Shear-Flow Modification of Interchange

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Introduction. Stability of interchange modes in presence of a sheared flow has been studied extensively in the past. It was found that all linearly unstable modes are tied either to extremum points of the flow profile or to plasma boundaries[1]. In case of a linear flow profile there are no unstable modes, except for the boundary ones. This result seems to indicate that even a very slow sheared flow can stabilize the local internal interchange. We show that such conclusion is incorrect. By solving the initial problem, we found ballistic-type modes that change form during evolution. They are growing exponentially for a period, which is inversely proportional to the shear-flow amplitude, and then go into the power-law growth phase. Their total kinetic energy grows with the same threshold as for the standard interchange, i.e., such modified interchange modes have the same stability condition as they would have without the flow! The difference is in the constantly changing form of each perturbation. The average radial wave-number of the ballistic interchange is growing with time (linearly, for large times). Thus, the interchange inevitably goes into the dissipative phase and ultimately decays, at least if no other small-scale effects (such as the FLR) are taken into account. This limitation of growth and the reduction of the radial wavelength are the likely causes of the observed favorable influence of shear flows on the convective transport.

Linearized equation for interchange modes. Equation describing the interchange modes in an axially symmetric plasma column with due account for the diamagnetic and the ExB rotation, is well known:

$$\frac{1}{r^2}\frac{d}{dr}S(r)\frac{d\psi}{dr}+\frac{1-m^2}{r^2}S(r)\psi+(\omega^2+m^2\gamma_0^2)\left(\frac{d}{dr}\frac{n_0}{B^2}\right)\psi=0. \tag{1}$$

Here m is the wavenumber, $\omega_0=\omega-m\omega_E$, $\psi=\tilde{\varphi}/\omega_0 r$,

$$S(r)=\omega_0\left(\omega_0-\omega_*\right)\frac{n_0r^3}{B^2}=\left(\left(\omega-m\omega_E-\frac{\omega_*}{2}\right)^2-\frac{\omega_*^2}{4}\right)\frac{n_0r^3}{B^2}, \tag{2}$$

and γ_0 is the instability growth rate of the standard MHD-interchange in the “local approximation”.

This equation should be solved in the whole plasma column with appropriate boundary conditions. However, most eigenmodes are highly localized, and if we are interested in turbulence at a given radius, we should consider a local approximation. In the following we try to analyze this equation, assuming linear radial profile for ω_E and keeping ω_* constant. This can be viewed as a local Taylor expansion of the problem.

Small Larmor radius approximation. Assume that ω_*=0 and ω_E is expanded around r_0

$$\omega_E = \omega_{E0} - x\omega_E', \tag{3}$$

as well as the rest of coefficients. Then the equation looks like

$$\frac{d^2\psi}{dx^2} - \left(\frac{m^2}{r_0^2} + \frac{m^2\lambda^2}{\left(\Omega - mx\omega_E'\right)^2}\right)\psi = 0, \tag{4}$$

where $\lambda \sim \gamma_0$ describes the interchange instability drive. This equation is singular, the singularity being a particular case of the hydrodynamic resonance. The expansion is designed to describe the vicinity of the singularity at small Ω.

The hydrodynamic resonance. According to Timofeev[1], various types of oscillations in sheared flows can be described by equations of the type

$$\varphi_{xx}'' - \left[k^2 - U(x)\right]\varphi = 0, \tag{5}$$

where the function $U(x)$ vanishes when $|x| \to \infty$, and around the hydrodynamic resonance point (where the phase velocity of the mode is equal to the flow velocity) has the form

$$U(x) \propto 1/\left[\omega - kV_0(x)\right]^n. \tag{6}$$

Our case corresponds to a particular choice of $n = 2$...

Timofeev presents a proof that there are no localized solutions, i.e., all possible exponentially growing modes are tied either to boundaries or to inflection points of the flow profile. It is usually interpreted as local stabilization, but such result runs counter to the common sense: a vanishingly weak sheared flow should not be able to affect the behavior of such a robust mode as the interchange.

The initial value problem. Let's find the time evolution of the initial interchange-like perturbation in the vicinity of its hydrodynamic resonance. For this purpose we Fourier-transform the equation in Ω and in mx/r_0:

$$\left(\frac{\partial}{\partial\tau} - u\frac{\partial}{\partial k}\right)^2 \left(k^2+1\right)\psi - \lambda^2\psi = 0, \tag{7}$$

where $u = r_0 \omega'_{E0}$ is the flow-velocity shear.

If we introduce new variables $l = (k+u\tau)/2,\ s = (k-u\tau)/2$, the equation becomes

$$\frac{\partial^2 \bar{\psi}}{\partial s^2} - \frac{\Lambda^2}{(l+s)^2+1}\bar{\psi} = 0, \tag{8}$$

where $\Lambda = \lambda/u$, and $\bar{\psi} = \psi\left(k^2+1\right)$ Now the variable l behaves as a parameter!

The derived time-dependent equation has simple asymptotic solutions in two limiting cases

1. 1. $k{<<}1$: $\bar{\psi}(l,s) = a_1 \exp[\Lambda s] + a_2 \exp[-\Lambda s]$,
2. 2. $k{>>}1$: $\bar{\psi}(l,s) = b_1 (l+s)^{\sigma_1} + b_2 (l+s)^{\sigma_2}$,

where

$$\sigma_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}+\Lambda^2}. \tag{9}$$

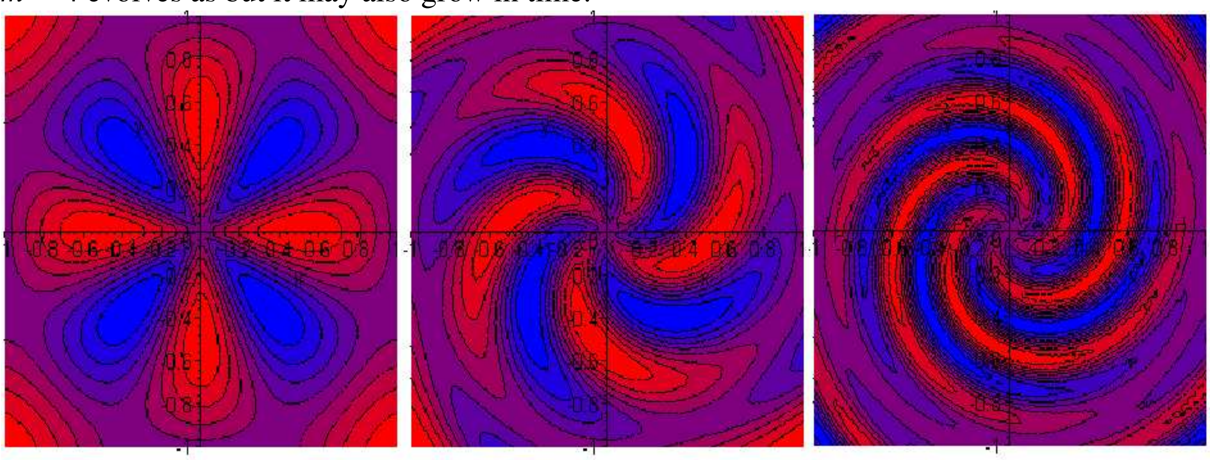
Assume that the initial perturbation $\psi_0(k) = \psi_0(2l)$ is localized at $k{<<}1$ (broad in radial extent). Since l behaves as a parameter, we can look for solutions of the form $\psi(l,s) \approx \psi_0(2l)g(s)$, where the amplitude $g(s)$ satisfies

$$\frac{\partial^2 g}{\partial s^2} - \frac{\Lambda^2}{s^2+1} g = 0, \tag{10}$$

while the location of the perturbation in k-space moves to large $|k|$ with constant velocity u (retaining its form, if it is sufficiently localized).

Conclusion: *For $\tau < 1/u = 1/r_0\omega'_{E0}$ the amplitude grows exponentially, since the $k << 1$ limit applies, and then changes to the power law.*

Evolution of initial perturbation. In the (x,t) coordinates the found asymptotic solution for $m = 4$ evolves as but it may also grow in time.



uτ=0 uτ=2 uτ=5

One can say that it is unstable, if its kinetic energy grows:

$$W \propto t^{2(\sigma_1 - 1)} \tag{11}$$

Thus, the instability threshold is $\sigma_1 > 1$ or $\gamma_0 > 0$, which is exactly the threshold for the standard interchange!

The ion viscosity. It is possible to take into account the finite value of ω_* in a similar manner:

$$\left[\left(\frac{\partial}{\partial\tau} - u\frac{\partial}{\partial k}\right)^2 + \frac{\omega_*^2}{4}\right]\left(k^2+1\right)\psi - \lambda^2\psi = 0. \tag{12}$$

Under the same transformations we get

$$u^2\frac{\partial^2\bar{\psi}}{\partial s^2} - \left(\frac{\lambda^2}{(l+s)^2+1} - \frac{\omega_*^2}{4}\right)\bar{\psi} = 0, \tag{13}$$

so that even a small ω_* drastically changes the asymptotic:

3. $k^2 >> (2\lambda/\omega_*)^2 - 1$: $\bar{\psi}(l,s) = c_1 \exp\left[i\omega_* s/2u\right] + c_2 \exp\left[-i\omega_* s/2u\right]$,

so that the growth stops, even if the mode is not stabilized by the FLR effects according to the usual criterion $(2\lambda/\omega_*)^2 < 1$! At this stage the fluctuation energy decreases.

Summary. We found that the evolution of an initial interchange-like perturbation in a sheared flow goes through several stages. Throughout all of them the radial wavenumber linearly grows with time.

If the flute mode is unstable without the flow, then it will also grow exponentially during the first stage, and with the same growth rate. Duration of this stage is inversely proportional to the flow shear.

The second stage starts when the radial wavenumber of the perturbation becomes larger than the azimuthal one. The growth slows to the power law.

In the third stage the collisionless ion viscosity stops the growth altogether.

It can be concluded that the interchange can be stabilized by a sheared flow only via the drift effects and the ion viscosity. Otherwise its threshold is unchanged, though the mode is of a transient type, and hence the transport estimates are smaller.

Some of the results are published in [2].

References

[1] A.V. Timofeev, in Reviews of Plasma Physics ed. by B.B. Kadomtsev 17, 193 (1992)

[2] A.D.Beklemishev, M.S.Chaschin, Plasma Physics Reports, 2008, Vol. 34, No. 5, pp. 422–430.