Recent progress on HQET lagrangian

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HQET lagrangian up to $1/m^3$ terms is discussed. Consequences of reparameterization invariance are considered. Results for the chromomagnetic interaction coefficient at two loops, and in all orders in the large–β_1 approximation, are presented.

1 HQET lagrangian

QCD problems with a single heavy quark staying approximately at rest can be conveniently treated in the heavy quark effective field theory (HQET) (see [1] for review and references). We shift the energy zero level: $E = m + \omega$, and consider the region where residual energies ω and momenta $\vec{p}$ are not large: $\omega \sim |\vec{p}| \sim \Lambda \ll m$. The effective field theory is constructed to reproduce QCD on–shell scattering amplitudes expanded to some order $(\Lambda/m)^n$. This is achieved by writing down the most general effective Lagrangian consistent with the required symmetries, and tuning the coefficients to reproduce QCD on-shell amplitudes. Terms with $D_0 Q$ can be eliminated by field redefinitions.
The most general lagrangian up to $1/m^3$ is [2]–[6]

$$
\begin{aligned}
&L = Q^+ iD_0 Q \\
&+\frac{C_k}{2m} Q^+ \vec{D}^2 Q + \frac{C_m}{2m} Q^+ \vec{B}\cdot\vec{\sigma} Q + \frac{iC_s}{8m^2} Q^+ (\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\cdot\vec{\sigma} Q + \frac{C_d}{8m^2} Q^+ [\vec{D}\cdot\vec{E}] Q \\
&+\frac{C_{k2}}{8m^3} Q^+ \vec{D}^4 Q + \frac{C_{w1}}{8m^3} Q^+ \{\vec{D}^2, \vec{B}\cdot\vec{\sigma}\} Q - \frac{C_{w2}}{4m^3} Q^+ D^i \vec{B}\cdot\vec{\sigma} D^i Q \\
&\quad+\frac{C_{p'p}}{8m^3} Q^+ (\vec{D}\vec{B}\cdot\vec{D} + \vec{D}\cdot\vec{B}\vec{D})\cdot\vec{\sigma} Q + \frac{iC_M}{8m^3} Q^+ (\vec{D}\cdot[\vec{D}\times\vec{B}] + [\vec{D}\times\vec{B}]\cdot\vec{D}) Q \\
&\quad+\frac{C_{a1}}{8m^3} Q^+ (\vec{B}^2 - \vec{E}^2) Q - \frac{C_{a2}}{16m^3} Q^+ \vec{E}^2 Q + \frac{C_{a3}}{8m^3} Q^+ \operatorname{Tr}(\vec{B}^2 - \vec{E}^2) Q - \frac{C_{a4}}{16m^3} Q^+ \operatorname{Tr}\vec{E}^2 Q \\
&\quad+\frac{iC_{b1}}{8m^3} Q^+ (\vec{B}\times\vec{B} - \vec{E}\times\vec{E})\cdot\vec{\sigma} Q - \frac{iC_{b2}}{8m^3} Q^+ (\vec{E}\times\vec{E})\cdot\vec{\sigma} Q + \cdots
\end{aligned}
\tag{1}
$$

where Q is 2–component heavy–quark field. Here heavy–light contact interactions are omitted, as well as operators involving only light fields.
HQET can be rewritten in relativistic notations. Momenta of all states are decomposed as $p = mv + k$ where residual momenta $k \sim \Lambda$. The heavy–quark field is now Dirac spinor obeying $\not{v} Q_v = Q_v$. The lagrangian is

$$
\begin{aligned}
L_v &= \overline{Q}_v iv\cdot D Q_v - \frac{C_k}{2m} \overline{Q}_v D_\perp^2 Q_v - \frac{C_m}{4m} \overline{Q}_v G_{\mu\nu}\sigma^{\mu\nu} Q_v \\
&+\frac{iC_s}{8m^2} \overline{Q}_v \{D_\perp^\mu, G^{\lambda\nu}\} v_\lambda \sigma_{\mu\nu} Q_v - \frac{C_d}{8m^2} \overline{Q}_v v^\mu [D_\perp^\nu G_{\mu\nu}] Q_v + \cdots
\end{aligned}
\tag{2}
$$

where $D_\perp = D - v(vD)$. The velocity v may be changed by an amount $\delta v \lesssim \Lambda/m$ without spoiling the applicability of HQET and changing its predictions. This reparameterization invariance relates coefficients of varying degrees in $1/m$ [7]–[13].
At the tree level, there are easier ways to find the coefficients C_i than QCD/HQET matching: Foldy–Wouthuysen transformation [14, 15], or using equations of motion [5]

(or integrating out lower components [16, 17]) followed by a field redefinition. The result is

$$C_k = C_m = C_d = C_s = C_{k2} = C_{w1} = C_{a1} = C_{b1} = 1\,,\qquad C_{w2} = C_{p'p} = C_M = C_{a2} = C_{a3} = C_{a4} = C_{b2} = 0\,. \tag{3}$$

However, these algebraic methods don't generalize to higher loops.
At $1/m$ level, the kinetic coefficient $C_k = 1$ due to the reparameterization invariance [7]. One–loop matching for the chromomagnetic coefficient C_m was done in [3]; two–loop anomalous dimension of the chromomagnetic operator in HQET was obtained in [18, 19], and two–loop matching was done in [19]; in [20], all orders of perturbation theory for C_m were summed at large β_1.
At $1/m^2$ level, the spin–orbit coefficient $C_s = 2C_m - 1$ due to the reparameterization invariance [21]–[24]. The Darwin term reduces to a contact interaction. One–loop matching for the heavy–light contact interactions was done in [24]. The one–loop anomalous dimension matrix of dimension 6 terms in the HQET lagrangian was obtained in [15], [22]–[25].
At $1/m^3$ level, one–loop matching was done in [6] for the terms involving the heavy–quark fields twice and the gluon field once. The one–loop renormalization of dimension 7 terms in the HQET lagrangian was recently considered [26].

2 Matching quark–quark vertex

Renormalized QCD on–shell quark–quark proper vertex

$$-\,\overline{u}(\not{p} - m)u \tag{4}$$

gets no correction in the on–shell renormalization scheme. QCD spinors are related to HQET spinors by the Foldy–Wouthuysen transformation

$$u = \left(1 + \frac{\not{k}}{2m} + \frac{k^2}{4m^2} + \cdots\right) u_v\,, \quad \not{v} u_v = u_v\,. \tag{5}$$

Expressing QCD proper vertex via HQET spinors, we obtain

$$\overline{u}_v \frac{\vec{k}^2}{2m} u_v + \cdots \tag{6}$$

Let's denote the sum of bare 1–particle–irreducible self–energy diagrams of the heavy quark in HQET at $1/m^0$ as $-i\frac{1+\not{v}}{2}\Sigma(\omega)$, $\omega = kv$. At the $1/m$ level, self–energy diagrams with a single chromomagnetic vertex vanish. Let the sum of bare diagrams with a single kinetic vertex be $-i\frac{C_k}{2m}\frac{1+\not{v}}{2}\Sigma_k(\omega, k_\perp^2)$. Consider variation of Σ at $v \to v + \delta v$ for an infinitesimal δv ($v\,\delta v = 0$). All factors $\frac{1+\not{v}}{2}$ can be combined into a single one, and the variation $\delta\not{v}$ in it provides the variation of the γ–matrix structure in front of Σ. There are two sources of the variation of Σ. Terms from the expansion of denominators of the propagators produce insertions $ik\delta v$. Terms from the vertices produce $igt^a\delta v^\mu$. Now consider variation of Σ_k at $k_\perp \to k_\perp + \delta k_\perp$ for an infinitesimal $\delta k_\perp$. Quark–quark kinetic vertices produce $i\frac{C_k}{m}k\delta k_\perp$; quark–quark–gluon kinetic vertices produce $i\frac{C_k}{m}gt^a\delta k_\perp^\mu$; two–gluon vertices produce nothing. Therefore,

$$\frac{\partial\Sigma_k}{\partial k_\perp^\mu} = 2\frac{\partial\Sigma}{\partial v^\mu}\,. \tag{7}$$

This is the Ward identity of the reparameterization invariance first derived in [10]. Taking into account $\frac{\partial\Sigma_k}{\partial k_\perp^\mu} = 2\frac{\partial\Sigma_k}{\partial k_\perp^2}k_\perp^\mu$ and $\frac{\partial\Sigma}{\partial v^\mu} = \frac{d\Sigma}{d\omega}k_\perp^\mu$, we obtain

$$\frac{\partial\Sigma_k}{\partial k_\perp^2} = \frac{d\Sigma}{d\omega}\,. \tag{8}$$

The right–hand side does not depend on $k_\perp^2$, and hence

$$\Sigma_k(\omega,k_\perp^2) = \frac{d\Sigma(\omega)}{d\omega}k_\perp^2 + \Sigma_{k0}(\omega)\,. \tag{9}$$

This result can also be understood in a more direct way. Only diagrams with a quark–quark kinetic vertex contain $k_\perp^2$; its coefficient is is $i\frac{C_k}{2m}$. The sum of diagrams with a unit insertion is $-i\frac{d\Sigma}{d\omega}$. Note that diagrams with a quark–quark–gluon kinetic vertex vanish because there is no preferred transverse direction.
On the mass shell ($\omega=0$), the renormalized HQET quark–quark proper vertex is $\frac{C_k}{2m}Z_Q\overline{u}_v\,[-k_\perp^2+\Sigma_k(0,k_\perp^2)]u_v = -\frac{C_k}{2m}Z_Q\left[1-\frac{d\Sigma}{d\omega}\right]_{\omega=0}k_\perp^2\overline{u}_v u_v$. On the mass shell, only diagrams with finite–mass particles in loops contribute (e.g., c–quark loops in b–quark HQET) (Fig. 1). Taking into account $Z_Q^{-1} = 1-\left.\frac{d\Sigma}{d\omega}\right|_{\omega=0}$ and comparing with (6), we finally obtain

$$C_k(\mu) = 1\,. \tag{10}$$

This argument works for an arbitrary μ; hence, the anomalous dimension of the kinetic–energy operator in HQET vanishes exactly. In a similar way, it is not difficult to prove that

$$C_{k2} = 1\,. \tag{11}$$

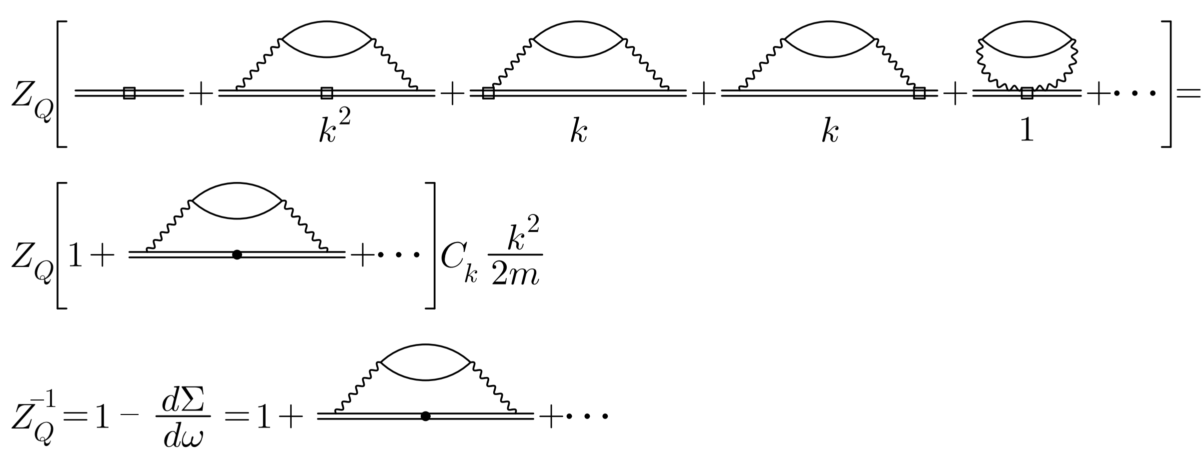


Figure 1: HQET quark–quark proper vertex on the mass shell

3 Matching quark–quark–gluon vertex

QCD on–shell proper vertex is characterized by 2 form factors:

$$\overline{u}(p')t^a \left(\varepsilon(q^2)\frac{(p+p')^\mu}{2m} + \mu(q^2)\frac{[\not{q},\gamma^\mu]}{4m} \right) u(p)\,, \tag{12}$$
$$\varepsilon(q^2) = 1 + \varepsilon' \frac{q^2}{m^2} + \cdots, \quad \mu(q^2) = \mu + \mu' \frac{q^2}{m^2} + \cdots$$

The total colour charge of a quark $\varepsilon(0) = 1$ due to the gauge invariance. Ward identities in the background field formalism [27] are shown in Fig. 2, where the large dot means convolution with the gluon incoming momentum q and colour polarization e^a, the second equalities are valid only for an infinitesimal q (or in the case of an abelian external field), and $(t^a)^{bc} = i f^{acb}$ in the adjoint representation. Therefore, the QCD proper vertex $\Lambda^a_\mu(p,q) = \Lambda_\mu t^a$ obeys $\Lambda^a_\mu q^\mu e^a = -\Sigma(p + q e^a t^a) + \Sigma(p)$ for infinitesimal q, or $\Lambda_\mu(p,0) = -\frac{\partial\Sigma(p)}{\partial p^\mu}$. The form factor is projected out by $\varepsilon(0) = Z_Q[1 + \frac{1}{4}\operatorname{Tr}\Lambda_\mu v^\mu(1+\not{v})]$. On the mass shell, $\frac{1}{4}\operatorname{Tr}\frac{\partial\Sigma}{\partial p^\mu} = (1 - Z_Q^{-1})v_\mu$, and hence $\varepsilon(0) = 1$.

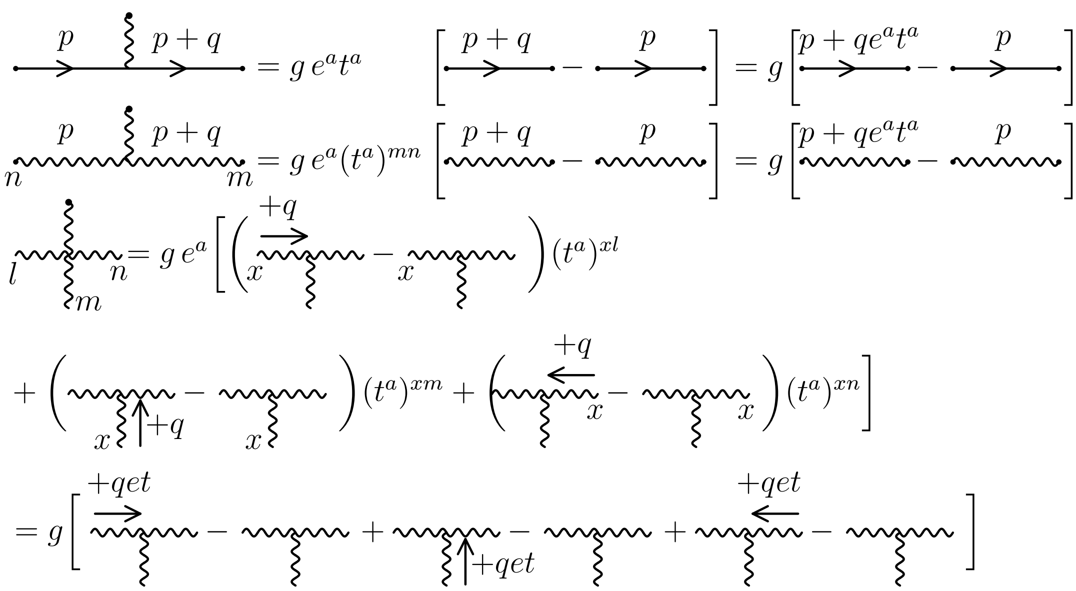


Figure 2: Ward identities in the background field formalism

Let's denote the sum of bare vertex diagrams in HQET at $1/m^0$ as $igt^a v^\mu \frac{1+\not{v}}{2}[1 + \Lambda(\omega,\Delta)]$, where $\Delta = qv = \omega' - \omega$. The Ward identity for the static quark propagator is the same as for the ordinary one (Fig. 2). Therefore, $\Delta e^a t^a \Lambda(\omega,\Delta) = -\Sigma(\omega + \Delta e^a t^a) + \Sigma(\omega)$ for infinitesimal Δ, or

$$\Lambda(\omega,0) = -\frac{d\Sigma(\omega)}{d\omega}\,. \tag{13}$$

It is interesting, that for an abelian external field $\Lambda(\omega,\Delta) = -\frac{\Sigma(\omega+\Delta)-\Sigma(\omega)}{\Delta}$ exactly. The total colour charge of a static quark $Z_Q[1 + \Lambda(0,0)] = 1$, as expected.
The $1/m$ HQET bare proper vertex has the form

$$i\frac{C_k}{2m} g t^a \frac{1+\not{v}}{2}\left[(1+\Lambda_k)(p+p')^\mu_\perp + (\Lambda_{k0} + \Lambda_{k1}p_\perp^2 + \Lambda'_{k1}p'^2_\perp + \Lambda_{k2}q_\perp^2)v^\mu\right]$$

$$+\, i\frac{C_m}{4m} g t^a \frac{1+\not{v}}{2}[\gamma^\mu,\not{q}]\frac{1+\not{v}}{2}(1+\Lambda_m)\,, \tag{14}$$

where all Λ_i depend on ω, Δ; $\Lambda'_{k1}(\omega,\Delta)=\Lambda_{k1}(\omega+\Delta,-\Delta)$; $\Lambda_k(\omega,\Delta)=\Lambda_k(\omega+\Delta,-\Delta)$, and similarly for Λ_{k0}, Λ_{k2}. Similarly to the previous Section, we can see that variation of the leading vertex function at $v\to v+\delta v$ coincides with that of the kinetic–energy vertex function at $p_\perp \to p_\perp+\delta p_\perp$, if $\delta v=\frac{C_k}{m}\delta p_\perp$. This requires

$$\Lambda_k(\omega,\Delta)=\Lambda(\omega,\Delta)\,,\quad \Lambda'_{k1}(\omega,\Delta)=\frac{\partial\Lambda(\omega,\Delta)}{\partial\Delta} \tag{15}$$

(and hence $\Lambda_{k1}(\omega,\Delta)=\left(\frac{\partial}{\partial\omega}-\frac{\partial}{\partial\Delta}\right)\Lambda(\omega,\Delta)$). The Ward identities of Fig. 2 result in

$$\Lambda_{k0}(\omega,0)=-\frac{d\Sigma_{k0}(\omega)}{d\omega}\,,\quad \Lambda_{k2}(\omega,0)=0 \tag{16}$$

(in an abelian external field, $\Lambda_{k0}(\omega,\Delta)=-\frac{\Sigma_{k0}(\omega+\Delta)-\Sigma_{k0}(\omega)}{\Delta}$, $\Lambda_{k2}(\omega,\Delta)=0$).
Reparameterization invariance relates the spin–orbit vertex function to the chromomagnetic one, but we shall not discuss details here.
The on–shell HQET vertex at the tree level is

$$\overline{u}_v(k')\left(v^\mu+C_k\frac{(k+k')^\mu}{2m}+C_m\frac{[\not{q},\gamma^\mu]}{4m}+C_d\frac{q^2}{8m^2}v^\mu+C_s\frac{[\not{k},\not{q}]}{8m^2}v^\mu+\cdots\right)u_v(k)\,. \tag{17}$$

As we have demonstrated above, there are no corrections to the first two terms. Other terms have corrections starting from two loops, if there is a finite–mass flavour (such as c in b–quark HQET). Expressing the on–shell QCD vertex via HQET spinors, we obtain

$$\begin{aligned}\overline{u}_v(k')\Bigg[\varepsilon(q^2)\left(v^\mu+\frac{(k+k')^\mu}{2m}-\frac{q^2+[\not{k},\not{q}]}{8m^2}v^\mu+\cdots\right)\\ +\mu(q^2)\left(\frac{[\not{q},\gamma^\mu]}{4m}+\frac{q^2+[\not{k},\not{q}]}{4m^2}v^\mu+\cdots\right)\Bigg]u_v(k)\,.\end{aligned} \tag{18}$$

Therefore, the coefficients in the HQET lagrangian are

$$C_k=1\,,\quad C_m=\mu\,,\quad C_d=8\varepsilon'+2\mu-1\,,\quad C_s=2\mu-1\,. \tag{19}$$

The first one has no corrections (10). The coefficients (19) are not independent:

$$C_s=2C_m-1\,. \tag{20}$$

Probably, reparameterization–invariance Ward identities yield relations among corrections from finite–mass loops in HQET which ensure the absence of corrections to (20). However, we shall not trace details here.
Similarly, at the $1/m^3$ level, the coefficients in the HQET lagrangian are

$$C_{w1}=4\mu'+\tfrac{1}{2}\mu+\tfrac{1}{2}\,,\quad C_{w2}=4\mu'+\tfrac{1}{2}\mu-\tfrac{1}{2}\,,\quad C_{p'p}=\mu-1\,,\quad C_M=-4\varepsilon'-\tfrac{1}{2}\mu+\tfrac{1}{2}\,. \tag{21}$$

They are not independent:

$$C_{w2}=C_{w1}-1\,,\quad C_{p'p}=C_m-1\,,\quad C_M=\tfrac{1}{2}\left(C_m-C_d\right)\,. \tag{22}$$

Calculation of C_a, C_b requires matching amplitudes with two gluons. Calculation of contact terms requires matching amplitudes with light quarks.

4 Chromomagnetic interaction at two loops

As we know, the kinetic coefficient $C_k(\mu) = 1$, and the only coefficient in the HQET lagrangian up to $1/m$ level which is not known exactly is the chromomagnetic coefficient $V_m(\mu)$. It is natural to find it from QCD/HQET matching at $\mu \sim m$ where no large logarithms appear. Renormalization group can be used to obtain C_m at $\mu \ll m$:

$$C_m(\mu) = C_m(m) \exp\left(- \int\limits_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{\gamma_m(\alpha)}{2\beta(\alpha)} \frac{d\alpha}{\alpha} \right) , \tag{23}$$

where $C_m(m) = 1 + C_1 \frac{\alpha_s(m)}{4\pi} + C_2 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$, $\gamma_m = \frac{d \log Z_m}{d \log \mu} = \gamma_1 \frac{\alpha_s}{4\pi} + \gamma_2 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$ is the anomalous dimension of the chromomagnetic operator in HQET, and the β–function is $\beta = -\frac{1}{2}\frac{d\log\alpha_s}{d\log\mu} = \beta_1 \frac{\alpha_s}{4\pi} + \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$ (where $\beta_1 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$). If $L = \log m/\mu$ is not very large, it is better to retain all two–loop terms and neglect higher loops:

$$C_m(\mu) = 1 + (C_1 - \gamma_1 L) \frac{\alpha_s(m)}{4\pi} + \left[C_2 - (C_1\gamma_1 + \gamma_2) L + \gamma_1 (\gamma_1 - \beta_1) L^2 \right] \left(\frac{\alpha_s}{4\pi}\right)^2 . \tag{24}$$

This approximation holds up to relatively large L because C_2 is numerically large. If L is parametrically large, then it is better to sum leading and subleading logarithms:

$$C_m(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(m)}\right)^{-\frac{\gamma_1}{2\beta_1}} \left[1 + C_1 \frac{\alpha_s(m)}{4\pi} - \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{2\beta_1^2} \frac{\alpha_s(\mu) - \alpha_s(m)}{4\pi} \right] . \tag{25}$$

In this case, we cannot utilize C_2 without knowing γ_3. In general, the solution of (23) can be written as

$$C_m(\mu) = \hat{C}_m K(\mu) , \quad \hat{C}_m = \alpha_s(m)^{\frac{\gamma_1}{2\beta_1}} (1 + \delta c) , \quad \delta c = c_1 \frac{\alpha_s(m)}{4\pi} + c_2 \left(\frac{\alpha_s(m)}{4\pi}\right)^2 + \cdots \tag{26}$$

where $\hat{C}_m$ is scale– and scheme–independent.
As a simple application, we consider B–B^* mass splitting [28, 29][1]

$$m_{B^*} - m_B = \frac{2C_m(\mu)}{3m} \mu_m^2(\mu) + \frac{1}{3m^2} \left[C_m(\mu)\rho_{km}^3(\mu) + C_m^2(\mu)\rho_{mm}^3(\mu) - C_s(\mu)\rho_s^3(\mu) \right] , \tag{27}$$

where $\mu_m^2(\mu)$ and $\rho_s^3(\mu)$ are local matrix elements of chromomagnetic interaction and spin–orbit one, while $\rho_{km}^3(\mu)$ and $\rho_{mm}^3(\mu)$ are kinetic–chromomagnetic and chromomagnetic–chromomagnetic bilocal matrix elements (in the later case, there are two γ–matrix structures, 1 and $\sigma_{\mu\nu}$; the coefficient of the second one is implied here). Introducing renormalization group invariants

$$\begin{aligned} &\hat{\mu}_m^2 = K(\mu)\mu_m^2(\mu) , \quad \hat{\rho}_{km}^3 = K(\mu)\rho_{km}^3(\mu) + [1 - K(\mu)]\, \rho_s^3(\mu) , \\ &\hat{\rho}_{mm}^3 = K^2(\mu)\rho_{mm}^3 , \quad \hat{\rho}_s^3 = \rho_s^3(\mu) , \end{aligned} \tag{28}$$

we can rewrite it as

$$m_{B^*} - m_B = \frac{2\hat{C}_m}{3m} \hat{\mu}_m^2 + \frac{1}{3m^2} \left[\hat{C}_m \left(\hat{\rho}_{km}^3 - 2\hat{\rho}_s^3 \right) + \hat{C}_m^2 \hat{\rho}_{mm}^3 + \hat{\rho}_s^3 \right] . \tag{29}$$

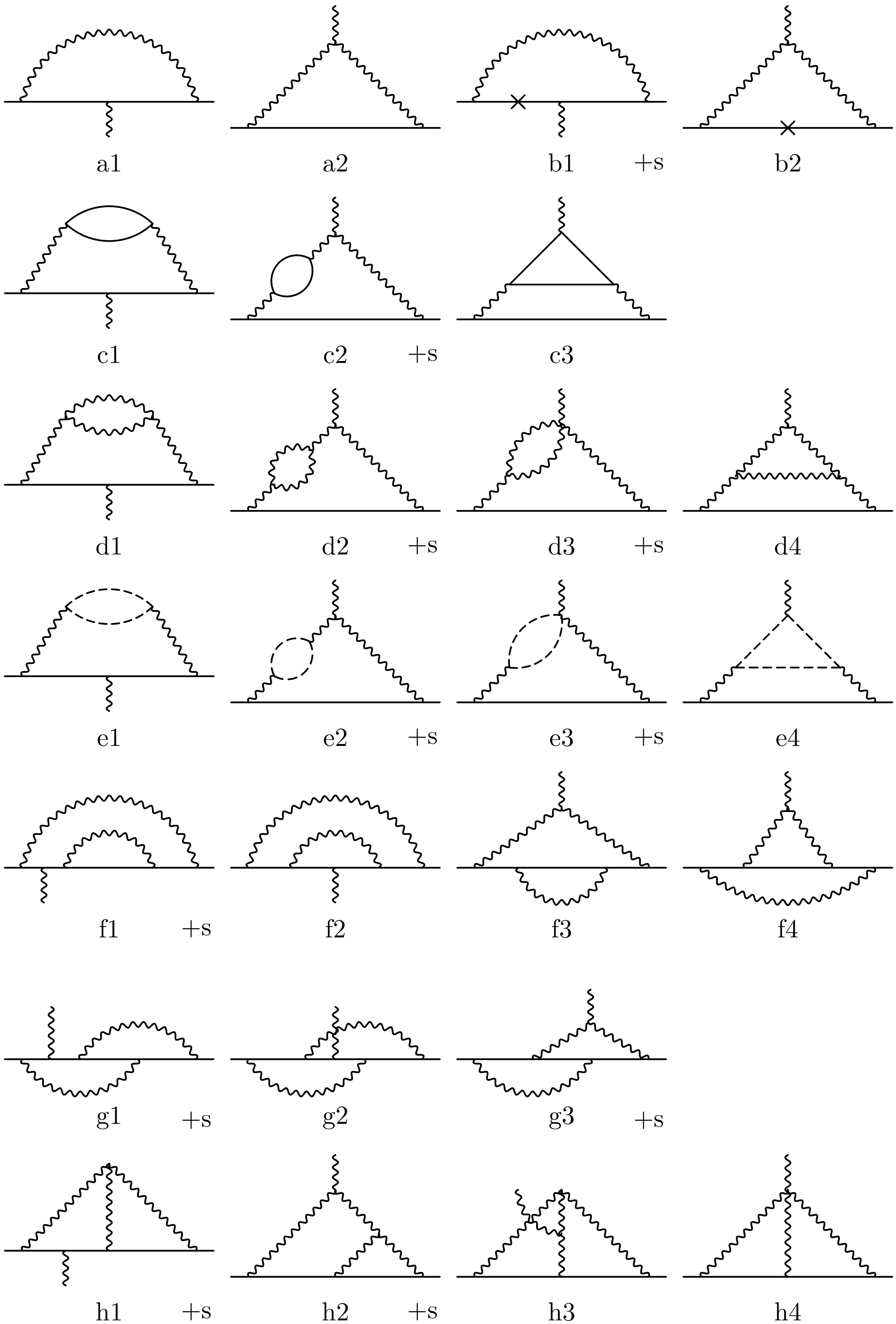
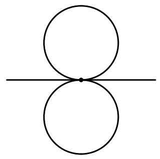
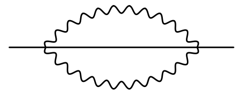
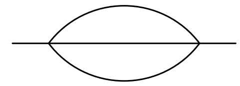


Figure 3: Diagrams for the QCD proper vertex

In order to obtain C_m, we should calculate the heavy–quark chromomagnetic moment μ (Fig. 3). All on–shell massive integrals can be reduced to 3 basis ones

$I_0^2 =$  , $I_1 =$  , $I_2 =$  (30)

using integration by parts [30]–[32]. I_0^2 and I_1 are expressed via Γ–functions of d; I_2 is expressed via I_0^2, I_1, and one difficult convergent integral [32]

$$I = \pi^2 \log 2 - \frac{3}{2}\zeta(3) + O(\varepsilon)\,. \tag{31}$$

The result has the structure

$$\begin{aligned}\mu &= 1 + \frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}(C_F, C_A) \times I_0 \\ &\quad + \frac{g_0^4 m^{-4\varepsilon}}{(4\pi)^d}(C_F^2, C_F C_A, C_A^2, C_F T_F n_l, C_A T_F n_l, C_F T_F, C_A T_F) \times (I_0^2, I_1, I_2)\,.\end{aligned} \tag{32}$$

Now we express it via $\alpha_s(\mu)$ and expand in ε. The coefficient of $1/\varepsilon$ gives the anomalous dimension

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \left(17 C_A - 13 T_F n_f\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \tag{33}$$

The chromomagnetic interaction coefficient at $\mu = m$ is

$$\begin{aligned}C_m(m) &= 1 + 2(C_F + C_A)\frac{\alpha_s(m)}{4\pi} \\ &+ \Bigg[C_F^2 \left(-8I + \frac{20}{3}\pi^2 - 31\right) + C_F C_A \left(\frac{4}{3}I + \frac{4}{3}\pi^2 + \frac{269}{9}\right) + C_A^2 \left(\frac{4}{3}I - \frac{17}{9}\pi^2 + \frac{805}{27}\right) \\ &\quad + C_F T_F n_l \left(-\frac{100}{9}\right) + C_A T_F n_l \left(-\frac{4}{9}\pi^2 - \frac{299}{27}\right) \\ &\quad + C_F T_F \left(-\frac{16}{3}\pi^2 + \frac{476}{9}\right) + C_A T_F \left(\pi^2 - \frac{298}{27}\right) \Bigg] \left(\frac{\alpha_s}{4\pi}\right)^2 \\ &= 1 + \frac{13}{6}\frac{\alpha_s(m)}{\pi} + (21.79 - 1.91 n_l)\left(\frac{\alpha_s}{\pi}\right)^2 .\end{aligned} \tag{34}$$

The coefficient of $(\alpha_s/\pi)^2$ is about 11 for $n_l = 4$ light flavours. It is 40% less than the expectation based on naive nonabelianization [33]. The contribution of the heavy quark loop to this coefficient is merely -0.1.

5 Chromomagnetic interaction at higher loops

Perturbation series for C_m can be rewritten via β_1 instead of n_f:

$$C_m(\mu) = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_{Ln} \beta_1^n \alpha_s^L = 1 + \frac{1}{\beta_1} f(\beta_1 \alpha_s) + O\left(\frac{1}{\beta_1^2}\right) . \tag{35}$$

[1] in [28], ρ_{mm}^3 is missing; in [29], the leading logarithmic running of $C_m(\mu)$ has a wrong sign.

There is no sensible limit of QCD in which β_1 may be considered a large parameter (except, may be, $n_f \to -\infty$). However, retaining only the leading β_1 terms often gives a good approximation to exact multi–loop results [33]. This limit is believed to provide information about summability of perturbation series [34]. At the first order in $1/\beta_1$, multiplicative renormalization amounts to subtraction of $1/\varepsilon^n$ terms;

$$\frac{\beta_1 g_0^2}{(4\pi)^2} = \bar{\mu}^{2\varepsilon} \frac{\beta}{1+\beta/\varepsilon}\,, \quad \beta = \frac{\beta_1 \alpha_s}{4\pi} = \frac{1}{2\log\mu/\Lambda_{\overline{\rm MS}}}\,. \tag{36}$$

The perturbation series (35) can be rewritten as

$$C_m(\mu) = 1 + \frac{1}{\beta_1} \sum_{L=1}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left(\frac{\beta}{\varepsilon+\beta}\right)^L - (\text{subtractions}) + O\left(\frac{1}{\beta_1^2}\right)\,. \tag{37}$$

Knowledge of the function $F(\varepsilon,u)$ allows one to obtain the anomalous dimension

$$\gamma_m = \frac{2\beta}{\beta_1} F(-\beta, 0) + O\left(\frac{1}{\beta_1^2}\right) \tag{38}$$

and the finite term

$$C_m(\mu) = 1 + \frac{1}{\beta_1} \int\limits_{-\beta}^{0} d\varepsilon \frac{F(\varepsilon,0)-F(0,0)}{\varepsilon} + \frac{1}{\beta_1} \int\limits_{0}^{\infty} du\, e^{-u/\beta} \frac{F(0,u)-F(0,0)}{u} + O\left(\frac{1}{\beta_1^2}\right) \tag{39}$$

(this method was used in [33]; see references in this paper). Renormalization group invariant (26) is

$$\delta c = \frac{1}{\beta_1} \int_0^\infty du\, e^{-\frac{4\pi}{\beta_1\alpha_s}u} S(u) + O\left(\frac{1}{\beta_1^2}\right)\,, \quad S(u) = e^{-\frac{5}{3}u}\, \left.\frac{F(0,u)-F(0,0)}{u}\right|_{\mu=m} \tag{40}$$

(here α_s is taken at $\mu = m$ in the V–scheme, $\exp(-\frac{4\pi}{\beta_1\alpha_s}u) = (\frac{\Lambda_V}{m})^{-2u}$).

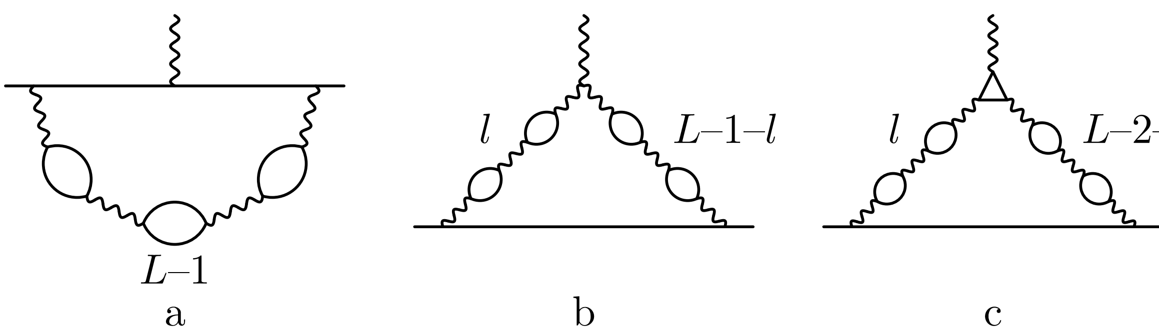


Figure 4: L–loop diagrams with the maximum number of quark loops.

The function $F(\varepsilon,u)$ is determined by the coefficient of the highest degree of n_f in the L–loop term, which is given by the diagrams in Fig. 4. Calculating them, we obtain

$$\begin{aligned}
&F(\varepsilon,u) = \left(\frac{\mu}{m}\right)^{2u} e^{\gamma\varepsilon} \frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u-\varepsilon)} D(\varepsilon)^{u/\varepsilon-1} N(\varepsilon,u)\\
&D(\varepsilon) = 6e^{\gamma\varepsilon}\Gamma(1+\varepsilon)B(2-\varepsilon,2-\varepsilon) = 1 + \tfrac{5}{3}\varepsilon + \cdots\\
&N(\varepsilon,u) = C_F 4u(1+u-2\varepsilon u) + C_A \frac{2-u-\varepsilon}{2(1-\varepsilon)}(2+3u-5\varepsilon-6\varepsilon u+2\varepsilon^2+4\varepsilon^2 u)\,.
\end{aligned} \tag{41}$$

This gives the anomalous dimension

$$\gamma_m = C_A \frac{\alpha_s}{2\pi} \frac{\beta(1+2\beta)\Gamma(5+2\beta)}{24(1+\beta)\Gamma^3(2+\beta)\Gamma(1-\beta)} \\ = C_A \frac{\alpha_s}{2\pi} \left[1 + \frac{13}{6}\frac{\beta_1\alpha_s}{4\pi} - \frac{1}{2}\left(\frac{\beta_1\alpha_s}{4\pi}\right)^2 + \cdots \right] . \tag{42}$$

This perturbation series is convergent with the radius $\beta_1|\alpha_s| < 4\pi$. The Borel image of δc

$$S(u) = \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \left[4u(1+u)C_F + \tfrac{1}{2}(2-u)(2+3u)C_A \right] - e^{-\frac{5}{3}u}\frac{C_A}{u} \tag{43}$$

has infrared renormalon poles at $u = \frac{n}{2}$. They produce ambiguities in the sum of the perturbation series for δc, which are of order of the residues $\sim (\Lambda_V/m)^n$. The leading ambiguity ($u=\frac{1}{2}$) is

$$\Delta\hat{C}_m = \left(1 + \frac{7}{8}\frac{C_A}{C_F} \right) \frac{\Delta m}{m} , \tag{44}$$

where Δm is the ambiguity of the heavy–quark pole mass [35, 36].
Physical quantities, such as the mass splitting (27), are factorized into short–distance coefficients and long–distance hadronic matrix elements. In regularization schemes without a hard momentum cut–off, such as $\overline{\mathrm{MS}}$, Wilson coefficients also contain large–distance contributions which produce infrared renormalon ambiguities. Likewise, hadronic matrix elements contain small–distance contributions which produce ultraviolet renormalon ambiguities. In other words, the separation into short– and long–distance contributions is ambiguous; only when they are combined to form a physical quantity, an unambiguous result is obtained. Cancellations between infrared and ultraviolet renormalon ambiguities in HQET were traced in [37].
Ultraviolet renormalon ambiguities in matrix elements ρ_i^3 don't depend on external states, and may be calculated at the level of quarks and gluons (Fig. 5). Note that there is an ultraviolet renormalon ambiguity in the wave function renormalization $\Delta Z_Q = \frac{3}{2}\frac{\Delta m}{m}$ (Fig. 5d). The result is

$$\Delta\rho_{km}^3 = -\frac{2}{3}\frac{C_A}{C_F}\mu_m^2\Delta m , \quad \Delta\rho_{mm}^3 = -\frac{19}{12}\frac{C_A}{C_F}\mu_m^2\Delta m , \quad \Delta\rho_s^3 = -\frac{1}{2}\frac{C_A}{C_F}\mu_m^2\Delta m . \tag{45}$$

The sum of ultraviolet ambiguities of the $1/m^2$ contributions to (27) cancels the infrared ambiguity of the leading term.
The requirement of cancellation of renormalon ambiguities in the mass splitting (28) for all m allows us to establish the structure of the leading infrared renormalon singularity in $S(u)$ at $u=\frac{1}{2}$ beyond the large β_1 limit. The ultraviolet ambiguity of the square bracket in (28) should be equal to $\hat{\mu}_m^2$ times

$$\Lambda_V = m\, e^{-\frac{2\pi}{\beta_1\alpha_s}} \alpha_s^{-\frac{\beta_2}{2\beta_1^2}} [1 + O(\alpha_s)] . \tag{46}$$

In order to reproduce the correct fractional powers of α_s, $S(u)$ in (40) should have the branch point at $u=\frac{1}{2}$ instead of a pole:

$$S(u) = \frac{1}{\left(\frac{1}{2}-u\right)^{1+\beta_2/2\beta_1^2}} \left[2C_F K_1 - \frac{1}{3}C_A K_2 + \frac{19}{12}\frac{C_A K_3}{\left(\frac{1}{2}-u\right)^{-\gamma_1/2\beta_1}} + \frac{1}{2}\frac{C_A K_4}{\left(\frac{1}{2}-u\right)^{\gamma_1/2\beta_1}} \right] , \tag{47}$$

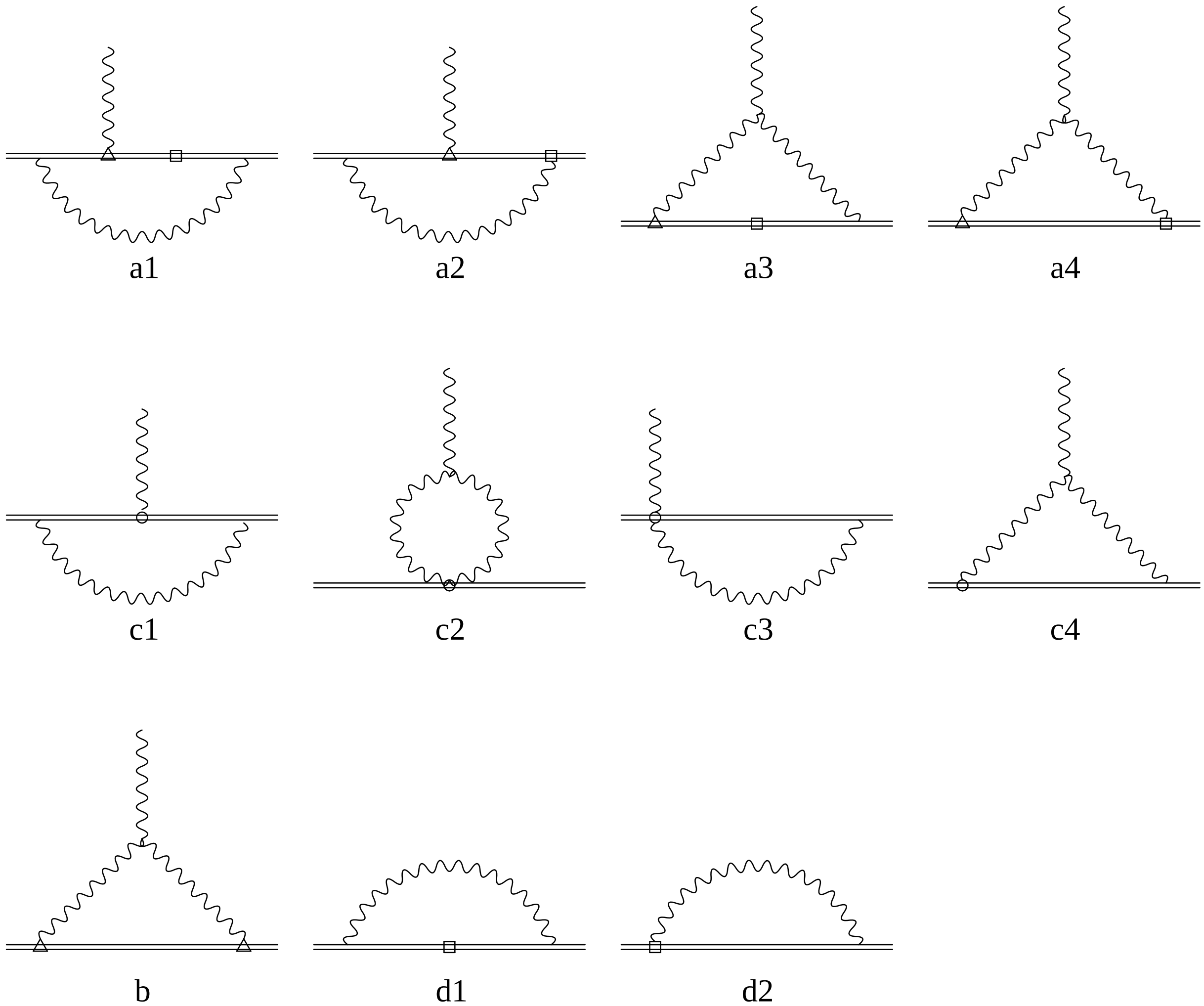


Figure 5: Diagrams for ρ_i^3; quark loops are inserted in all possible ways.

where omitted terms are suppressed as $\frac{1}{2} - u$ compared to the displayed ones. Normalization constants are known in the large β_1 limit only: $K_i = 1 + O(1/\beta_1)$. The large–order behaviour of the perturbation series for δc is

$$c_{n+1} = n!\,(2\beta_1)^n\, n^{\beta_2/2\beta_1^2}\,\left[4C_F K_1 - \tfrac{2}{3}C_A K_2 + \tfrac{19}{6}C_A K_3 n^{-\gamma_1/2\beta_1} + C_A K_4 n^{\gamma_1/2\beta_1}\right]\,, \qquad (48)$$

where omitted terms are suppressed as $1/n$ compared to the displayed ones.
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