NONTHERMAL RADIATION OF BLACK HOLES

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A b s t r a c t

The charged particle production by a charged black hole is due to the tunneling of created particles through an effective Dirac gap. Nonthermal radiation of a rotating black hole is also described in an analogous way.

НЕТЕМПЕРАТУРНОЕ ИЗЛУЧЕНИЕ ЧЕРНЫХ ДЫР

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А н н о т а ц и я

Рождение заряженных частиц заряженной черной дырой обусловлено туннелированием рождающихся частиц сквозь эффективную дираковскую щель. Аналогичным образом описывается нетемпературное излучение вращающейся черной дыры.

1 Introduction

Particle production by charged (Reissner-Nordstrem) and rotating (Kerr) black holes was predicted simultaneously with or even somewhat earlier than the famous thermal radiation. Though the problems of the nonthermal particle creation are pretty old, there are some aspects of them which were elucidated only recently. These aspects are discussed in the present article.

The problem of particle production by the electric field of a black hole has been discussed repeatedly [1-7]. The probability of this process was estimated in [1-6] using in one way or another the result obtained previously [8-10] for the case of an electric field constant all over the space. This approximation might look quite natural with regard to sufficiently large black holes, for which the gravitational radius exceeds essentially the Compton wave length of the particle $\lambda = 1/m$. (We use the units with $\hbar = 1, c = 1$; the Newton gravitational constant k is written down explicitly.) However, in fact, as will be demonstrated below, the constant-field approximation, generally speaking, is inadequate to the present problem, and does not reflect a number of its essential peculiarities. A consistent semiclassical solution of the problem was given in [11].

The investigation of particle production by Kerr black holes started with the prediction [12, 13] of amplification of an electromagnetic wave at the reflection from a rotating black hole, so called superradiation. The effect was studied in detail in [14, 15] for electromagnetic and gravitational waves. It looks rather obvious that if the amplification of a wave at the reflection is possible, then its generation by a rotating black hole is possible as well. Indeed, direct calculation [16] has demonstrated that the discussed, nonthermal radiation does exist, and not only for bosons, photons and gravitons, but for neutrinos as well. The last result looks rather mysterious since for fermions there is no superradiation.

In [17] the nonthermal radiation of Kerr black holes was considered from another point of view: as tunneling of quanta being created through the Dirac gap. Certainly, this approach by itself can be valid for fermions only. It is clear however that in the leading semiclassical approximation the production of fermions and bosons is described by the same, up to the statistical weight, relations.

Let us note that in [18] an analogous mechanism was considered for the description of the friction experienced by a body rotating in superfluid liquid at $T = 0$: the quantum tunneling of quasiparticles to the region where their energy in the rotating frame is negative.

2 Radiation of charged black holes

2.1 Particle production by constant electric field

It is convenient to start the discussion just from the problem of particle pr duction by a constant electric field. We restrict ourselves to the consideratio of the production of electrons and positrons, primarily because the probabili of emitting these lightest charged particles is the maximum one. Besides, t picture of the Dirac sea allows one in the case of fermions to manage wit out the second-quantization formalism, thus making the consideration mo transparent. To calculate the main, exponential dependence of the effect, is sufficient to use a simple approach due to [8] (see also [19, 20]). In t

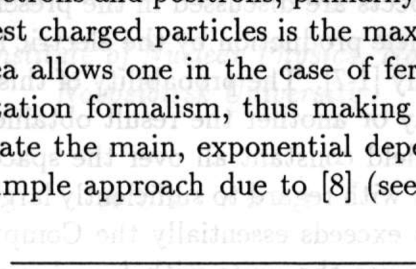

Figure 1: Dirac gap

potential $-eEz$ of a constant electric field E the usual Dirac gap (Fig. 1) til (see Fig. 2). As a result, a particle, having a negative energy in the absen of the field, can now tunnel through the gap (see the horizontal dashed li

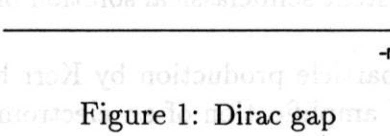

Figure 2: Dirac gap in electric field

in Fig. 2) and go to infinity as a usual particle. The hole created in this wa is nothing but an antiparticle. The exponential factor in the probability the particle creation obviously depends only on the action inside the barrier. This action does not change under a shift of the dashed line in Fig. 2 up or down, i.e., under a shift by $\Delta\mathcal{E}$ of the energy $\mathcal{E}$ of the created particle. Being obviously an integral of motion, $\mathcal{E}$ is also the energy of the initial particle of the Dirac sea. If we put for instance $\mathcal{E} = -m$, so that the particle enters the barrier at $z = 0$, the squared four-dimensional momentum

$$(\mathcal{E} - e\phi)^2 - p^2 = m^2$$

becomes

$$(-m + eEz)^2 - p^2 = m^2.$$

For the time being we assume that the transverse part of the particle momentum $\mathbf{p}_\perp = (p_x, p_y)$, which is also an integral of motion, is equal to zero. Inside the barrier the modulus of the momentum $p(z) = p_z(z)$ is

$$|p(z)| = \sqrt{m^2 - (m - eEz)^2}.$$

The action inside the barrier equals:

$$S = \int_0^{2m/eE} dz|p(z)| = \frac{\pi m^2}{2eE}.$$

Finally, the exponential factor in the probability W is [8]:

$$W \sim \exp(-2S) \sim \exp(-\pi m^2/eE). \tag{1}$$

One can easily take into account in the exponent (1) the transverse momentum $\mathbf{p}_\perp$. This integral of motion will, clearly, enter all the previous formulae in the only combination $m^2 + p_\perp^2$. So, expression (1) in this case demands the substitution $m^2 \to m^2 + p_\perp^2$, thus changing to

$$W \sim \exp[-\pi(m^2 + p_\perp^2)/eE]. \tag{2}$$

Let us calculate now the pre-exponential factor in the probability of particle creation, as it was done in [21]. The obtained exponential (2) is the probability that a particle of the Dirac sea approaching the potential barrier from the left (see Fig. 2), will tunnel through it to the right, thus becoming a real electron. To obtain the total number of pairs created per unit volume per unit time, the exponential (2) should be multiplied by the current density of the particles of the Dirac sea

$$j_z = \rho v_z\,. \tag{3}$$

For the velocity we use the common relation

$$v_z = \frac{\partial \mathcal{E}}{\partial p}$$

(the subscript z of the longitudinal momentum p is again omitted here and below). The particle density is as usual

$$\rho = 2\,\frac{d^2 p_\perp dp}{(2\pi)^3}, \tag{4}$$

the factor 2 being due to two possible orientations of the electron spin.

For a fixed coordinate z and fixed $\mathbf{p}_\perp$ the identity holds:

$$\frac{\partial \mathcal{E}}{\partial p} dp = d\mathcal{E}. \tag{5}$$

On the other hand, it is obvious that the interval of energies $d\mathcal{E}$ of the tunneling particles is directly related to the interval dz between longitudinal coordinates of the points where the particles enter the barrier: $d\mathcal{E} = eEdz$ (up to an inessential sign). Being interested in the probability per unit volume in general, and per unit longitudinal distance in particular, we should delete the arising factor dz when calculating the effect. So, the total number of pairs created per unit volume per unit time is

$$W_{1/2} = 2eE \int \frac{d^2 p_\perp}{(2\pi)^3} \exp[-\pi(m^2 + p_\perp^2)/eE]. \tag{6}$$

Now the trivial integration over the transverse momenta gives the final result

$$W_{l/2} = \frac{e^2 E^2}{4\pi^3} \exp(-\pi m^2/eE). \tag{7}$$

The probability W in the above formulae is supplied with the subscript 1/2 to indicate that the result refers to particles of spin one half. Obviously, the notion of the Dirac sea, and hence the above derivation by itself, does not apply to boson pair creation. However, in the semiclassical approximation, the creation rate for particles of spin zero is almost the same. The only difference is that these particles do not have two polarization states, so the their rate is two times smaller than (7):

$$W_0 = \frac{e^2 E^2}{8\pi^3} \exp(-\pi m^2/eE). \tag{8}$$

The corresponding exact results for a constant electric field are [10]

$$W_{l/2} = \frac{e^2E^2}{4\pi^3}\sum_0^\infty \frac{1}{n^2}\exp(-n\pi m^2/eE), \tag{9}$$

$$W_0 = \frac{e^2E^2}{8\pi^3}\sum_0^\infty \frac{(-1)^{n-1}}{n^2}\exp(-n\pi m^2/eE). \tag{10}$$

Obviously, the account for higher terms, with $n \geq 2$, in the sums (9), (10) makes sense only for very strong electric fields, for $eE \sim m^2$. For smaller fields, when $eE \ll m^2$, simple formulae (7) and (8) are quantitatively correct.

The above straightforward derivation clearly explains some important properties of the phenomenon. First of all, at the constant electric field the action inside the barrier does not change under a shift of the dashed line in Fig. 2 up or down. Owing to this property expressions (1) and (7) are independent of the energy of created particles. Then, for the external field to be considered as a constant one, it should change weakly along the path inside the barrier. Obviously, the length of this path $l \sim m/eE$ essentially differs from the Compton wave length $\lambda = 1/m$ of the particle. The ratio l/λ is of the same order of magnitude as the action S inside the barrier, and therefore should be large for the semiclassical approximation to be applicable at all.

The considered case of a constant electric field has one more peculiarity. The same criterion of the semiclassical approximation, $l/\lambda \gg 1$, means also that the tilt of the Dirac gap is very small. Therefore, the vicinity of the turning point, where the classical picture is inapplicable, is anomalously large in the (formally) classically accessible region. That is why the formation length for the electron positron pairs is in this case not m/eE, as one may expect naïvely, but much larger, $m/eE(m^2/eE)^{1/2}$, as was demonstrated by direct calculations in [21, 22].

2.2 Particle production by charged black holes. Exponential dependence

It is clear now that, generally speaking, the constant-field approximation is not applicable to the problem of a charged black hole radiation, and that the probability of particle production in this problem is strongly energy-dependent. The explicit form of this dependence will be found below.

We start the solution of the problem with calculating the action inside the barrier. The metric of a charged black hole is well-known:

$$ds^2 = f dt^2 - f^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{11}$$

where

$$f = 1 - \frac{2kM}{r} + \frac{kQ^2}{r^2}, \quad (12)$$

M and Q being the mass and charge of the black hole, respectively. The equation for a particle 4-momentum in these coordinates is

$$f^{-1}(\varepsilon - \frac{eQ}{r})^2 - fp^2 - \frac{l^2}{r^2} = m^2. \quad (13)$$

Here ε and p are the energy and radial momentum, respectively, of the particle. We assume that the particle charge e is of the same sign as the charge of the hole Q, ascribing the charge $-e$ to the antiparticle.

Clearly, the action inside the barrier is minimum for the vanishing orbital angular momentum l. It is rather evident therefore (and will be demonstrated below explicitly) that after the summation over l just the s-state defines the exponential in the total probability of the process. So, we restrict for the moment to the case of a purely radial motion. The equation for edges of the Dirac gap for $l = 0$ is

$$\varepsilon_{\pm}(r) = \frac{eQ}{r} \pm m\sqrt{f}, \quad (14)$$

which is presented in Fig. 3. It is known [23] that at the horizon of a black hole, for $r = r_+ = kM + \sqrt{k^2M^2 - kQ^2}$, the gap vanishes. Then, with the increase of r, the lower boundary $\varepsilon_-(r)$ of the gap decreases monotonically, tending asymptotically to $-m$. The upper branch $\varepsilon_+(r)$ at first, in general, increases, and then decreases, tending asymptotically to m.

It is clear from Fig. 3 that those particles of the Dirac sea whose coordinate r exceeds the gravitational radius r_+ and whose energy ε belongs to the interval $\varepsilon_-(r) > \varepsilon > m$, tunnel through the gap to infinity. In other words, a black hole loses its charge due to the discussed effect, by emitting particles with the same sign of the charge e, as the sign of Q. Clearly, the phenomenon takes place only under the condition

$$\frac{eQ}{r_+} > m. \quad (15)$$

For an extremal black hole, with $Q^2 = kM^2$, the Dirac gap looks somewhat different (see Fig. 4): when Q^2 tends to kM^2, the location of the maximum of the curve $\varepsilon_+(r)$ tends to r_+, and the value of the maximum tends to eQ/r_+. It is obvious however that the situation does not change qualitatively due to it. Thus, though an extremal black hole has zero Hawking temperature and,

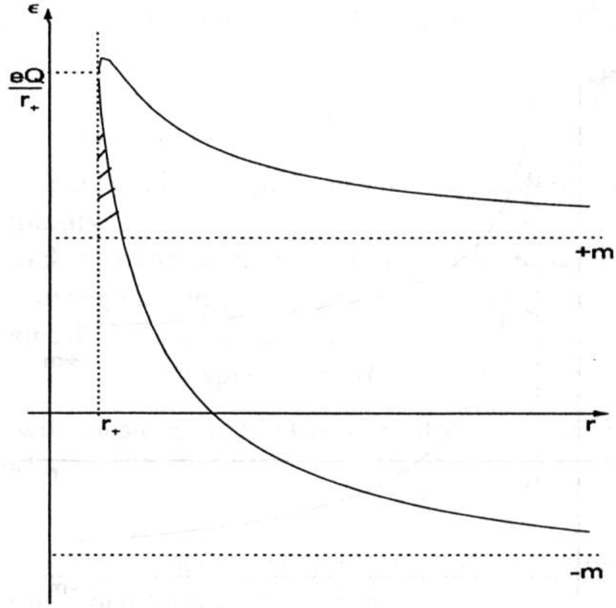

Figure 3: Dirac gap for nonextremal black hole

correspondingly, gives no thermal radiation, it still creates charged particles due to the discussed effect.

In the general case $Q^2 \leq kM^2$ the doubled action inside the barrier entering the exponential for the radiation probability is

$$2|S| = 2\int_{r_1}^{r_2} dr|p(r,\varepsilon)| =$$

$$= 2\int_{r_1}^{r_2} \frac{drr}{r^2 - 2kMr + kQ^2}\sqrt{-p_0^2r^2 + 2(\varepsilon eQ - km^2M)r - (e^2 - km^2)Q^2}. \tag{16}$$

Here $p_0 = \sqrt{\varepsilon^2 - m^2}$ is the momentum of the emitted particle at infinity, and the turning points $r_{1,2}$ are as usual the roots of the quadratic polynomial under the radical; we are interested in the energy interval $m \leq E \leq eQ/r_+$. Of course, the integral can be found explicitly, though it demands somewhat tedious calculations. However, the result is sufficiently simple:

$$2|S| = 2\pi \frac{m^2}{(\varepsilon + p_0)p_0}[eQ - (\varepsilon - p_0)kM]. \tag{17}$$

(Previously, this exponent was obtained in [7] from the solution of the Klein-Gordon equation in the Reissner-Nordstrem metric.) Certainly, the expression

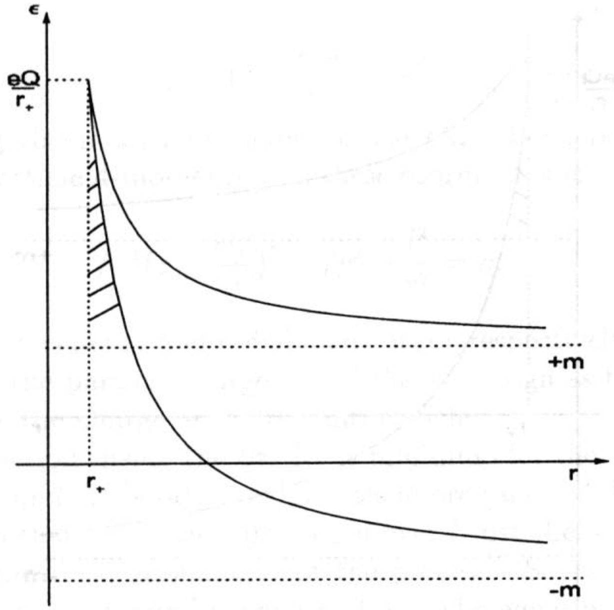

Figure 4: Dirac gap for extremal black hole

(17), as distinct from the exponent in formula (1), depends on the energy quite essentially.

Let us note that the action inside the barrier does not vanish even for the limiting value of the energy $\varepsilon_m = eQ/r_+$. For a nonextremal black hole it is clear already from Fig. 3. For an extremal black hole this fact is not as obvious. However, due to the singularity of $|p(r,\varepsilon)|$, the action inside the barrier is finite for $\varepsilon = \varepsilon_m = eQ/r_+$ for an extremal black hole as well. In this case the exponential factor in the probability is

$$\exp[-\pi(\sqrt{k}m/e)kmM]. \tag{18}$$

Due to the extreme smallness of the ratio

$$\frac{\sqrt{k}m}{e} \sim 10^{-21}, \tag{19}$$

the exponent here is large only for a very heavy black hole, with a mass M exceeding that of the Sun by more than five orders of magnitude. And since the total probability, integrated over energy, is dominated by the energy region $\varepsilon \sim \varepsilon_m$, the semiclassical approach is applicable in the case of extremal black holes only for such very heavy objects. Let us note also that for the particles itted by an extremal black hole, the typical values of the ratio ε/m are very rge:

$$\frac{\varepsilon}{m} \sim \frac{\varepsilon_m}{m} = \frac{eQ}{kmM} = \frac{e}{\sqrt{k}m} \sim 10^{21}.$$

other words, an extremal black hole in any case radiates highly ultrarela-istic particles mainly.

Let us come back to nonextremal holes. In the nonrelativistic limit, when $/r_+ \to m$ and, correspondingly, the particle velocity $v \to 0$, the exponential of course very small:

$$\exp(-2\pi kmM/v). \tag{20}$$

Therefore, we will consider mainly the opposite, ultrarelativistic limit where e exponential is

$$\exp[-\pi(m^2/e^2)eQ]. \tag{21}$$

course, here also the energies $\varepsilon \sim \varepsilon_m \sim eQ/kM$ are essential, so that the trarelativistic limit corresponds to the condition

$$eQ > kmM. \tag{22}$$

it then the semiclassical result (21) is applicable (i.e., the action inside the rrier is large) only under the condition

$$kmM > 1. \tag{23}$$

is last condition means that the gravitational radius of the black hole $_+ \sim kM$) is much larger than the Compton wave length of the electron m. In other words, the result (21) refers to macroscopic black holes. Com-ning (22) with (23), we arrive at one more condition for the applicability of rmula (21):

$$eQ \gg 1. \tag{24}$$

e shall return to this relation later.

Let us note that in [4] the action inside the barrier was being calculated der the same assumptions as formula (21). However, the answer presented [4], $2|S| = \pi m^2 r_+^2/eQ$, is totally independent of energy (and corresponds to rmula (2) which refers to the case of a constant electric field). Obviously, is cannot be correct for the discussed integral in the general case $\varepsilon \neq \varepsilon_m$.

2.3 Particle production by charged black holes. Pre-exponential factor

Now the radiation intensity is conveniently calculated in the following way. At $r \to \infty$ the radial current density of free particles in the energy interval $d\varepsilon$ is

$$j_r(\varepsilon, l)d\varepsilon = 2\sum \frac{d^3p}{(2\pi)^3}\frac{\partial\varepsilon}{\partial p_r} = 2\,\frac{2\pi\,(2l+1)dp_r}{(2\pi)^3 r^2}\frac{\partial\varepsilon}{\partial p_r}. \tag{25}$$

The summation over the directions of the angular momentum reduces in fact to the multiplication by the number $2l+1$ of possible projections of the orbital angular momentum **l** onto the z-axis and to the integration over the azimuth angle of the vector **l**, which gives 2π. By means of the identity

$$\frac{\partial\varepsilon}{\partial p_r}dp_r = d\varepsilon,$$

we obtain in the result that the total flux of free particles at $r \to \infty$ is

$$4\pi r^2 j_r(\varepsilon, l) = \frac{2(2l+1)}{\pi}. \tag{26}$$

One can easily see that in our problem the total flux of radiated particles differs from the last expression only by the barrier penetration factor . Thus, the number of particles emitted per unit time is

$$\frac{dN}{dt} = \frac{2}{\pi}\int d\varepsilon \sum_l (2l+1)\exp[-2|S(\varepsilon,l)|]. \tag{27}$$

In the most interesting, ultrarelativistic case dN/dt can be calculated explicitly. Let us consider the expression for the momentum in the region inside the barrier for $l \neq 0$:

$$|p(\varepsilon,l,r)| = f^{-1}\sqrt{\left(m^2+\frac{l^2}{r^2}\right)f + \left(\varepsilon - \frac{eQ}{r}\right)^2}. \tag{28}$$

The main contribution to the integral over energies in formula (27) is given by the region $\varepsilon \to \varepsilon_m$. In this region the functions $f(r)$ and $\varepsilon - eQ/r$, entering expression (28), are small and change rapidly. As to the quantity

$$\mu^2(r,l) = m^2 + l^2/r^2, \tag{29}$$

one can substitute in it for r its average value, which lies between the turning points r_1 and r_2. Obviously, in the discussed limit $\varepsilon \to \varepsilon_m$ the near turning point coincides with the horizon radius, $r_1 = r_+$. And the expression for the distant turning point is in this limit

$$r_2 = r_+ \left[1 + \frac{2\mu^2}{\varepsilon^2 - \mu^2} \frac{\sqrt{k^2M^2 - kQ^2}}{r_+} \right]. \tag{30}$$

Assuming that for estimates one can put in formula (29) $r \sim r_+$, one can easily show that the addition to 1 in the square bracket is bounded by the ratio $l^2/(eQ)^2$. Assuming that this ratio is small (we will see below that this assumption is self-consistent), we arrive at the conclusion that $r_2 \approx r_+$, and hence μ^2 can be considered independent of r: $\mu^2(r,l) \approx m^2 + l^2/r_+^2$. As a result, we obtain

$$2|S(\varepsilon, l)| = \pi e Q \left(\frac{m^2}{\varepsilon^2} + \frac{l^2}{r_+^2 \varepsilon^2} \right). \tag{31}$$

Now we easily find

$$dN/dt = m \left(\frac{eQ}{\pi m r_+} \right)^3 \exp(-\pi m^2 r_+^2 / eQ). \tag{32}$$

Let us note that the range of orbital angular momenta, contributing to the total probability (32), is effectively bounded by the condition $l^2 \lesssim eQ$. Since $eQ \gg 1$, this condition allows one to change from the summation over l in formula (27) to the integration. On the other hand, this condition justifies the used approximation $\mu^2(r,l) \approx m^2 + l^2/r_+^2$.

2.4 Applicability of the semiclassical approximation

However, up to now we have not considered one more condition necessary for the derivation of formula (32). We mean the applicability of the semiclassical approximation to the left of the barrier, for $r_+ \leq r \leq r_1$. This condition has the usual form

$$\frac{d}{dr} \frac{1}{p(r)} < 1. \tag{33}$$

In other words, the minimum size of the initial wave packet should not exceed the distance from the horizon to the turning point. Using the estimate

$$p(r) \sim \frac{r_+(eQ - \varepsilon r_+)}{(r - r_+)(r - r_-)}, \quad r_- = kM - \sqrt{k^2M^2 - kQ^2}$$

for the momentum in the most essential region, one can check that for an extremal black hole the condition (33) is valid due to the bound $eQ \gg 1$. In a non-extremal case, for $r_+ - r_- \sim r_+$, the situation is different: the condition (33) reduces to

$$\varepsilon < \frac{eQ-1}{r_+} \sim \frac{eQ}{r_+}. \tag{34}$$

Thus, for a non-extremal black hole in the most essential region $\varepsilon \to \varepsilon_m$ the condition of the semiclassical approximation does not hold. Nevertheless, the semiclassical result (32) remains true qualitatively, up to a numerical factor in the pre-exponential.

Now we give some comments on the radiation of light charged black holes, for which $kmM < 1$, i.e., for which the gravitational radius is less than the Compton wave length of the electron. In this case the first part,

$$\varepsilon < \frac{eQ-1}{r_+},$$

of inequality (34), which guarantees the localization of the initial wave packet in the region of a strong field, means in particular that

$$eQ = Z\alpha > 1 \tag{35}$$

(we have introduced here $Z = Q/e$). It is well-known (see, e.g., [24, 25]) that the vacuum for a point-like charge with $Z\alpha > 1$ is unstable, so that such an object loses its charge by emitting charged particles. It is quite natural that for a black hole whose gravitational radius is smaller than the Compton wave length of the electron, the condition of emitting a charge is the same as in the pure quantum electrodynamics. (Let us note that 1 in the right-hand-side of (35) should not be taken too literally: even in quantum electrodynamics, where the instability condition for the vacuum of particles of spin 1/2 is just $Z\alpha > 1$ for a point-like nucleus, it changes for a finite-size nucleus [24, 25] to $Z\alpha > 1.24$. On the other hand, for the vacuum of scalar particles in the field of a point-like nucleus the instability condition is [26, 27]: $Z\alpha > 1/2$.) As has been already mentioned , for a light black hole with $kmM < 1$, the discussed condition $eQ > 1$ leads to a small action inside the barrier and to the inapplicability of the semiclassical approximation used in the present article. The problem of the radiation of a charged black hole with $kmM < 1$ was investigated numerically in [28].

2.5 Discussion of previous results. Comparison with the Hawking radiation

The exponential

$$\exp(-\pi m^2 r_+^2/eQ)$$

in our formula (32) coincides with the expression arising from formula (1), which refers to a constant electric field E, if one plugs in for this field its value Q/r_+^2 at the black hole horizon. An approach based on formulae for a constant electric field was used in Refs. [1-6]. Thus, our result for the main, exponential dependence of the probability integrated over energies, coincides with the corresponding result of these papers. Moreover, our final formula (32) agrees with the corresponding result of [6] up to an overall factor 1/2. (This difference is of no interest by itself: as has been noted above, for a non-extremal black hole the semiclassical approximation cannot guarantee an exact value of the overall numerical factor.)

As to the corresponding result of [7] (see formula (36) in [7]), the exponential therein is $\exp[-4\pi(kmM)^2/eQ]$ instead of $\exp(-\pi m^2 r_+^2/eQ)$, and the pre-exponential factor is proportional to $(eQ)^2/kM$ instead of $m(eQ/mr_+)^3$.

We believe that our analysis of the phenomenon, which demonstrates its essential distinctions from the particle production by a constant external field, is useful. First of all, it follows from this analysis that the probability of the particle production by a charged black hole has absolutely nontrivial energy spectrum. Then, in no way are real particles produced by a charged black hole all over the whole space: for a given energy ε they are radiated by a spherical surface of the radius $r_2(\varepsilon)$, this surface being close to the horizon for the maximum energy. (It implies, for instance, that the derivation of the mentioned result of [6] for dN/dt has no physical grounds: this derivation reduces to plugging $E = Q/r^2$ to the Schwinger formula (7), obtained for a constant field, with subsequent integrating all over the space outside the horizon.)

Let us compare now the radiation intensity I due to the effect discussed, with the intensity I_H of the Hawking thermal radiation. Introducing additional weight ε in the integrand of formula (27), we obtain

$$I = \pi m^2 \left(\frac{eQ}{\pi m r_+}\right)^4 \exp(-\pi m^2 r_+^2/eQ). \tag{36}$$

As to the Hawking intensity, the simplest way to estimate it, is to use dimensional arguments, just to divide the Hawking temperature

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2}$$

by a typical classical time of the problem r_+. Thus,

$$I_H \sim \frac{l}{4\pi r_+^2}. \tag{37}$$

More accurate answer for I_H differs from this estimate by a small numerical factor $\sim 2\times10^{-2}$, but for qualitative estimates one can neglect this distinction. The intensities (36) and (37) get equal for

$$eQ \sim \frac{\pi}{6}\frac{(mr_+)^2}{\ln mr_+} \sim \frac{\pi}{6}\frac{(kmM)^2}{\ln kmM}. \quad (38)$$

(One cannot agree with the condition $eQ \sim 1/(4\pi)$ for the equality of these intensities, derived in [6] from the comparison of $\varepsilon_m = eQ/r_+$ with $T_H = (r_+ - r_-)/(4\pi r_+)$.)

2.6 Change of the horizon area

In conclusion of this section let us consider the change of the horizon surface of a black hole, and hence of its entropy, due to the discussed non-thermal radiation. To this end, it is convenient to introduce, following [29], the so-called irreducible mass M_0 of a black hole:

$$2M_0 = M + \sqrt{M^2 - Q^2}, \quad (39)$$

here and below we put $k = 1$. This relation can be conveniently rewritten also as

$$M = M_0 + \frac{Q^2}{4M^0}. \quad (40)$$

Obviously, $r_+ = 2M_0$, so that the horizon surface and the black hole entropy are proportional to M_0^2.

When a charged particle is emitted, the charge of a black hole changes by $\Delta Q = -e$, and its mass by $\Delta M = -eQ/r_+ + \xi$, where ξ is the deviation of the particle energy from the maximum one. Using the relation (40), one can easily see that, as a result of the radiation, the irreducible mass M_0 and hence the horizon surface and entropy of a non-extremal black hole do not change if the particle energy is the maximum, eQ/r_+. In other words, such a process, which is the most probable one, is adiabatic. For $\xi > 0$, the irreducible mass, horizon surface, and entropy increase.

As usual, an extremal black hole, with $M = Q = 2M_0$, is a specific case. Here for the maximum energy of an emitted particle $\varepsilon_m = e$, we have $\Delta M = \Delta Q = -e$, so that the black hole remains extremal after the radiation. In this case $\Delta M_0 = -e/2$, i.e., the irreducible mass and the horizon surface decrease. In a more general case, $\Delta M = -e + \xi$, the irreducible mass changes as follows:

$$\Delta M_0 = -\frac{e-\xi}{2} + \sqrt{\left(M_0 - \frac{e}{2} + \frac{\xi}{4}\right)\xi}. \tag{41}$$

Clearly, in the case of an extremal black hole of a large mass, already for a small deviation ξ of the emitted energy from the maximum one, the square root is dominating in this expression, so that the horizon surface increases.

3 Radiation of rotating black holes

3.1 Scalar field

We will start the discussion of radiation of rotating black holes with a problem of a methodological rather than direct physical interest, with the radiation of scalar massless particles.

The semiclassical solution of the problem is started from the Hamilton-Jacobi equations for the motion of a massless particle in the Kerr field (see, for instance, [30]):

$$\left(\frac{\partial S_r(r)}{\partial r}\right)^2 = -\frac{\kappa^2}{\Delta} + \frac{[(r^2+a^2)\varepsilon - a\,l_z]^2}{\Delta^2}, \tag{42}$$

$$\left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2 = \kappa^2 - \left(a\,\varepsilon\,\sin\theta - \frac{l_z}{\sin\theta}\right)^2. \tag{43}$$

Here $S_r(r)$ and $S_\theta(\theta)$ are the radial and angular actions, respectively;

$$\Delta = r^2 - r_g r + a^2, \quad r_g = 2kM;$$

$\mathrm{a} = \mathbf{J}/M$ is the angular momentum of the black hole in the units of its mass M; l_z is the projection of the particle angular momentum onto $\mathbf{a}$.

As to the constant κ^2 of the separation of variables, in the spherically symmetric limit $a \to \infty$ it is equal to the particle angular momentum squared l^2, or to $l(l+1)$ in quantum mechanics. The influence of the black hole rotation, i.e. of the finite a, upon κ^2 is taken into account by means of the perturbation theory applied to equation (43). The result is [14]

$$\kappa^2 = l(l+1) - 2\omega\,\alpha\,l_z + \frac{2}{3}\omega^2\,\alpha^2\left[1 + \frac{3l_z^2 - l(l+1)}{(2l-1)(2l+3)}\right]. \tag{44}$$

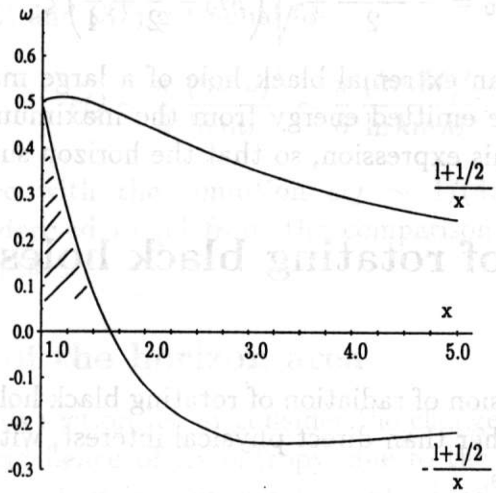

Figure 5: Gap for $l = 1$

Here and below we use the dimensionless variables $\omega = \varepsilon kM$, $x = r/kM$, $\alpha = a/kM$. Let us recall that in the semiclassical approximation the substitution

$$l(l+1) \to (l+1/2)^2$$

should be made.

Besides, in the exact quantum mechanical problem, under the reduction of the radial wave equation to the canonical form

$$R'' + p^2(r)R = 0,$$

the expression for $p^2(r)$ contains additional (as compared to the right-hand side of equation (42)) nonclassical terms, which should be, strictly speaking, included for $l \simeq 1$. However, for the simplicity sake we neglect here and below these nonclassical corrections to $p^2(r)$, which should not influence qualitatively the results obtained.

The dependence of the classically inaccessible region, where the radial momentum squared p^2 is negative, on the distance x is presented for different an-

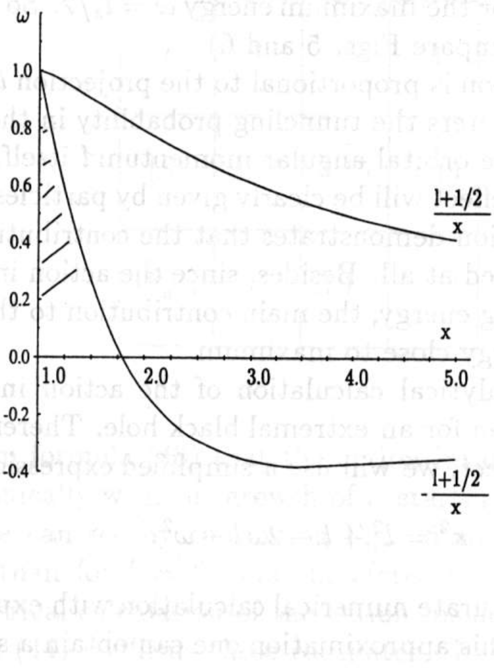

Figure 6: Gap for $l > 1$

gular momenta in Figs. 5, 6. At the horizon the gap vanishes [23]. For $r \to \infty$ the boundaries of the classically inaccessible region behave as $\pm(l + 1/2)/r$. In other words, the centrifugal term for massless particles plays in a sense the role of the mass squared. Let us note that for $l > 1$ both branches of the equation $p^2(r) = 0$ fall down, but for $l = 1$ one branch near the horizon grows up, and the second one falls down. Thus, the radiation mechanism consists in tunneling, i.e., in particles going out of the dashed region to the infinity.

One should note the analogy between the emission of charged particles by a charged black hole and the effect discussed. In the first case the radiation is due to the Coulomb repulsion, and in the second case it is due to the repulsive interaction between the angular momenta of the particle and black hole [31].

The action inside the barrier for the radial equation (42) is:

$$|S_r| = \int dx \sqrt{\frac{k^2}{(x-1)^2} - \frac{[\omega(x^2+1) - l_z]^2}{(x-1)^4}}, \tag{45}$$

the integral is taken between two turning points. For simplicity sake we confine for the time being to the case of an extremal black hole, $a = kM$. Let us note that due to a singular dependence of p on x, the action inside the barrier does not vanish at $l > 1$ even for the maximum energy $\omega = l_z/2$. So much the more, it stays finite at $l = 1$ (compare Figs. 5 and 6).

The repulsive interaction is proportional to the projection l_z of the particle angular momentum and enters the tunneling probability in the exponent, but the barrier depends on the orbital angular momentum l itself. Therefore, the main contribution to the effect will be clearly given by particles with l_z close to l. The numerical calculation demonstrates that the contribution of the states with $l_z \neq l$ can be neglected at all. Besides, since the action inside the barrier decreases with the growing energy, the main contribution to the effect is given by the particles with energy close to maximum.

Unfortunately, an analytical calculation of the action inside the barrier does not look possible even for an extremal black hole. Therefore, to obtain a qualitative idea of the effect, we will use a simplified expression for κ^2:

$$\kappa^2 = l^2 + l - 2\omega\, l + \omega^2. \tag{46}$$

(The results of a more accurate numerical calculation with expression (44) will be presented below.) In this approximation one can obtain a simple analytical formula for the action inside the barrier for all angular momenta but $l = 1$. Let us assume that $\omega = (1 - \delta)\, l/2$ with $\delta \ll 1$; just this range of energies dominates the radiation. Then, the turning points of interest to us, which are situated to the right of the horizon, are:

$$x_{1,2} = 1 + \frac{2\delta}{2 \pm \sqrt{1 + 4/l}}. \tag{47}$$

Now one finds easily

$$|S_{an}| = \frac{\pi\, l}{2}\left(2 - \sqrt{3 - \frac{4}{l}}\right). \tag{48}$$

One can see from this equation that the term l (following l^2) in formula (46) is quite essential even for large angular momenta: it generates the terms $4/l$ in formulae (47) and (48), thus enhancing $|S|$ for $l \gg 1$ by $\pi/\sqrt{3}$. Correspondingly, the transmission factor $D = \exp(-2|S|)$ gets about 40 times smaller. Let us note that even the transition in κ^2 from $l(l+1)$ to $(l+1/2)^2$ makes the effect considerably smaller for l comparable to unity; but this suppression dies out for large angular momenta.

Table 1: Action inside the barrier for scalar particles

l	1	2	3
$\|S\|$	3.45	3.15	3.33
$\|S_{an}\|$		3.14	3.34

It follows from formula (48) that the action inside the barrier is large, it increases monotonically with the growth of l, starting with $|S| = \pi$ for $l = 2$. As to $l = 1$, one can see by comparing Fig. 5 with Fig. 6 that here the barrier is wider than for $l = 2$, and therefore the action should be larger. Indeed, the numerical calculation of the action inside the barrier $|S|$ with κ^2 given by formula (44) confirms these estimates. Its results are presented in Table 1 where for the comparison sake we present also the analytical estimates $|S_{an}|$ with formula (46). By the way, this comparison demonstrates that the approximate analytical formula (46) works very well. The numbers presented in the table refer to an extremal black hole and maximum energy of emitted particles. It is clear however that the transition to nonextremal black holes, lower energies, and larger l will result in the growth of the action inside the barrier. Since it proves here to be always considerably larger than unity, the use of the semiclassical approximation inside the barrier is quite reasonable.

Let us check now whether it applies to the left of the barrier. Here, near the horizon, one can neglect in the expression for the momentum p the term related to the centrifugal barrier, so that condition (33) transforms to

$$\frac{d}{dx}\frac{1}{p(x)} \approx \frac{d}{dx}\frac{(x-1)^2}{\omega(x^2+\alpha^2)-\alpha l}. \tag{49}$$

One can see easily that for not so large l, which are of importance in our case, this expression is comparable to unity and condition (49) does not hold. Nevertheless, despite of this circumstance and of the neglect of the nonclassical corrections to $p^2(x)$ mentioned above, the results of the semiclassical calculation, presented below, should be correct at least qualitatively.

Let us come back to the calculation of the radiation intensity. The line of reasoning used previously demonstrates that here the total flux of free particles at $r \to \infty$ is

$$4\pi r^2 j_r(\varepsilon, l) = 4\pi r^2 \sum_{l_z} \frac{2\pi}{(2\pi)^3 r^2} \to \frac{1}{\pi}. \tag{50}$$

Again, the total flux of radiated particles differs from this expression by the barrier penetration factor only. Thus, in our semiclassical approximation we obtain for the loss of mass by a black hole in the unit time the following expression:

$$\frac{dM}{dt} = -\frac{1}{\pi} \sum_{l=1}^{\infty} \int_0^{\varepsilon_{max}} \varepsilon \exp(-2\,|S(\varepsilon, l)|)\, d\varepsilon\,. \tag{51}$$

Here the maximum energy of radiated quanta is

$$\varepsilon_{max} = \frac{a\,l}{r_h^2 + a^2}; \tag{52}$$

$r_h = km + \sqrt{k^2 M^2 - a^2}$ is the radius of the horizon of a Kerr black hole. The analogous expression for the loss of the angular momentum is

$$\frac{dJ}{dt} = -\frac{1}{\pi} \sum_{l=1}^{\infty} \int_0^{\varepsilon_{max}} l \exp(-2\,|S(\varepsilon, l)|)\, d\varepsilon\,. \tag{53}$$

The results of the numerical calculations with formulae (51) and (53) of the loss by a black hole of its mass and angular momentum for different values of the rotation parameter α are presented in Table 2. We present here and below, for spinning particles, results of calculations only for sufficiently rapid rotation, $\alpha \approx 1$. The point is that with further decrease of α, not only the thermal radiation grows rapidly, but the effect discussed falls down even more rapidly. For smaller α this effect becomes much smaller than the thermal one, and thus its consideration there does not make much sense.

As one can see from Table 2, the rate of loss of the angular momentum is higher, in the comparable units, than the rate of loss of the mass. In fact, it follows immediately from expression (52). Already from this expression one can see that even for the maximum possible energy the ratio of the corresponding numbers is 2:1. Real ratios are even larger. Hence an important conclusion follows: extremal black holes do not exist. Even if an extremal hole is formed somehow, in the process of radiation it looses the extremality immediately.

Table 2: Loss of mass (in the units of $10^{-3}\pi M^2$) and angular momentum (in the units of $10^{-3}\pi M$), due to the radiation of scalar particles

α	$\|dM/dt\|$	$\|dJ/dt\|$
0.999	2.6	6.4
0.9	0.19	0.77

3.2 Radiation of photons and gravitons

The investigation of the radiation of real particles we start with the electromagnetic field. Photon has two modes of opposite parity: the so called electric mode, with $l = j \pm 1$, and magnetic one, with $l = j$ [19]. It follows from the duality invariance that the radiation intensities for these two modes are equal. Thus one can confine to the solution of the problem for the magnetic mode, and then just double the result.

One can demonstrate that the situation with the gravitational waves is analogous. Again, there are two modes which, due to a special duality, contribute equally to the radiation, and for one of these modes $l = j$.

Obviously, for a mode with $l = j$ the radial equation in the semiclassical approximation is the same as for the scalar field, but with different value of κ^2. It can be demonstrated also starting from the so called Teukolsky equation [32] (neglecting again nonclassical corrections to $p^2(r)$). The corresponding eigenvalues of the angular equation for particles of spin s, found again in the perturbation theory, are [14]:

$$\kappa^2 = j(j+1) + \frac{1}{4} - 2\alpha\,\omega\,j_z - \frac{2\alpha\,\omega\,j_z\,s}{j(j+1)} +$$

$$+\alpha^2\omega^2\left\{\frac{2}{3}\left[1 + \frac{3j_z^2 - j(j+1)}{(2j-1)(2j+3)}\right] - \frac{2s^2}{j(j+1)}\frac{3j_z^2 - j(j+1)}{(2j-1)(2j+3)} + \right. \tag{54}$$

$$\left. +2\,s^2\left[\frac{(j^2 - s^2)(j^2 - j_z^2)}{j^2(2j-1)(2j+1)} - \frac{((j+1)^2 - j_z^2)((j+1)^2 - s^2)}{(j+1)^3(2j+1)(2j+3))}\right]\right\}.$$

Table 3: Action inside the barrier for photons and gravitons

	s=1		s=2	
j	1	2	2	3
$\|S\|$	1.84	2.17	1.0	1.7

We have included into this expression the term 1/4, necessary for the correct semiclassical description. Let us note that, as follows from the consideration of the helicity of a massless particle, the restriction $j \geq s$ holds. Correspondingly, for a photon $j \geq 1$, for a graviton $j \geq 2$. As well as in the scalar case, the main contribution into the radiation is given by the states with the maximum projection of the angular momentum, $j_z = j$.

Let us discuss first of all whether the semiclassical approximation is applicable here. As to the situation to the left of the barrier, it does not differ qualitatively from the scalar case. The situation inside the barrier is different. As one can see from equation (54), the presence of spin makes κ^2 smaller, and correspondingly, makes smaller the centrifugal repulsion. In result, both the barrier and the action inside it decrease. This qualitative argument is confirmed by a numerical calculation of $|S|$ for photons and gravitons with the maximum projection of the angular momentum $j_z = j$ and maximum energy for the case of an extremal black hole (see Table 3). Therefore in the present case, one should expect that the accuracy of semiclassical results is lower than in the scalar case.

The semiclassical formulae for electromagnetic and gravitational radiation differ formally from the corresponding scalar ones (51) and (53) by extra factor 2 only, which reflects the existence of two modes. The results of this calculation are presented in Table 4. In it we indicate in brackets for comparison the results of the complete quantum mechanical calculation [16] which takes into account as well the thermal radiation.

It is clear from Table 4 that even for $\alpha = 0.999$, when the thermal radiation is negligibly small, our semiclassical calculation agrees with the complete one qualitatively only. It is quite natural if one recalls that the semiclassical action in the present problem exceeds unity not so much, if any. This explanation

Table 4: Loss of mass (in the units of $10^{-3}\,\pi M^2$) and angular momentum (in the units of $10^{-3}\,\pi M$), due to the radiation of photons and gravitons

	s=1		s=2	
α	$\lvert dM/dt\rvert$	$\lvert dJ/dt\rvert$	$\lvert dM/dt\rvert$	$\lvert dJ/dt\rvert$
0.999	16.5(9.6)	39(24)	66(228)	148(549)
0.9	0.72(2.26)	2.8(8.2)	0.58(12.9)	2(48)

is supported by the fact that for photon, where $|S|$ is considerably larger (see Table 3), the semiclassical calculation agrees better with the complete one.

3.3 Radiation of neutrino

Let us consider at last the radiation by a rotating black hole of neutrinos, massless particles of spin 1/2. The wave function of a 2-component neutrino is written as (see, for instance [16]):

$$\psi = \exp(-i\varepsilon\, t + i j_z \phi) \begin{pmatrix} R_1\, S_1 \\ R_2\, S_2 \end{pmatrix}. \tag{55}$$

It is essential that the wave equations for neutrino in the Kerr metrics allow for the separation of variables as well [32]. The radial equations in dimensionless variables are:

$$\begin{aligned} \frac{d\,R_1}{d\,x} - i\,\frac{\omega(x^2+\alpha^2) - j_z\,\alpha}{\Delta}\, R_1 &= \frac{\kappa}{\sqrt{\Delta}}\, R_2, \\ \frac{d\,R_2}{d\,x} + i\,\frac{\omega(x^2+\alpha^2) - j_z\,\alpha}{\Delta}\, R_2 &= \frac{\kappa}{\sqrt{\Delta}}\, R_1. \end{aligned} \tag{56}$$

The angular equations are:

$$\frac{d\,S_1}{d\,\theta} + \left(\omega\,\alpha\,\sin\theta - \frac{j_z}{\sin\theta}\right) S_1 = \kappa\, S_2,$$

Table 5: Loss of mass (in the units of $10^{-3}\,\pi M^2$) and angular momentum (in the units of $10^{-3}\,\pi M$), due to the radiation of neutrinos

α	$\lvert dM/dt \rvert$	$\lvert dJ/dt \rvert$
0.99	4.4 (2.1)	11 (5.65)
0.9	0.7 (1)	2.7 (3.25)

$$\frac{d\,S_2}{d\,\theta} - \left(\omega\,\alpha\,\sin\theta - \frac{j_z}{\sin\theta}\right)\,S_2 = -\kappa\,S_1. \qquad (57)$$

For κ^2 the same formula (54) takes place, but now of course with $s = 1/2$. As well as for bosons, it is sufficient practically to consider states with $j_z = j$.

It is essential that R_1 corresponds at infinity, for $x \to \infty$, and at the horizon, for $x \to 1$, to the wave running to the right, and R_2 corresponds for $x \to \infty$ and for $x \to 1$ to the wave running to the left. (For this classification, it is convenient to use the so called "tortoise" coordinate $\xi(x)$; for $x \to \infty$ $\xi \approx x \to +\infty$, for $x \to 1$ $\xi \approx \ln(x-1) \to -\infty$.) It is quite natural therefore that here the radial current density is

$$j_r = |R_1|^2 - |R_2|^2.$$

We are interested in the barrier penetration factor for the state which is an outgoing wave at infinity. For a neutrino or antineutrino such a state has a fixed helicity, but has no definite parity. Meanwhile, the potential barrier depends in our problem, roughly speaking, on the orbital angular momentum, and therefore is much more transparent for the states of $l = j - 1/2$, than for the states of $l = j + 1/2$. (These states of given l have definite parity, and are superpositions of neutrino and antineutrino.) Moreover, at $l = j - 1/2$ for small j, which give the main contribution to the radiation, the action either has no imaginary part at all, or its imaginary part is small, so that our above approach is inapplicable. Therefore, we will solve numerically the exact problem of the neutrino radiation.

Technically, it is convenient to find the reflection coefficient R in the problem of the neutrino scattering off a black hole, and then use the obvious relation for the transmission coefficient D:

$$D = 1 - R.$$

Here the expressions for the loss of mass and angular momentum by a black hole are:

$$\frac{dM}{dt} = -\frac{1}{\pi} \sum_{j=1/2}^{\infty} \int_0^{\varepsilon_{max}} \varepsilon D(\varepsilon, j)\, d\varepsilon\,; \tag{58}$$

$$\frac{dJ}{dt} = -\frac{1}{\pi} \sum_{j=1/2}^{\infty} \int_0^{\varepsilon_{max}} j D(\varepsilon, j)\, d\varepsilon\,; \tag{59}$$

$$\varepsilon_{max} = \frac{aj}{r_h^2 + a^2}. \tag{60}$$

The results, obtained by numerical solution of the system of radial equations (56), are presented in Table 5. In brackets we present the results of [16], which include the Hawking radiation contribution. For a black hole close to extremal one, at $\alpha = 0.99$, where the thermal radiation is practically absent, our results are about twice as large as previous ones.

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