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Quantum Chaos: Degree of Reversibility of Quantum Dynamics of Classically Chaotic Systems

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We present a quantitative analysis of the reversibility properties of classically chaotic quantum motion by relating the degree of reversibility to the rate at which a quantum state acquires a more and more complicated structure during its time evolution. This complexity can be characterized by the number $\mathcal{M}(t)$ of harmonics of the (initially isotropic, i.e. $\mathcal{M}(0) = 0$) Wigner function, which are generated by the time t. We show that, in contrast to the classical exponential increase, this number can grow after the Ehrenfest time t_E not faster than linearly. It follows that if the motion is reversed at some arbitrary moment T immediately after applying an instant perturbation with intensity described by the parameter ξ, then there exists a critical perturbation strength, $\xi_c \approx \sqrt{2}/\mathcal{M}(T)$, below which the initial state is well recovered, whereas reversibility disappears when $\xi \gtrsim \xi_c(T)$. In the classical limit the number of harmonics proliferates exponentially with time and the motion becomes practically irreversible. The above results are illustrated in the example of the kicked quartic oscillator model.

Keywords: Quantum chaos; instability; complexity; reversibility.

1. Introduction

Extreme sensitivity of the phase space trajectories to initial conditions and system's parameters, which is characterized by positive Lyapunov exponent or, in more general terms, positive *algorithmic complexity*, is the very essence of the classical dynamical chaos. Such an exponential instability of classical motion results in extraordinary complexity of random and unpredictable classical trajectories. In computer simulations of classical motion this leads to rapid loss of memory and practical irreversibility even though the exact equations of motion are reversible. Any, however small, imprecision such as computer round-off errors, is magnified by the exponential instability of the classical orbits to the extent that the memory of the initial conditions is effaced. In other words, due to this instability smaller and smaller scales of the classical phase space are explored during evolution of a phase space distribution exponentially fast with time. These fine details are lost due to the finite accuracy of numerical simulations, and therefore the reversibility of the evolution is destroyed.

In contrast, almost exact reversion is observed in numerical simulations of the quantum motion of classically chaotic systems, even in the regime in which statistical phenomena such as deterministic diffusion take place.[1,2] Qualitatively, this crucial difference is explained by a much simpler structure of quantum states as compared to the complexity of random classical trajectories.[3,4] Unfortunately, the concept of complexity formulated in terms of the exponential instability of the phase trajectories cannot be immediately transferred to quantum mechanics, where the very notion of trajectory is irrelevant and there is no quantum analogue to the Lyapunov exponent. At first glance, there is no means to measure comparative complexity of classical and quantum states of motion.[5]

However, individual classical trajectories are, in essence, of minor interest if the motion is chaotic. They all are alike in this case and rather behavior of manifolds of them carries really valuable information. Therefore the methods of the phase space and the Liouville form of the classical mechanics become the most adequate. It is very important that, opposite to the classical trajectories, the classical phase space distribution and the Liouville equation have direct quantum analogs. Hence, a comparison between classical and quantum dynamics can be made by studying the evolutions in time of the classical and quantum phase space distributions expressed in similar canonical action I and angle θ variables and both ruled by linear equations.

The exponentially fast structuring of the system's phase space on finer and finer scales, which is the paramount property of the classical chaotic dynamics, is restricted in quantum case by the quantization of the phase space. This makes quantal phase-space quasi-distribution, e.g. the quantum Wigner function,"simpler" in comparison with its classical counterpart. In particular, while in the case of chaotic motion the mean number $\mathcal{M}(t)$ of θ-Fourier components of the classical phase-space distribution grows exponentially in time, similar growth in the case of the quantum Wigner function turns out, as we will show below, to be impossible. We notice in this connection that this number also increases much slower (only power-like) if the classical motion is regular.

It is intuitive that the degree of reversibility of a motion should depend on the ratio with which complexity of the state grows. Nevertheless, a rigorous link between the intuitively expected different degrees of reversibility of quantum and classical motion and the structure developed by the phase-space distributions during dynamical evolution has never been established before. We aim to clarify this problem.

2. The Phase Space Approach

Let $\hat{H} \equiv H(\hat{a}^\dagger, \hat{a}; t) = H^{(0)}(\hat{n} = \hat{a}^\dagger \hat{a}) + H^{(1)}(\hat{a}^\dagger, \hat{a}; t)$ be the Hamiltonian of a generic nonlinear system with a bounded below discrete energy spectrum $E_n^{(0)} \geq 0$, which is driven by a time-dependent force of such a kind that the classical motion exhibits a transition from integrable to chaotic behavior when the strength of the driving force is increased. Here $\hat{a}^\dagger, \hat{a}$ are the bosonic, creation-annihilation operators: $[\hat{a}, \hat{a}^\dagger] = 1$. In our analytical study and numerical simulations we considered as an illustrative example the kicked quartic oscillator with Hamiltonian $\hat{H}^{(0)}(\hat{n}) = \hbar\omega_0\hat{n} + \hbar^2\,\hat{n}^2$ driven by periodic kicks $\hat{H}^{(1)}(\hat{a}^\dagger, \hat{a}) = -\sqrt{\hbar}\, g(t)(\hat{a}+\hat{a}^\dagger)$ where $g(t) = g_0 \sum_s \delta(t-s)$. In our units, the time and parameters $\hbar, \omega_0$ as well as the strength of the driving force are dimensionless. The period of the driving force is set to one. The classical dynamics of such a non-linear oscillator becomes chaotic when the kicking strength g_0 exceeds a critical value $g_{0,c} \approx 1$. The angular phase correlations decay in this case exponentially and the mean action grows diffusively with the diffusion coefficient $D = g_0^2$.

We use the method of c-number α-phase space borrowed from the quantum optics (see for example [6–8]). It is, basically, built upon the basis of the coherent states $|\alpha\rangle = \hat{D}\left(\frac{\alpha}{\sqrt{\hbar}}\right)|0\rangle$ obtained from the ground state $|0\rangle$ of the unperturbed Hamiltonian with the help of the unitary displacement operator $\hat{D}(\lambda) = \exp(\lambda\,\hat{a}^\dagger - \lambda^*\hat{a})$. Here α is a complex phase space variable independent of the effective Planck's constant $\hbar$.

The Wigner function W in the α-phase plane is defined by the following Fourier transformation

$$W(\alpha^*, \alpha; t) = \frac{1}{\pi^2\hbar}\int d^2\eta\, e^{(\eta^*\frac{\alpha}{\sqrt{\hbar}} - \eta\frac{\alpha^*}{\sqrt{\hbar}})}\mathrm{Tr}\left[\hat{\rho}(t)\,\hat{D}(\eta)\right], \tag{1}$$

where $\hat{\rho}$ is the density operator and the integration runs over the complex η-plane. The Wigner function is normalized to unity, $\int d^2\alpha W(\alpha^*, \alpha; t) = 1$ and is real though, in general, not positive definite. It satisfies the evolution equation

$$i\frac{\partial}{\partial t}W(\alpha^*, \alpha; t) = \hat{\mathcal{L}}_q\, W(\alpha^*, \alpha; t), \tag{2}$$

with the Hermitian "quantum Liouville operator" $\hat{\mathcal{L}}_q$. This equation reduces in the case $\hbar = 0$ to the classical Liouville equation with respect to the canonical pair $\alpha, i\alpha^*$ with the classical Hamiltonian function being given by the diagonal matrix elements $H_c(\alpha^*, \alpha; t) = \langle\alpha|\hat{H}^{(N)}(\hat{a}^\dagger, \hat{a})|\alpha\rangle$ of the normal form $\hat{H}^{(N)}$ of the quantum Hamiltonian operator. In other words, this function is obtained from the quantum Hamiltonian by substituting $\hat{a} \to \alpha/\sqrt{\hbar},\ \hat{a}^\dagger \to \alpha^*/\sqrt{\hbar}$.

We define the harmonic's amplitudes $W_m(I)$ as the Fourier components of the Wigner function with respect to the angle variable θ introduced by the canonical transformation $\alpha = \sqrt{I}\,e^{-i\theta}$. The normalization condition reduces then to $\int_0^\infty dI\, W_0(I; t) = 1$ whereas the amplitudes of other harmonics are expressed in terms of the matrix elements $\langle n{+}m|\hat{\rho}|n\rangle$ of the density operator along the mth collateral diagonal as

$$\begin{gathered} W_m(I; t) = \tfrac{2}{\hbar}\, e^{-\frac{2}{\hbar}I}\sum_{n=0}^{\infty}(-1)^n\sqrt{\tfrac{n!}{(n+m)!}}\times \\ (4I/\hbar)^{\frac{m}{2}}\, L_n^m\,(4I/\hbar)\,\langle n+m|\hat{\rho}(t)|n\rangle,\quad m \geq 0, \end{gathered} \tag{3}$$

with L_n^m Laguerre polynomials and $W_{-m} = W_m^*$. The inverted relation reads

$$\begin{gathered} \langle n+m\big|\hat{\rho}(t)\big|n\rangle = (-1)^n\, 2\sqrt{\tfrac{n!}{(n+m)!}}\times \\ \int_0^\infty dI\, e^{-2\frac{I}{\hbar}}\ (4I/\hbar)^{\frac{m}{2}}\ L_n^m\,(4I/\hbar)\ W_m\,(I; t)\ . \end{gathered} \tag{4}$$

These formulae allow us to freely translate all relations given below from the language of the density matrix into the language of the Wigner function and back.[9,10]

3. Reversibility and Peres Fidelity

Aiming to connect the reversibility of the motion with the complexity of the Wigner function, we follow the approach developed in [11]. We consider first the forward evolution $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$ of a simple initial (generally mixed) state $\hat{\rho}(0)$ up to some time $t = T$. An instantaneous Hermitian perturbation $\xi\hat{V}\,\delta(t-T)$ with the intensity ξ is then applied which transforms the state $\hat{\rho}(T)$ into $\hat{\tilde{\rho}}(T,\xi) = \hat{P}(\xi)\hat{\rho}(T)\hat{P}^\dagger(\xi)$. The resulting transformation $\hat{P}(\xi) = e^{-i\xi\hat{V}}$ is unitary. For example this transformation is equivalent to the global rotation $W(I,\theta;T) \to W(I,\theta+\xi;T)$ by the angle ξ in the phase plane if the operator $\hat{V} = \hat{n}$. In particular, we use below an infinitesimal perturbation of such a kind to reveal complexity of the Wigner function at the instant T.

The new state $\hat{\tilde{\rho}}(T,\xi)$ serves as the initial condition for the backward evolution $\hat{U}(-T) = \hat{U}^\dagger(T)$ during the same time T, after which the reversed state $\hat{\tilde{\rho}}(0|T,\xi) = \hat{U}^\dagger(T)\hat{\tilde{\rho}}(T,\xi)\hat{U}(T) = \hat{P}(\xi,T)\hat{\rho}(0)\hat{P}^\dagger(\xi,T)$ is finally obtained. Here $\hat{P}(\xi,T) \equiv e^{-i\xi\hat{V}(T)}$, with $\hat{V}(T) \equiv \hat{U}^\dagger(T)\hat{V}\hat{U}(T)$ being the Heisenberg evolution of the perturbation during the time T. At last, to characterize the degree of reversibility we consider the distance between the reversed $\hat{\tilde{\rho}}(0|T,\xi)$ and the initial $\hat{\rho}(0)$ states, as measured by the Peres fidelity[12]

$$F_{\mathrm{rev}}(\xi;T) = \frac{\mathrm{Tr}[\hat{\tilde{\rho}}(0|T,\xi)\hat{\rho}(0)]}{\mathrm{Tr}[\hat{\rho}^2(0)]} = \left.\frac{\mathrm{Tr}[\hat{\tilde{\rho}}(t,\xi)\hat{\rho}(t)]}{\mathrm{Tr}[\hat{\rho}^2(t)]}\right|_{t=T} = F\,(\xi;t)|_{t=T}\,. \quad (5)$$

The fidelity is bounded in the interval $[0,1]$ and is the closer to unity the more similar are the initial and reversed states. The second equality in (5) is a consequence of the unitary time evolution. The fidelity $F(\xi;T)$ measures the *complexity* of the Wigner function at the moment $t = T$ (see below). Both the functions $F_{\mathrm{rev}}(\xi;T)$ and $F(\xi;T)$ are numerically identical but expressed in different variables. The relation (5) plays the key role in our analyses. It allows us not only to relate the degree of reversibility to the complexity of the state at the reversal time T but also *to establish a strong restriction on the upgrowth* of the number $\mathcal{M}(t)$ of harmonics of the Wigner function. The crucial point is that while in classical mechanics the number of Fourier components has no direct physical meaning, in quantum mechanics the number of the components of the Wigner function at any given time is related to the *degree of excitation* of the system (see for example eq. (10) below) and therefore unrestricted exponential growth of this number is not allowed.[13–16]

4. Peres Fidelity and the Number of Harmonics

First of all, let us establish a connection between fidelity and complexity of the Wigner function at an arbitrary moment t. To do this we make the instant rotation in the phase plane $e^{-i\xi\hat{n}}$ at this moment and utilize then the second form of fidelity in Eq. (5). This yields[9,10]

$$F(\xi;t) = 1 - 2\sum_{m=1}^{\infty} \sin^2(\xi m/2)\,\mathcal{W}_m(t) = 1 - \frac{1}{2}\xi^2\,\langle m^2\rangle_t + O(\xi^4)\,, \tag{6}$$

where $\langle m^2\rangle_t = \sum_{m=0}^{+\infty} m^2\,\mathcal{W}_m(t)$ and

$$\mathcal{W}_{m\geqslant 0}(t) = \frac{(2-\delta_{m0})\sum_{n=0}^{\infty}\left|\langle n+m|\hat{\rho}(t)|n\rangle\right|^2}{\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\left|\langle n+m|\hat{\rho}(t)|n\rangle\right|^2}\,. \tag{7}$$

We define then the number of harmonics of the Wigner function at the time t as $\mathcal{M}(t) = \sqrt{\langle m^2\rangle_t}$. This is in line with [14–16]. The set of positive definite quantities $\mathcal{W}_m$ is normalized to unity, $\sum_{m=0}^{+\infty}\mathcal{W}_m = 1$, and can be, therefore, given a probabilistic interpretation. According to (6) the number of harmonics at arbitrary time t is found as

$$\mathcal{M}^2(t) \equiv \langle m^2\rangle_t = -\frac{d^2F(\xi;t)}{d\xi^2}\Big|_{\xi=0}\,. \tag{8}$$

This formula remains valid also in classical limit provided that the fidelity is calculated with the help of the corresponding classical phase-space distribution function.

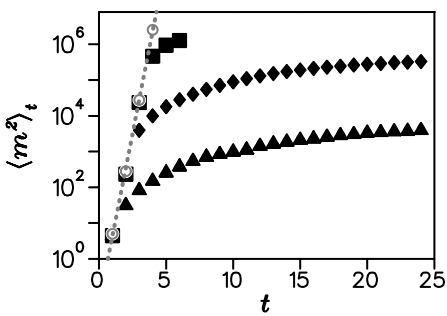

Fig. 1. Root-mean-square $\langle m^2\rangle_t$ versus time t at $g_0 = 1.5$. Squares, diamonds and triangles correspond to $\hbar = 0.01, 0.1$ and 1. Empty circles refer to classical dynamics and the dashed line fits these data.

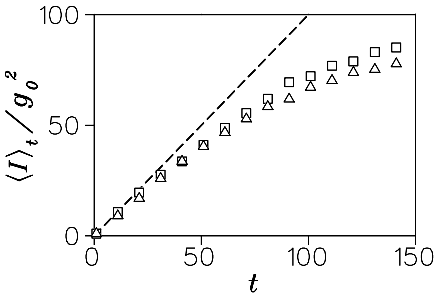

Fig. 2. Quantum diffusion: mean value $\langle I\rangle_t/g_0^2 = \hbar\,\langle n\rangle_t/g_0^2$ in function of time t. Squares and triangles correspond to $(\hbar, g_0)=(1,2)$ and $(2,3)$. The straight line shows the classical diffusion law $\langle I\rangle_t = g_0^2 t$.

The time evolution of $\mathcal{M}^2(t)$ is illustrated for different values of the effective Planck constant $\hbar$ in Fig.1. The initial state is chosen to be a pure ground state $\hat\rho(0) = |0\rangle\langle 0|$ which corresponds to the isotropic Wigner function $W(\alpha^*, \alpha; 0) = 2\,e^{-2|\alpha|^2}$ with the size 1/2. This size is kept constant throughout all calculations whereas the quantum Liouville equation is solved for a number of decreasing values of the Planck's constant thus approaching the classical dynamics. It is clearly seen that the exponential increase of $\langle m^2\rangle_t$ takes place only up to the Ehrenfest time scale $t_E \propto \ln\hbar$, consistently with the findings reported in [14–16]. Then, a much slower power-law increase follows.

To reveal the reason for such a behavior we will analyse now the fidelity $F_{\rm rev}(\xi; t)$ expressed in terms of the density matrices (or, equivalently, the Wigner functions) at the initial time of the forward – backward cycle of evolution. This yields[9,10] $F_{\rm rev}(\xi; t) = |f(\xi, t)|^2$ where

$$f(\xi, t) = \langle 0|e^{-i\xi\hat n(t)}|0\rangle = \mathrm{Tr}\left[e^{-i\xi\hat n}\hat\rho(t)\right] = \sum_{n=0}^{\infty} w_n(t)\, e^{-i\xi n} \tag{9}$$

with $w_n(t) \equiv \langle n|\hat\rho(t)|n\rangle = \left|\langle n|\hat U(t)|0\right|^2$ being the excitation number probability distribution. In such a way we relate the behavior of fidelity to evolution of the excitation numbers. Equating now the ξ^2 terms in expansions of the both possible representations of fidelity we arrive at the following *significant exact relation* between the number of harmonics and the root-mean-square deviation of the action

$$\langle m^2\rangle_t = 2\,\chi_2(t), \quad \chi_2(t) \equiv \langle 0|\hat n(t)^2|0\rangle - \langle 0|\hat n(t)|0\rangle^2\,. \tag{10}$$

Thorough numerical study[9] convince us that after proper averaging over strong irregular fluctuations (*coarse graining*) the excitation number distribution w_n decays exponentially with n,

$$w_n(t) \approx \frac{1}{\langle n\rangle_t + 1}\left[\frac{\langle n\rangle_t}{\langle n\rangle_t + 1}\right]^n, \quad \langle n\rangle_t \equiv \langle 0|\hat{n}(t)|0\rangle\,. \tag{11}$$

It follows then[9,10] that after the Ehrenfest time

$$\mathcal{W}_m(t) = (2-\delta_{m0})\sum_{n=0}^{\infty} w_{n+m}(t)\, w_n(t) \approx \frac{2-\delta_{m0}}{2\langle n\rangle_t + 1}\left[\frac{\langle n\rangle_t}{\langle n\rangle_t + 1}\right]^m, \tag{12}$$

and

$$\langle m^2\rangle_t = 2\,\chi_2(t) \approx 2\langle n\rangle_t^2\,. \tag{13}$$

The time dependence of the mean action $\langle I\rangle_t$ (the deterministic quantum diffusion) is shown in Fig. 2. Thus the relation (13) implies that the number of harmonics grows after the Ehrenfest time not faster than linearly.

5. Reversibility Versus the Number of Harmonics

One can readily show now that for any finite $\xi \ll 1$ fidelity equals in the approximation (11)

$$F_{\text{rev}}(\xi;T) = F(\xi;t=T) = \frac{1}{1+\frac{1}{2}\xi^2\langle m^2\rangle_T}\,. \tag{14}$$

More than that, in fact this formula remains valid for any time including the times $T < T_E$.[9] This formula directly relates reversibility of the motion to complexity of the Wigner function at the reversal T.

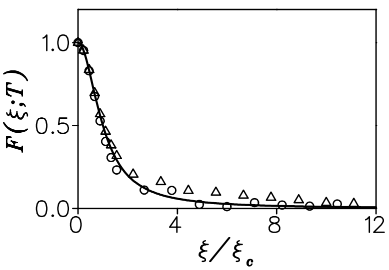

Fig. 3. Fidelity $F(\xi;T)$ versus the scaled variable $\xi/\xi_c(T)$. Data correspond to: (a) $\hbar = 1, g_0 = 2$; circles: $T = 10$; triangles: $T = 50$. The full curve shows the theoretical prediction Eq. (14). The deviations on the tail of the curve are due to fluctuations neglected in (14).

According to Eq. (14) a crossover takes place near the *critical value* $\xi_c(T) \equiv \sqrt{2}/\mathcal{M}(T)$, from good, $F(\xi;T) \approx 1$, to broken, $F(\xi;t) \approx (\xi/\xi_c(T))^2 \ll 1$, reversibility. Our numerical simulations (Fig. 3) nicely confirm this formula.

6. Summary

We have established a quantitative relation between complexity of the Wigner function and degree of reversibility of motion of a classically chaotic quantum system. We have analytically proved that the number of harmonics $\mathcal{M}(t)$ of this function, which can serve as a natural measure of the complexity, increases after the Ehrenfest time *not faster than linearly* in striking contrast with classical dynamics, where the number of harmonics of phase space distribution proliferates exponentially. We have shown that if a quantum motion has been perturbed at some moment T by an external force with intensity ξ and then reversed, its initial state is recovered with the accuracy $\sim (\xi/\xi_c)^2$ as long as the strength is restricted to the interval $0 < \xi < \xi_c(T) = \sqrt{2}/\mathcal{M}(T)$. This interval decreases with the time T at most linearly beyond the semi-classical domain but shrinks exponentially due to the classical exponential instability when this domain is approached.

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