ADVANCED STATISTICAL TECHNIQUES IN MUON G-2 EXPERIMENT AT BNL

S.I.Redin (for the muon g-2 collaboration)*
Budker Institute of Nuclear Physics, Novosibirsk, Russia
and Department of Physics, Yale University, New Haven, CT 06511, USA

Abstract
We present some analytical statistical techniques used for the data analysis in muon g-2 experiment at Brookhaven National Laboratory.

1 INTRODUCTION

In our muon g-2 experiment we intend to measure the muon g-2 value to about 0.3 ppm (parts per million) accuracy. There are two quantities in this experiment which have to be measured precisely: the first is the magnetic field in muon storage ring, measured by a system of NMR probes in terms of the frequency of the proton spin resonance, ω_p, and the second is the frequency of g-2 oscillations from the time distribution of electrons emerging from muon decays, ω. Technically, ω is found by chi-squared optimization of the parameters of a fit function approximating the time distribution (histogram) of decay electrons. This is a standard data analysis technique in high energy physics, however the precision requirements in our experiment are exceptionally high. For this reason all statistical properties of the fitted distribution, such as statistical errors and correlations among parameters, must be well understood, all statistical tests must be well justified and all possible systematic effects must be extensively studied.

A simple procedure was developed and applied to calculate the statistical errors and correlations of parameters analytically. It was found that the correlation between the frequency and phase of g-2 oscillations can be used to improve the statistical error of ω by use of information on phase, which we possess in our experiment. A similar procedure was developed for the analytical estimation of the systematic shift of ω due to the possible presence of a small unidentified background. This allows us to study a number of possible sources of systematic errors including such a fundamental systematic effect as finite binwidth, which might be important for any distribution in physics and elsewhere.

Some of the ω stability checks in the g-2 experiment are based on comparison of results obtained for the full set of data and that obtained for some subset of data.These are, in particular, checks for stability with respect to change of (1) the histogram fit start time and (2) energy threshold E_{thr} of decay electrons. For such problems we have derived equations (20) and (21) for statistical fluctuations of fit parameters as a function of fluctuations in individual channels of the histogram. Using this equation we has shown that the difference in ω for the first case should be within $\pm\sqrt{\sigma^2_{subset} - \sigma^2_{full\ set}}$ for one standard deviation. We have found that this formula is actually valid for any distribution no matter of how many parameters it has and whether or not they correlate to each other. However, for the case of energy threshold change, some parameters of the g-2 distribution (the phase and amplitude of g-2 oscillations) depend on the energy threshold and therefore this formula can not be applied. Nonetheless, eqs. (20) and (21) allowed us to derive an appropriate formula for this case too.

* Muon g-2 collaboration: H. N. Brown, G. Bunce, R. M. Carey, P. Cushman, G. T. Danby, P. T. Debevec, M. Deile, H. Deng, W. Deninger, S. K. Dhawan, V. P. Druzhinin, L. Duong, E. Efstathiadis, F. J. M. Farley, G. V. Fedotovich, S. Giron, F. Gray, D. Grigoriev, M. Grosse-Perdekamp, A. Grossmann, M. F. Hare, D. W. Hertzog, V. W. Hughes, M. Iwasaki, K. Jungmann, D. Kawall, M. Kawamura, B. I. Khazin, J. Kindem, F. Krienen, I. Kronkvist, R. Larsen, Y. Y. Lee, I. Logashenko, R. McNabb, W. Meng, J. Mi, J. P. Miller, W. M. Morse, D. Nikas, C. J. G. Onderwater, Y. Orlov, C. S. Ozben, J. M. Paley, C. Polly, J. Pretz, R. Prigl, G. zu Putlitz, S. I. Redin, O. Rind, B. L. Roberts, N. Ryskulov, S. Sedykh, Y. K. Semertzidis, Yu. M. Shatunov, E. P. Sichtermann, E. Solodov, M. Sossong, A. Steinmetz, L. R. Sulak, C. Timmermans, A. Trofimov, D. Urner, P. von Walter, D. Warburton, D. Winn, A. Yamamoto, D. Zimmerman

2 STATISTICAL ERRORS AND CORRELATIONS OF FIT PARAMETERS

In a typical data analysis procedure, the fit parameters values x_i are obtained by minimization of $\chi^2(\boldsymbol{x})$:

$$\chi^2(\boldsymbol{x}) \equiv \sum_n \frac{(f(\boldsymbol{x};t_n) - \mathcal{N}_n)^2}{f(\boldsymbol{x};t_n)} \tag{1}$$

where $f(\boldsymbol{x};t_n)$ is a fit function and t_n and $\mathcal{N}_n$ are center of the n^{th} channel of histogram and its content, respectively. Statistical errors and correlations of fit parameters can be found from equation:

$$\Delta\chi^2(\delta\boldsymbol{x}) = 1\,, \qquad \text{where} \quad \Delta\chi^2(\delta\boldsymbol{x}) = \chi^2(\boldsymbol{x}) - \chi^2(\boldsymbol{x}_\circ) \quad \text{and} \quad \delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_\circ. \tag{2}$$

This equation defines an ellipsoid of errors : $$\Delta\chi^2(\delta\boldsymbol{x}) \approx \sum_i \sum_j a_{ij}\,\delta x_i\,\delta x_j = 1 \tag{3}$$

$$\text{where} \quad a_{ij} = \sum_n \left(f^{-1} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right)_{\boldsymbol{x} = \boldsymbol{x}_\circ} \quad \text{or} \quad a_{ij} \approx \frac{N}{\int f\,dt} \int f^{-1} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}\,dt \tag{4}$$

are coefficients of error matrix $\mathcal{A}$. Here N is total number of events in the histogram. From the definition of statistical errors of fit parameters $\sigma_i \equiv \sigma(x_i)$, illustrated in Fig.1, one can prove that

$\sigma_i = \sqrt{\dfrac{M_{ii}}{det(\mathcal{A})}}$, where M_{ii} is the ii^{th} minor of matrix $\mathcal{A}$.

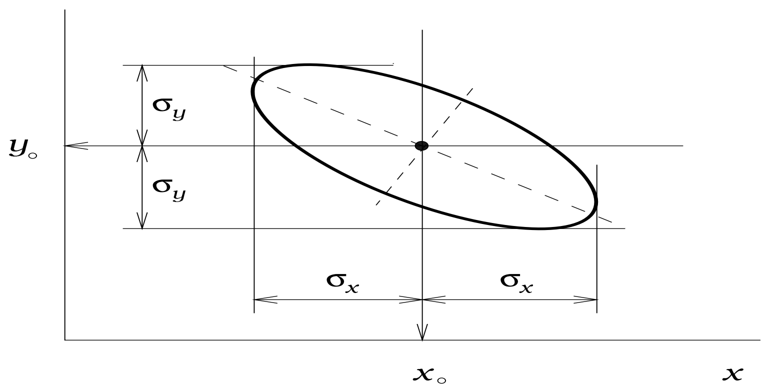

Fig. 1: Ellipsoid of errors and definition of statistical errors σ_x and σ_y

In this presentation we analyse the properties of five parameter distribution $G(t)$:

$$G(t) = N_\circ\, e^{-t/\tau}[1 + A\cos(\omega t + \phi)] \tag{5}$$

which (with or without minor corrections) is the time distribution of high energy electrons ($E > E_{thr}$) from muon decays in the muon g-2 experiment. This distribution is shown schematically in Fig.2(left). Three special points are marked in this plot: moment of injection, which is often chosen as the time origin $t = 0$ and at which we may have some knowledge of phase ϕ of g-2 oscillations; data taking start time T and the center of gravity of distribution $(T + \tau)$. For technical reasons it is convenient sometimes to set the time origin $t = 0$ at the center of gravity. In such a case $T = -\tau$.

For this g-2 distribution, eq.(3) for the ellipsoid of errors can be written in a simplified matrix form as

$$\left(\frac{\sqrt{N}}{N_\circ}\,\delta N_\circ \;\; \frac{\sqrt{N}}{\tau}\,\delta\tau \;\; \frac{\sqrt{N}}{\sqrt{2}}\,\delta A \;\; \frac{\sqrt{N}}{\sqrt{2}} A\tau\,\delta\omega \;\; \frac{\sqrt{N}}{\sqrt{2}} A\,\delta\phi \right) \times \mathcal{A} \times \begin{pmatrix} \sqrt{N}/N_\circ & \delta N_\circ \\ \sqrt{N}/\tau & \delta\tau \\ \sqrt{N/2} & \delta A \\ \sqrt{N/2}\,A\tau & \delta\omega \\ \sqrt{N/2}\,A & \delta\phi \end{pmatrix} = 1 \tag{6}$$

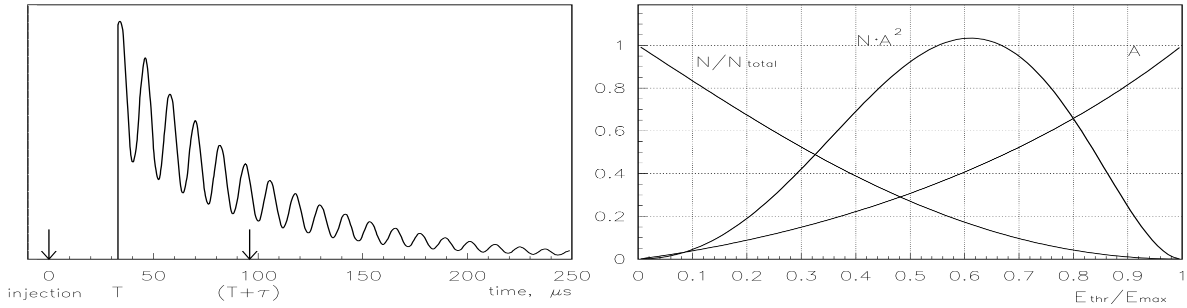

Fig. 2: **Left:** Distribution of decay electrons. Marked are injection point, data taking start time T and distribution's center of gravity $(T+\tau)$; **Right:** Number of events N, asymmetry A and $(N\cdot A^2)$ as function of energy threshold E_{thr}

where the error matrix $\mathcal{A}$ (in general and for $T=-\tau$) is:

$$\mathcal{A}=\begin{pmatrix} 1 & \frac{T}{\tau}+1 & 0 & 0 & 0 \\ \frac{T}{\tau}+1 & (\frac{T}{\tau}+1)^2+1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (\frac{T}{\tau}+1)^2+1 & \frac{T}{\tau}+1 \\ 0 & 0 & 0 & \frac{T}{\tau}+1 & 1 \end{pmatrix}, \quad \text{for } T=-\tau \quad \mathcal{A}=\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \tag{7}$$

Equations (6) and (7) give a solution for the statistical errors of the parameters:

$$\sigma_{N_\circ}=\frac{N_\circ}{\sqrt{N}}\sqrt{(T/\tau+1)^2+1} \quad \sigma_\tau=\frac{\tau}{\sqrt{N}} \quad \sigma_A=\frac{\sqrt{2}}{\sqrt{N}} \quad \sigma_\omega=\frac{\sqrt{2}}{\tau A\sqrt{N}} \quad \sigma_\tau=\frac{\sqrt{2}}{A\sqrt{N}}\sqrt{(T/\tau+1)^2+1} \tag{8}$$

from which σ_ω is most important in this experiment. As follows from eq.(8), the figure of merit for σ_ω as a function of energy threshold is $N(E_{thr})\cdot A^2(E_{thr})$. A plot of $N(E_{thr})\cdot A^2(E_{thr})$ is shown in Fig.2(right), it gets the maximal value at about 0.6 of maximal energy $E_{max}=3.1\,$GeV.

As follows from eq.(7), there is statistical correlation between the g-2 frequency ω and the phase ϕ (as well as between $N_\circ$ and τ), unless $T=-\tau$. This correlation can improve σ_ω if some external knowledge of phase is available. In the muon g-2 experiment such a knowledge might come from the polarization of the muon beam at injection. An exact knowledge of ϕ is equivalent to fixing it, i.e. excluding it from the list of fit parameters. That gives

$$\sigma_\omega=\frac{\sqrt{2}}{\tau A\sqrt{N}}\times\frac{1}{\sqrt{(T/\tau+1)^2+1}} \tag{9}$$

i.e. improvement by factor of $\sqrt{(T/\tau+1)^2+1}$, which is $\sim\sqrt{3.25}$ for our experiment where $T/\tau\approx 1/2$. In reality, the phase ϕ is known at injection time with some precision σ_F. In this case the equation

$$\frac{(\delta\phi)^2}{\sigma_F^2}+\Delta\chi^2(\delta N_\circ,\delta\tau,\delta A,\delta\omega,\delta\phi)=1 \tag{10}$$

should be considered instead of eqs.(2) and (3). The result for σ_ω in this case is

$$\sigma_\omega=\frac{\sigma_{\omega\circ}}{\sqrt{\frac{\sigma_F^{-2}}{(\tau\sigma_{\omega\circ})^{-2}+\sigma_F^{-2}}(T/\tau+1)^2+1}}, \qquad \text{where} \quad \sigma_{\omega\circ}\equiv\frac{\sqrt{2}}{\tau A\sqrt{N}} \tag{11}$$

3 SYSTEMATIC SHIFT OF FIT PARAMETERS DUE TO LOW LEVEL BACKGROUND

The systematic shift of fit parameters $\delta\boldsymbol{x}$ due to the presence of an unidentified low level background $h(t)$ can be found from the $\chi^2(\boldsymbol{x})$ minimization:

$$0 \;=\; \frac{\partial\chi^2}{\partial x_i}=\frac{\partial}{\partial x_i}\sum_n\frac{(f-\mathcal{N}_n)^2}{f}\approx 2\sum_n\frac{f-\mathcal{N}_n}{f}\,f_i'\approx 2\sum_n\frac{f-f_\circ-h}{f}\,f_i'\approx$$

$$\approx \quad 2 \sum_n \frac{\sum_{j=1}^{L}\left(f_j'\,\delta x_j\right)-h}{f}\; f_i' = 2\sum_{j=1}^{L}\delta x_j \sum_n \frac{f_j'\,f_i'}{f} - 2\sum_n \frac{h}{f} f_i' = 0 \tag{12}$$

where $f{=}f(\boldsymbol{x};t_n)$, $f_\circ{=}f(\boldsymbol{x}_\circ;t_n)$, $f_i' = (\partial f/\partial x_i)$ and $h{=}h(t_n)$. Thus

$$\sum_{j=1}^{L}\delta x_j \int \frac{f_i'\,f_j'}{f}\,dt = \sum_{j=1}^{L} s_{ij}\;\delta x_j = \int h\,\frac{f_i'}{f}\,dt\,, \qquad i=1,2,...,L \tag{13}$$

Here $s_{ij} = \int \frac{f_i'\,f_j'}{f}\,dt$ are the elements of the shift matrix $\mathcal{S}$ and L is the number of fit parameters, i.e. the number of components of vector $\boldsymbol{x}$. For the g-2 function $G(t)$ and for $T=-\tau$, the matrix $\mathcal{S}$ is diagonal:

$$\begin{pmatrix} \frac{e\,\tau}{N_\circ} & 0 & 0 & 0 & 0 \\ 0 & \frac{N_\circ\,e}{\tau} & 0 & 0 & 0 \\ 0 & 0 & \frac{N_\circ\,e\,\tau}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{N_\circ\,A^2 e\,\tau^3}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{N_\circ\,A^2 e\,\tau}{2} \end{pmatrix} \times \begin{pmatrix} \delta N_\circ \\ \delta\tau \\ \delta A \\ \delta\omega \\ \delta\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{N_\circ}\int_{-\tau}^{\infty} h(t)\,dt \\ \frac{1}{\tau^2}\int_{-\tau}^{\infty} t\,h(t)\,dt \\ \int_{-\tau}^{\infty}\frac{h(t)\,\cos(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \\ -A\int_{-\tau}^{\infty}\frac{t\,h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \\ -A\int_{-\tau}^{\infty}\frac{h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \end{pmatrix} \tag{14}$$

As an example of application of eq.(14) we consider the systematic shift of ω due to finite binwidth. Numerical routines like PAW/MINUIT usually set the fit function value at the center of the histogram channel, $f(t_c)$, rather than the average function over the bin width b, $\overline{f(t)}$. This effectively introduces a systematic shift by $h(t) = \overline{f(t)} - f(t_c)$:

$$\begin{aligned} h(t) \quad &= \quad \overline{f(t)} - f(t_c) = \left(\frac{1}{b}\int_{-b/2}^{b/2} f(t_c+\xi)\,d\xi\right) - f(t_c) \approx \\ &\approx \quad \left(\frac{1}{b}\int_{-b/2}^{b/2}\left[f(t_c) + \xi\,\dot f(t_c) + \frac{1}{2}\,\xi^2\,\ddot f(t_c)\right]d\xi\right) - f(t_c) = \frac{b^2}{24}\ddot f(t_c) \end{aligned} \tag{15}$$

For the g-2 function $G(t)$, the background function is : $h(t) = \frac{b^2}{24}\ddot G(t) = \frac{b^2}{24}\,\omega^2 N_\circ e^{-t/\tau}\times$

$$\times\left[\epsilon^2 - A(1-\epsilon^2)\cos(\omega t+\phi) + 2A\epsilon\,\sin(\omega t+\phi)\right] \approx -\frac{b^2}{24}\,\omega^2\,N_\circ\,A\,e^{-t/\tau}\,\cos(\omega t+\phi) \tag{16}$$

Here we neglect terms of relative order $\epsilon = (\omega\tau)^{-1} = 0.01$. The systematic shift in g-2 frequency is:

$$\begin{aligned} \frac{\delta\omega}{\omega} \quad &\approx \quad \frac{1}{\omega}\,\frac{2}{e\,N_\circ\,A^2\,\tau^3}\,\frac{b^2}{24}\,\omega^2\,N_\circ\,A\int_{-\tau}^{\infty} A\,t\,e^{-t/\tau}\,\frac{\cos(\omega t+\phi)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt = \\ &= \quad \frac{\omega\,b^2}{12\,e\,\tau^3}\times\frac{e\tau}{2\omega}\left[-\frac{1}{2}\cos 2(\omega\tau-\phi) + \frac{A}{2}\left(\cos(\omega\tau-\phi) + \frac{1}{3}\cos 3(\omega\tau-\phi)\right)\right] \end{aligned}$$

Here we neglect terms of relative order $(A/2)^2 \approx 0.04$. Finally we have

$$\left|\frac{\delta\omega}{\omega}\right| \le \frac{b^2}{24\,\tau^2}\,(0.5+0.2+0.07) = 3.2\cdot 10^{-2}\times\frac{b^2}{\tau^2} \tag{17}$$

For a typical binwidth $b = 0.15\,\mu$s and for $\tau = 64\,\mu$s, $|\delta\omega/\omega| \le 0.17$ ppm

4 FLUCTUATIONS OF FIT PARAMETERS VERSUS FLUCTUATIONS OF $\mathcal{N}_n$

In this section we denote $\boldsymbol{x}_\circ$ to be the "true" values of fit parameters and $\boldsymbol{x}$ be their "specific" values for some particular histogram. These "specific" values differ from the "true" ones by statistical fluctuations $\Delta\boldsymbol{x}$ which ultimately arise from statistical fluctuations of the number of events in individual histogram channels, $\mathcal{N}_n$. For χ^2 in this case we have:

$$\chi^2(\boldsymbol{x}_\circ + \Delta\boldsymbol{x}) = \sum_n \frac{[f(\boldsymbol{x}_\circ + \Delta\boldsymbol{x}; t_n) - \mathcal{N}_n]^2}{\sigma_n^2} \approx \sum_n \frac{(f_\circ + \sum_i f_i' \, \Delta x_i - \mathcal{N}_n)^2}{f_\circ} \tag{18}$$

where $f_i' = (\partial f/\partial x_i)_{\boldsymbol{x}=\boldsymbol{x}_\circ}$, $f_\circ = f(\boldsymbol{x}_\circ; t)$ and $\sigma_n^2 = f \approx f_\circ$. From $\partial\chi^2/\partial x_i = 0$ we have a system of L linear equations for Δx_i :

$$\sum_n f_i' \, \frac{f_\circ + \left(\sum_j f_j' \, \Delta x_j\right) - \mathcal{N}_n}{f_\circ} = 0 \tag{19}$$

which can be written in matrix form as $\mathcal{A} \times \Delta\boldsymbol{x} = \boldsymbol{b}$ with matrix elements a_{ij} of $\mathcal{A}$ and components b_i of vector $\boldsymbol{b}$ being

$$a_{ij} = \sum_n \frac{f_i' \, f_j'}{f_\circ} \quad \text{and} \quad b_i = \sum_n \frac{f_i'}{f_\circ} \, (\mathcal{N}_n - f_\circ) \tag{20}$$

The solution for eq. (19) is : $$\Delta x_j = \frac{\sum_{i=1}^L A_{ij} \, b_i}{det(\mathcal{A})}, \quad A_{ij} = (-1)^{i+j} \, M_{ij} \tag{21}$$

where A_{ij} are the cofactors of elements a_{ij} of matrix $\mathcal{A}$ and M_{ij} are its minors. Equations (20) and (21) for $\Delta\boldsymbol{x}$ versus $\mathcal{N}_n$ are extremely useful. In particular, such fundamental relations as

$$\sigma_i = \sqrt{\langle(\Delta x_i)^2\rangle} = \sqrt{\frac{M_{ii}}{det(\mathcal{A})}} \quad \text{and} \quad \rho_{ij} \equiv \frac{\langle\Delta x_i \, \Delta x_j\rangle}{\sigma_i \, \sigma_j} = \frac{A_{ij}}{\sqrt{M_{ii} \, M_{jj}}} = (-1)^{i+j} \frac{M_{ij}}{\sqrt{M_{ii} \, M_{jj}}} \tag{22}$$

can be derived with use of these equations. Here $\langle ... \rangle$ is an average over ensemble of similar histograms.

4.1 Application of eqs. (20) and (21) for set−subset relations for muon g-2 experiment

For systematic studies in the muon g-2 experiment we compare results of parameter optimization for the full set of data and for a subset, for which we subtract some particular energy bin (rise E_{thr}) from the full set. χ^2 optimization gives us $\Delta\omega_1$ for the first case (full set) and $\Delta\omega_2$ for the second case (subset). Five parameter functions $G_1(t)$ and $G_2(t)$, describing the set and subset, have different normalization constants ($N_{\circ 1}$, $N_{\circ 2}$), asymmetries (A_1, A_2), phases (ϕ_1, ϕ_2), total number of events (N_1, N_2) and statistical errors ($\sigma_{\omega 1}$, $\sigma_{\omega 2}$, etc.) but have the same parameters ω and τ. We also introduce the function $G_3(t) = G_1(t) - G_2(t)$ which describes this subtracted energy bin alone. We assume $T = -\tau$ for simplicity (no correlations). Let us construct $\Delta\omega_1 - \Delta\omega_2 =$

$$= \sum_n \left[\sigma_{\omega 1}^2 \, \frac{\partial G_1}{\partial\omega} \, \frac{1}{G_1} - \sigma_{\omega 2}^2 \, \frac{\partial G_2}{\partial\omega} \, \frac{1}{G_2}\right] (\mathcal{N}_{n2} - G_2) + \sigma_{\omega 1}^2 \sum_n \frac{\partial G_1}{\partial\omega} \, \frac{\mathcal{N}_{n3} - G_3}{G_1} \tag{23}$$

and evaluate $\left\langle(\Delta\omega_1 - \Delta\omega_2)^2\right\rangle =$

$$= \sum_n \left[\sigma_{\omega 1}^2 \, \frac{\partial G_1}{\partial\omega} \, \frac{1}{G_1} - \sigma_{\omega 2}^2 \, \frac{\partial G_2}{\partial\omega} \, \frac{1}{G_2}\right]^2 \left\langle(\mathcal{N}_{n2} - G_2)^2\right\rangle + \sigma_{\omega 1}^4 \sum_n \left(\frac{\partial G_1}{\partial\omega}\right)^2 \frac{\langle(\mathcal{N}_{n3} - G_3)^2\rangle}{G_1^2} =$$

$$= \sum_n \left[\sigma_{\omega 1}^2 \, \frac{\partial G_1}{\partial\omega} \, \frac{1}{G_1} - \sigma_{\omega 2}^2 \, \frac{\partial G_2}{\partial\omega} \, \frac{1}{G_2}\right]^2 G_2 \;+\; \sigma_{\omega 1}^4 \sum_n \left(\frac{\partial G_1}{\partial\omega}\right)^2 \frac{G_3}{G_1^2} =$$

$$= \sum_n \left[\sigma_{\omega 1}^4 \left(\frac{\partial G_1}{\partial \omega}\right)^2 \frac{G_2}{G_1^2} - 2\,\sigma_{\omega 1}^2\,\sigma_{\omega 2}^2 \left(\frac{\partial G_1}{\partial \omega}\right)\left(\frac{\partial G_2}{\partial \omega}\right) \frac{1}{G_1} + \sigma_{\omega 2}^4 \left(\frac{\partial G_2}{\partial \omega}\right)^2 \frac{1}{G_2} + \sigma_{\omega 1}^4 \left(\frac{\partial G_1}{\partial \omega}\right)^2 \frac{G_3}{G_1^2} \right] =$$

$$= \sigma_{\omega 1}^4 \sum_n \left(\frac{\partial G_1}{\partial \omega}\right)^2 \frac{1}{G_1} - 2\,\sigma_{\omega 1}^2\,\sigma_{\omega 2}^2 \sum_n \left(\frac{\partial G_1}{\partial \omega}\right)\left(\frac{\partial G_2}{\partial \omega}\right) \frac{1}{G_1} + \sigma_{\omega 2}^4 \sum_n \left(\frac{\partial G_2}{\partial \omega}\right)^2 \frac{1}{G_2} =$$

$$= \sigma_{\omega 1}^2 - 2\,\sigma_{\omega 1}^2\,\sigma_{\omega 2}^2 \sum_n \left(\frac{\partial G_1}{\partial \omega}\right)\left(\frac{\partial G_2}{\partial \omega}\right) \frac{1}{G_1} + \sigma_{\omega 2}^2 \tag{24}$$

Evaluating the sum in the correlation term explicitly:

$$\sum_n \left(\frac{\partial G_1}{\partial \omega}\right)\left(\frac{\partial G_2}{\partial \omega}\right) \frac{1}{G_1} \approx \int_{-\tau}^{\infty} \frac{N_{\circ 2}\,A_1\,A_2\,t^2\,e^{-t/\tau}\,\sin(\omega t + \phi_1)\,\sin(\omega t + \phi_2)}{1 + A_1\,\cos(\omega t + \phi_1)} \frac{dt}{b} \approx$$

$$\approx \frac{N_{\circ 2}\,A_1\,A_2\,\cos(\phi_1 - \phi_2)}{2\,b} \int_{-\tau}^{\infty} t^2\,e^{-t/\tau}\,dt = \frac{N_{\circ 2}\,A_1\,A_2\,\cos(\phi_1 - \phi_2)}{2\,b}\,\tau^3\,e =$$

$$= \frac{A_1}{A_2}\,\cos(\phi_1 - \phi_2)\,\frac{N_2\,A_2^2\,\tau^2}{2} = \frac{A_1}{A_2}\,\cos(\phi_1 - \phi_2)\;\sigma_{\omega 2}^{-2} \tag{25}$$

Finally : $$\left\langle (\omega_1 - \omega_2)^2 \right\rangle = \left\langle (\Delta\omega_1 - \Delta\omega_2)^2 \right\rangle = \sigma_{\omega 2}^2 - \sigma_{\omega 1}^2 \left(2\,\frac{A_1}{A_2}\,\cos(\phi_1 - \phi_2) - 1 \right) \tag{26}$$

Note that $$\left\langle (\omega_1 - \omega_2)^2 \right\rangle = \sigma_{\omega 2}^2 - \sigma_{\omega 1}^2 \quad \text{when} \;\; A_1 = A_2 \;\; \text{and} \;\; \phi_1 = \phi_2 \tag{27}$$

An even more common check for systematic errors in the muon g-2 data analysis is a test for stability of the g-2 frequency ω versus histogram fit start time. For that test we compare our result ω_1, obtained from the "standard" largest possible fitting time interval, with another result ω_2, obtained from fitting over a smaller time interval. The "standard" time interval starts from some time T_1 after muon beam injection and lasts about 10 lifetimes. For the "smaller" time interval we shift the fit start time forward to some time T_2. Evaluation of $\langle (\omega_1 - \omega_2)^2 \rangle$ for this case is more elaborate as compared to eqs.(23)-(26) and is not given in this presentation. The result is simple though:

$$\left\langle (\omega_1 - \omega_2)^2 \right\rangle = \left\langle (\Delta\omega_1 - \Delta\omega_2)^2 \right\rangle = \sigma_{\omega 2}^2 - \sigma_{\omega 1}^2 \tag{28}$$

Fig.3 shows the g-2 frequency ω as a function of the fit start time. The solid line represents the one sigma band $\omega(30\,\mu s) \pm \sqrt{\sigma^2(t) - \sigma^2(30\,\mu s)}$. Apparently, the plot reveals no deviation more than one sigma.

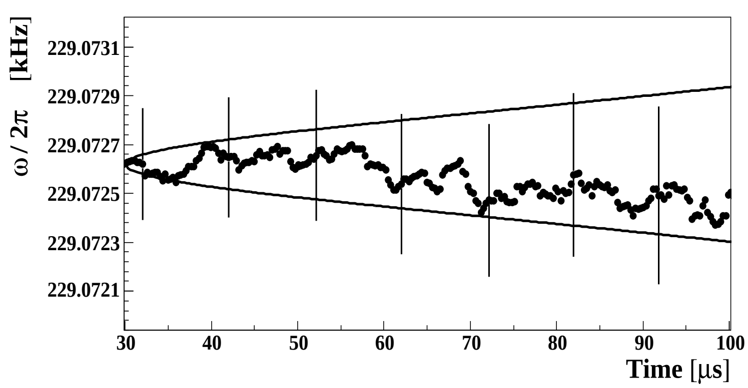

Fig. 3: Frequency ω versus fit start time

We have found that for fit start time scan and for similar situations, eq.(28) is actually valid for any distribution no matter how many parameters it has and whether or not they correlate to each other. More details of all derivations, given and mentioned in this presentation, can be found in our upcoming publications in (tentatively) Nucl. Instr. and Methods.