Mann–Low function. However, the usua proof of this fact do not refer on the direct counting of the Feynman graphs.

The generally considered renormalon chain of graphs is formed by dressing o one gluon(photon). In the present paper we would like to estimate the role of th arbitrary high order insertions to the dressed gluon line. It will be shown both b diagrammatic consideration and by direct analytical calculation that each new typ of insertions generates the correction to renormalon of the order of ~ 1. Howeve the k-th correction to the asymptotics for large k is not expected to have any k enhancement. Thus at least the series of corrections to the amplitude of renormalo asymptotics is not the asymptotic series.

It is a tradition now to consider the renormalon for QED. In this paper we wil also discuss only the QED–type diagrams of the perturbation theory, without th self-interaction of gluons. The heuristic way for extending this result for QCD ma be given by the so called 'naive nonabelization'.

Our approach is equally valid for both infrared and ultraviolet renormalons. However, in order not to interfere with the recent results[10, 15] for ultraviolet (UV renormalon, consider the infrared (IR) one. The contribution of the diagrams wit exchange of one soft gluon(photon) to some "physical" quantity has the generic form

$$R = \int_{k \ll Q} \alpha(k) \frac{k^2 dk^2}{Q^4} \quad . \tag{1}$$

The Feynman graphs corresponding to this value are shown in fig. 1. In (1) we have written down the effective running coupling constant $\alpha(k) = \alpha_{eff}(k)$, which is trivially connected with the transverse part of the gluon propagator. The function $\alpha(k)$ satisfies the RG equation:

$$\frac{d\alpha}{dx} = b_0\alpha^2 + b_1\alpha^3 + b_2\alpha^4 + ... , \quad x = \ln\left(Q^2/k^2\right) \quad . \tag{2}$$

It is to be noted here that we have fixed the renormalization scheme by considering the effective charge. Thus our coefficients $b_2, b_3, ...$ are neither the free parameters, nor the known, say, for $\overline{MS}$ scheme, $b_2(\overline{MS})$, $b_3(\overline{MS})$. At first stage one may

Fig.1

Figure 1: The renormalon graphs with exchange of one soft gluon. The internal gluon line will be dressed in the following figures.

neglect b_1, b_2, ... in (2)

$$R = \frac{2}{\alpha_0}\int_0^\infty \alpha(x)e^{-2x}dx = \int_0^\infty \frac{e^{-2x}}{1-b_0\alpha_0 x}2dx = \sum_{N=0}\left(\frac{b_0\alpha_0}{2}\right)^N N! \quad . \tag{3}$$

Here the first equality is also the explicit definition of our renormalon. We will consider only the asymptotics of the perturbation theory and will not concern the issue of the nonperturbatve ambiguity of the integral (3) due to the Landau pole. The integral (1) describes adequately the contribution of a certain chain of Feynman diagrams only for $k \ll Q$. It is seen from (3) that the main contribution to the N-th order of the expansion comes from $k^2 \sim Q^2e^{-N/2}$. Thus the renormalon contributions to the first few terms of perturbation theory are completely irrelevant. On the other hand, for sufficiently large Q a lot of terms of the expansion (3) come from the region $\Lambda^2_{QCD} \ll k^2 \ll Q^2$ where the effective charge is small and the perturbative approach for calculation of α_{eff} (2) seems to be useful.

2. Before passing to straightforward but rather formal manipulations with the RG equation (2) let us illustrate the role of complicated contributions to renormalon by the explicit estimate of Feynman graphs. The fig. 2 shows the chain of diagrams corresponding to (3). We show only the QED–type diagrams without gluon self-interaction. Each of the N bubbles from fig. 2 generates the factor $b_0\alpha_0 \ln\left(\frac{Q^2}{k^2}\right)$ in

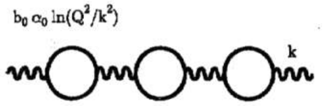

Fig.2

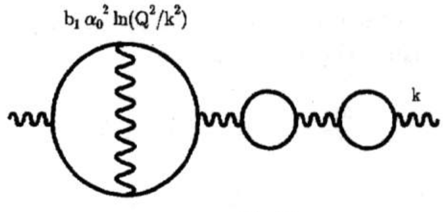

Fig.3

Figure 2: The simplest chain of diagrams corresponding to the renormalization of the soft gluon line. Each bubble generates the factor $b_0\alpha_0 \ln(Q^2/k^2)$.

Figure 3: The example of diagram with two – loop insertion into soft gluon line.

the integrand of (1),(3). The difference between QCD and QED may be thought to be hidden in the factor b_0, accompanying the single bubble.

Now let us replace two of the simple bubbles by the more complicated diagram of fig. 3. The two loop bubbles generate the factor $b_1\alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right)$ in the integrand which has one power of large logarithm less (or one α_0 more) than the leading order contribution (3). However, a large combinatorical factor N appears due to a number of permutations of the second order bubble among the simple bubbles, leading to

$$N b_1 \alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right)\left[b_0\alpha_0 \ln\left(\frac{Q^2}{k^2}\right)\right]^{N-2} \to \left(\frac{b_0\alpha_0}{2}\right)^N N! \frac{2b_1}{b_0^2} \,. \qquad (4)$$

Thus we see that taking into account one second order insertion into the soft gluon line leads to the correction of the order of one to the trivial asymptotics (3).

Consider now the more complicated diagram of fig. 4 with dressing of the internal gluon line of the second order bubble. To this end it is natural to write down explicitly the last integration over internal momentum of the two loop diagram

$$b_1\alpha_0^2 \int_{k^2}^{Q^2} \left[b_0\alpha_0 \ln\left(\frac{Q^2}{q^2}\right)\right]^n \frac{dq^2}{q^2} = \frac{1}{n+1} b_1\alpha_0^2 \ln\left(\frac{Q^2}{k^2}\right)\left[b_0\alpha_0 \ln\left(\frac{Q^2}{k^2}\right)\right]^n \,. \qquad (5)$$

Thus up to the overall factor $\frac{1}{n+1}$ the contribution of diagram of fig. 4 coincides with that of fig. 3. Summation over n naturally leads to $\ln(N)$. Taking into account a number of large bubbles of fig. 4 allows to exponentiate the correction

$$\left(\frac{b_0\alpha_0}{2}\right)^N N! \exp\left(\frac{2b_1}{b_0^2}\ln(N)\right) = \left(\frac{b_0\alpha_0}{2}\right)^N N^{\frac{2b_1}{b_0^2}} N! \,. \qquad (6)$$

This is the generally recognized expression for the IR renormalon. Our argumentation up to this stage repeats the line of reasoning of the paper [5]. However, the argument

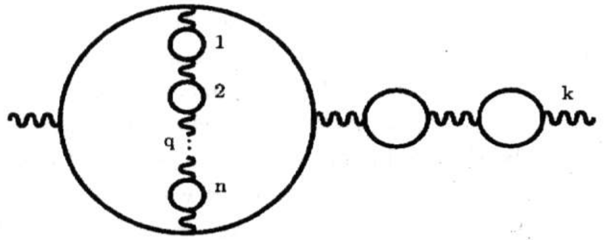

Fig.4

Figure 4: The dressing of internal gluon line of the second order bubble by n simple bubbles.

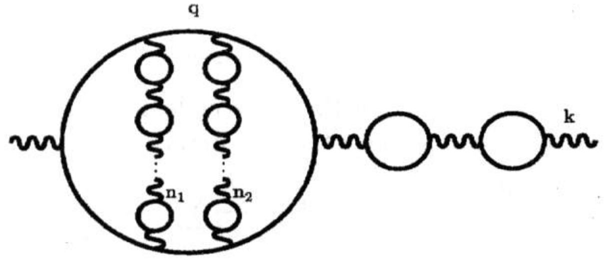

Fig.5

Figure 5: Three loop insertion with dressing of two internal gluon lines by the simple chains of bubbles. The summation over n_1 and n_2 allows to compensate all extra α-s.

of the exponent in (6) was found with the $\sim 1/\ln(N)$ accuracy and therefore the nontrivial overall factor as well as the function of N, weaker than N^{γ}, may appear in (6), as we consider in the following section.

Now let us consider the three loop correction (fig. 5 with $n_1 = n_2 = 0$). This contribution generates the factor $b_2\alpha_0^3 \ln\left(\frac{Q^2}{k^2}\right)$ in the integrand of (3). Thus here we have two extra α_0 which at first glance could not be compensated by one combinatorical N and hence the diagram of fig. 5 seems to generate only the $\sim 1/N$ correction to renormalon. In particular such conclusion was drown by Zakharov [5]. However, let us see, what happens if one dresses the internal gluon lines of the three loop diagram. Now summation over the number of trivial insertions n_1, n_2 gives:

$$b_2\alpha_0 \ln\left(\frac{Q^2}{k^2}\right) \sum_{n_1,n_2} \alpha_0^2 \frac{1}{n_1+n_2+1}(N-n_1-n_2-2) \sim b_2\alpha_0 \ln\left(\frac{Q^2}{k^2}\right) \times (\alpha_0 N)^2 \,. \quad (7)$$

Here the factor $(n_1+n_2+1)^{-1}$ appears after integration over the internal momentum of the large bubble, while $(N - n_1 - n_2 - 2)$ accounts for the combinatorics. We see that after dressing of all gluon lines the three loop ($\sim b_2$) diagram generates the correction to renormalon of the order of ~ 1. One can easily show that four loop ($\sim b_3$), five loop ($\sim b_4$) etc. diagrams generate the corrections of the same order of magnitude. Previously the analogous proof of the importance of the high loop corrections was done by Mueller[13] but this result was not published.

3. Although the corrections to renormalon generated by the high order contributions to the RG equation $b_2\alpha^4, b_3\alpha^5 ...$ (2) are not small, the correction induced by the second term $b_1\alpha^3$ still plays an outstanding role due to the additional enhancement by $\ln(N)$ (6). Moreover, the fractal iterations of the second order diagram of fig. 3 may be shown to have the same $\ln(N)$ enhancement. Therefore at the first stage let us omit $b_2, b_3, b_4 \dots$ in (2) and consider the truncated effective charge:

$$\frac{\alpha}{\alpha_0} = \frac{1}{1 - at - \beta a \ln(\alpha/\alpha_0)} \ , \tag{8}$$

where $\beta = 2b_1/b_0^2$, $a = b_0\alpha_0/2$. This is the transcendental equation for α which may be solved iteratively. It is easy to make the first iteration

$$\{R_1\}_N = \int \frac{e^{-t}dt}{1 - at + \beta a \ln(1 - at)} = \int \frac{e^{-t}dt}{1 - at} \sum_p \left[\frac{\beta a}{1 - at} \ln \frac{1}{1 - at}\right]^p = \tag{9}$$

$$= a^N N! \sum_p \frac{1}{p!} \left[\beta \ln\left(\frac{N}{p}\right)\right]^p = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta \ln N}\right]^\beta N! \left(1 + O\Big(\frac{1}{\ln N}\Big)\right) .$$

The asymptotics (9) should be compared with (6). One can see that the consistent treatment of the diagrams of fig. 4, which in fact we have done in (9), results in the nontrivial small factor $(\ln N)^{-\beta}$ as compared to the naive result (6).

Consider now the iteration of two loop correction which in terms of α means

$$\left(\frac{\alpha_0}{\alpha}\right)_{second\ iteration} = 1 - at + \beta a \ln(1 - at + \beta a \ln(1 - at)) . \tag{10}$$

The direct calculation with this α gives [14]

$$\{R_2\}_N \approx \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta \ln(\ln N)}\right]^\beta N! \left(1 + O\Big(\frac{1}{\ln(\ln N)}\Big)\right) . \tag{11}$$

Thus we see that making the second iteration (10) in the transcendental equation (8), leads again to parametrically large modification of the renormalon. Moreover, all third, fourth, etc. iterations in (8) are also of the 100% importance.

Finally, for k iterations in (8) the asymptotic formula takes the form [14]

$$\{R_k\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{k-1}} dx_k \frac{\beta - x_1}{\Gamma(1 + \beta - x_1)} \tag{12}$$

$$\times \frac{(x_1 - x_2)^{\beta - x_1}}{\Gamma(1 + x_1 - x_2)} \frac{(x_2 - x_3)^{x_1 - x_2}}{\Gamma(1 + x_2 - x_3)} \dots \frac{(x_{k-1} - x_k)^{x_{k-2} - x_{k-1}}}{\Gamma(1 + x_{k-1} - x_k)} \frac{x_k^{x_{k-1} - x_k}}{x_k} e^{-x_k \ln(\ln N)} .$$

Looking at this result, one may even doubt, whether it has any finite limit for large k? However, the direct integration over x_k [14] shows that this limit do exists.

We see that if one performs only one (a few) iterations while solving the transcendental equation for the truncated effective charge (8), the asymptotics of the perturbation theory differs drastically (9,11) from the generally recognized renormalon (6). Nevertheless, after taking into account the infinite number of iterations (12) the usual asymptotics is restored.

4. The formal solution of the RG equation (2) may be found in the form

$$\alpha = \alpha_1\Big(1 + \beta_2\alpha_1^2 + \beta_3\alpha_1^3 + \dots\Big)\,, \tag{13}$$

where α_1 is the solution of the eq. (8). Coefficients β_k may be expressed through b_n (2) with $n \le k$. Moreover for large k $b_k \sim n!$ and $\beta_k = b_k/(kb_0)(1 + O(1/k))$.

The analog of the equation (12) now has the form

$$\{R_{k+1}\}_N = \left(\frac{b_0\alpha_0}{2}\right)^N \left[\frac{N}{\beta}\right]^\beta N! \int_0^\beta dx_1 \tag{14}$$
$$\left(1 + \frac{\beta_2 2^2}{2!b_0^3}(\beta - x_1)^2 + \frac{\beta_3 2^3}{3!b_0^4}(\beta - x_1)^3 + \dots\right)\int_0^{x_1} dx_2 \dots\,,$$

where the part of the formula after integration over dx_2 simply repeats the corresponding part of the equation (12). Keeping in mind the finiteness of both (12) and (14) one can see that all $\beta_2, \beta_3, \dots$ make contributions of the order of ~ 1 to the renormalon.

On the other hand, all the high order ($\sim \beta_n$) terms in (14) have appeared in the combination $\beta_n/n!$. Thus at least the series of corrections to the renormalon is not the asymptotic one[1].

To summarize, we have shown that any complication of the renormalon chain of diagrams leads to $\sim 100\%$ correction to the asymptotics of the perturbation theory. This result is equally relevant for both IR and UV renormalon. These large corrections in terms of integration over the internal momentum k (fig. 1) still correspond to the domain where the effective running coupling is small ($\alpha_{eff}(k) \ll 1$). By considering the Feynman diagrams we have shown how the loss of large log-s for more complicated insertions to the renormalon chain is compensated by the pure combinatorical growth of the number of diagrams. In principle this approach may be useful for finding the most important Feynman diagrams in practical multiloop calculations.

The following promising problem will be the asymptotic estimate of the corrections to the renormalon asymptotics. First of all, such calculation may be performed for the nonleading IR renormalon. As we know the IR renormalon contribution to the asymptotics of perturbation theory has the form $R_N \sim (b_0/2)^N N!$. The coefficients β_n in (13,14) are determined by the first UV renormalon $\beta_n \sim (-b_0)^n n!$. This means

[1] Analogous observation for the UV has been done in [10]

that, the multiloop corrections to IR renormalon will grow up like $(-2)^n$ due to 'interference' between the IR and UV renormalons. Physically this huge interference of two renormalons shows that the consistent treatment of the nonleading IR renormalon is not possible without the explicit (Borel) summation of the UV renormalon.

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