Leading and next to leading large n_f terms in the cusp anomalous dimension and the quark–antiquark potential

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I discuss 3 related quantities: the cusp anomalous dimension, the HQET heavy-quark field anomalous dimension, and the quark–antiquark potential. Leading large n_f terms can be calculated to all orders in α_s. Next to leading terms with the abelian color structure C_F^2 also can be found to all orders (but not non-abelian $C_F C_A$ terms). This talk is based on Appendices C and D in [1].

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1. Introduction

The one-loop cusp anomalous dimension

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth\varphi - 1) \tag{1.1}$$

follows from the soft radiation function in classical electrodynamics: when a charge suddenly changes its velocity, it emits electromagnetic waves; integrating the intensity over directions, one obtains [2] $\varphi\coth\varphi - 1$. This result is probably known for more than 100 years, and should be included in The Guinness Book of Records as the anomalous dimension known for a longest time. The two-loop term has been calculated 30 years ago [3] (and rewritten via $\mathrm{Li}_{2,3}$ in [4]). The three-loop term has been calculated recently [5, 6, 1].

The HQET heavy-quark field anomalous dimension (or the anomalous dimension of a straight Wilson line) is known up to 3 loops. At 2 loops, after a wrong calculation [7], the correct result has been obtained in [8], and later in [9, 10, 11, 12]. The three-loop result has been obtained in [13, 14] (in the first paper [13] it has been found as a by-product of the calculation of the QCD on-shell heavy-quark field renormalization constant, from the requirement that the QCD/HQET matching coefficient for the heavy-quark field [15] is finite; at 2 loops this has been done in [11]).

The quark–antiquark potential is known at two [16, 17] and three [18, 19, 20] loops.

Some terms in perturbative series for these quantities can be obtained to all orders in α_s.

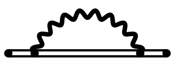
2. Large n_f terms

The terms with the highest power of n_f at each order of perturbation theory for the cusp anomalous dimension Γ have the structures $C_F (T_F n_f)^{L-1} \alpha_s^L$ ($L \geq 1$). They are known to all orders in α_s. The terms with next to highest power of n_f have the structures $C_F^2 (T_F n_f)^{L-2} \alpha_s^L$ and $C_F C_A (T_F n_f)^{L-2} \alpha_s^L$ ($L \geq 3$). The abelian ones (without C_A) can be also found to all orders in α_s. For this purpose it is sufficient to consider QED with n_f massless lepton flavors: $C_F = T_F = 1$, $C_A = 0$, $\beta_0 = -\frac{4}{3} n_f$. Let's introduce

$$b = \beta_0 \frac{\alpha}{4\pi} \,. \tag{2.1}$$

We assume $b \sim 1$ and take into account all powers of b; $1/\beta_0 \ll 1$ is our small parameter, and we consider only a few terms in expansions in $1/\beta_0$.

At the leading and next-to-leading large-β_0 orders (Lβ_0 and NLβ_0), the coordinate-space Wilson line of any shape is equal to

$\log W =$  , (2.2)

where the thick photon line is the full photon propagator with the NLβ_0 accuracy. This simple exponentiation formula is first broken at NNLβ_0 order by the light-by-light diagram (figure 1).

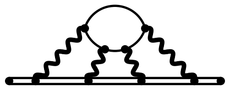
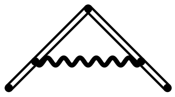
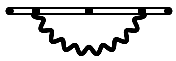


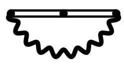
Figure 1: The light-by-light diagram is $n_f\alpha^4$, and hence NNLβ_0.

With the NLβ_0 accuracy the renormalization constant Z of the heavy-to-heavy current (the cusp) is given by

$$\log W(t,t';\varphi) - \log W(t,t';0) =$$

$$-$$

$$= \log Z + \text{finite} \qquad (2.3)$$

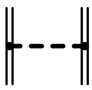
(diagrams where both photon-interaction vertices are before the cusp, or after the cusp, cancel in this difference). Going to momentum space, we can express it via the vertex function $V(\omega,\omega';\varphi)$ (it is convenient to set $\omega' = \omega$, in order to have a single-scale problem):

$$V(\omega,\omega;\varphi) - V(\omega,\omega;0) =$$

$$-$$

$$= \log Z + \text{finite}\,. \qquad (2.4)$$

The HQET field renormalization can be obtained from $V(\omega,\omega;0)$.

The static quark–antiquark potential can be considered similarly. The terms with the highest power of n_f in each order of perturbation theory have the structures $C_F(T_F n_f)^L \alpha_s^{L+1}$ ($L \ge 0$). The terms with next to highest power of n_f have the structures $C_F^2(T_F n_f)^{L-1}\alpha_s^{L+1}$ and $C_F C_A (T_F n_f)^{L-1}\alpha_s^{L+1}$ ($L \ge 2$); we'll consider only the abelian ones. In the Coulomb gauge, up to NLβ_0 the potential is given by the full Coulomb photon propagator

$$V(\vec{q}) =$$

$$= -\frac{e_0^2}{\vec{q}^{\,2}} \frac{1}{1-\Pi(-\vec{q}^{\,2})} \qquad (2.5)$$

($\Pi(q^2)$ is gauge invariant in QED, and can be taken from covariant-gauge calculations). This simple equality is first broken at NNLβ_0 order by the light-by-light diagram (figure 2).

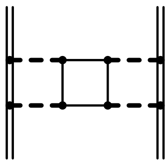


Figure 2: The light-by-light diagram is $n_f\alpha^4$, and hence NNLβ_0.

As discussed in [1], conformal symmetry leads to the relation between $\Gamma(\pi-\delta)$ at $\delta \to 0$ and $V(\vec{q})$:

$$\Delta \equiv \left[\delta\Gamma(\pi-\delta;\alpha_s)\right]_{\delta\to 0} - \frac{\vec{q}^{\,2} V(\vec{q};\alpha_s)}{4\pi} = 0 \qquad (2.6)$$

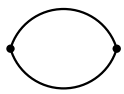
(this relation has been observed in [21] at 2 loops). In QCD (and QED) conformal symmetry is anomalous (thus leading to non-zero β function), and [1]

$$\Delta = \frac{\pi}{108}\beta_0 C_F \left(\frac{\alpha_s}{\pi}\right)^3 (47C_A - 28T_F n_f) + \mathcal{O}(\alpha_s^4)\,. \tag{2.7}$$

3. Leading β_0 order

The photon self energy at the Lβ_0 order is ~ 1:

$$\Pi_0(k^2) = $$



$$= \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \frac{D(\varepsilon)}{\varepsilon} (-k^2)^{-\varepsilon}\,,$$

$$D(\varepsilon) = e^{\gamma\varepsilon} \frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(1-2\varepsilon)(1-\frac{2}{3}\varepsilon)\Gamma(1-2\varepsilon)} = 1 + \frac{5}{3}\varepsilon + \cdots \tag{3.1}$$

The charge renormalization in the $\overline{\text{MS}}$ scheme is

$$\beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} = b Z_\alpha(b) \mu^{2\varepsilon}\,. \tag{3.2}$$

At the Lβ_0 order we can solve the RG equation

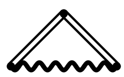
$$\frac{d\log Z_\alpha}{d\log b} = -\frac{b}{\varepsilon + b}$$

and obtain

$$Z_\alpha = \frac{1}{1 + b/\varepsilon}\,. \tag{3.3}$$

The vertex $V(\omega,\omega;\varphi)$ is given by the one-loop diagram with the factor $1/(1-\Pi(k^2))$ inserted in the integrand. At the Lβ_0 order (figure 3) the result can be written in the form

$$V(\omega,\omega;\varphi) = $$



$$= \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon;\varphi)}{L} \Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{3.4}$$

where L is the number of loops and Π_0 (3.1) is taken at $-k^2 = (-2\omega)^2$. Reduction of such integrals to master ones, as well as evaluation of these master integrals, has been considered in [22]. In

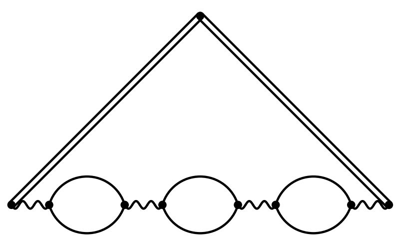


Figure 3: The L-loop vertex diagram at the Lβ_0 order contains $L-1$ Π_0 insertions.

Landau gauge we obtain

$$f(\varepsilon,u;\varphi) = -\frac{(1-\frac{2}{3}\varepsilon)\Gamma(2-2\varepsilon)\Gamma(1-u)\Gamma(1+2u)}{(1-\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(2+u-\varepsilon)} \times \left[\bigl((2+u-2\varepsilon)\cos\varphi - u\bigr)\, {}_2F_1\left(\left. \begin{array}{c} 1, 1-u \\ 3/2 \end{array} \right| \frac{1-\cos\varphi}{2} \right) + 1 \right] \tag{3.5}$$

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added). The function $f(\varepsilon,u;\varphi)$ is regular at the origin:

$$f(\varepsilon,u;\varphi) = \sum_{n,m=0}^{\infty} f_{nm}(\varphi)\varepsilon^n u^m \,. \tag{3.6}$$

The renormalization constant Z can be written as

$$\log Z = \frac{Z_1}{\varepsilon} + \frac{Z_2}{\varepsilon^2} + \cdots\,, \quad Z_n = \mathcal{O}(b^n)\,.$$

Only Z_1 is needed in order to obtain

$$\Gamma(b;\varphi) = -2\frac{dZ_1(b;\varphi)}{d\log b}\,;$$

higher Z_n contain no new information, and are uniquely reconstructed from Z_1 using self-consistency conditions. Choosing

$$\mu^2 = D(\varepsilon)^{-1/\varepsilon}(-2\omega)^2 \to e^{-\frac{5}{3}\varepsilon}(-2\omega)^2$$

we have

$$V(\omega,\omega;\varphi) - V(\omega,\omega;0) = \frac{1}{\beta_0}\sum_{L=1}^{\infty} \frac{\bar{f}(\varepsilon,L\varepsilon;\varphi)}{L}\left(\frac{b}{\varepsilon+b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{3.7}$$

where $\bar{f}(\varepsilon,u;\varphi) = f(\varepsilon,u;\varphi) - f(\varepsilon,u;0)$. We expand in b, expand $\bar{f}(\varepsilon,u;\varphi)$ in ε and u and select only ε^{-1} terms in order to obtain Z_1. All coefficients but f_{n0} cancel:

$$Z_1(b;\varphi) = 2\frac{\varphi\cot\varphi - 1}{\beta_0}\sum_{n=0}^{\infty}\frac{\hat{f}_n}{n+1}(-b)^{n+1}\,,$$

where

$$\bar{f}(\varepsilon,0;\varphi) = -2\hat{f}(\varepsilon)(\varphi\cot\varphi - 1)\,, \quad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty}\hat{f}_n\varepsilon^n\,.$$

Therefore at the Lβ_0 we obtain [23]

$$\begin{aligned}
\Gamma(b;\varphi) &= 4\frac{b}{\beta_0}\gamma_0(b)(\varphi\cot\varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,,\\
\gamma_0(b) &= \hat{f}(-b) = \frac{(1+\frac{2}{3}b)\Gamma(2+2b)}{(1+b)\Gamma^3(1+b)\Gamma(1-b)}\\
&= 1 + \frac{5}{3}b - \frac{1}{3}b^2 - \left(2\zeta_3 - \frac{1}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{10}{3}\zeta_3 - \frac{1}{3}\right)b^4 + \cdots
\end{aligned} \tag{3.8}$$

As a free bonus, we can obtain the HQET field anomalous dimension. The vertex function V at $\varphi = 0$ is related to the HQET propagator S by the Ward identity

$$V(\omega,\omega';0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}\,, \quad V(\omega,\omega;0) = \frac{dS^{-1}(\omega)}{d\omega}\,. \tag{3.9}$$

Therefore the renormalization constant of the HQET quark field Z_h is given by

$$\log V(\omega,\omega';0) = -\log Z_h + \text{finite}\,.$$

Using

$$f(\varepsilon,u;0) = -3\frac{(1-\frac{2}{3}\varepsilon)^2\Gamma(2-2\varepsilon)\Gamma(1-u)\Gamma(1+2u)}{(1-\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(2+u-\varepsilon)}\,,$$

we obtain in the Landau gauge [24]

$$\begin{aligned}
&\gamma_h(b) = 2\frac{b}{\beta_0}\gamma_{h0}(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,,\\
&\gamma_{h0}(b) = f(-b,0;0) = \frac{\left(1+\frac{2}{3}b\right)^2\Gamma(2+2b)}{(1+b)^2\Gamma^3(1+b)\Gamma(1-b)}\\
&\quad = 1 + \frac{4}{3}b - \frac{5}{9}b^2 - \left(2\zeta_3 - \frac{2}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3}\zeta_3 - \frac{7}{9}\right)b^4 + \cdots
\end{aligned} \tag{3.10}$$

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added).

Now we consider the potential $V(\vec{q}\,)$ at the Lβ_0 order. Choosing $\mu^2 = \vec{q}^{\,2}$ we have

$$V(\vec{q}\,) = -\frac{(4\pi)^{D/2}e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon)(\vec{q}^{\,2})^{1-\varepsilon}}\varepsilon\sum_{L=1}^{\infty}\left(D(\varepsilon)\frac{b}{\varepsilon+b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,.$$

The sum here can be written as

$$\sum_{L=1}^{\infty} g(\varepsilon,L\varepsilon)\left(\frac{b}{\varepsilon+b}\right)^L\,, \quad g(\varepsilon,u) = D(\varepsilon)^{u/\varepsilon} = \sum_{n,m=0}^{\infty} g_{nm}\varepsilon^n u^m\,.$$

This sum is equal to

$$\frac{b}{\varepsilon}\sum_{n=0}^{\infty} n!\,g_{0n}b^n + \mathcal{O}(\varepsilon^0)$$

($1/\varepsilon^n$ terms with $n > 1$ vanish, so that $V(\vec{q}\,)$ is automatically finite), where

$$g(0,u) = e^{\frac{5}{3}u}\,, \quad g_{0n} = \frac{1}{n!}\left(\frac{5}{3}\right)^n\,. \tag{3.11}$$

Therefore

$$V(\vec{q}\,) = -\frac{(4\pi)^2}{\vec{q}^{\,2}}\frac{b}{\beta_0}V_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \quad V_0(b) = \frac{1}{1-\frac{5}{3}b}\,. \tag{3.12}$$

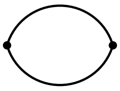
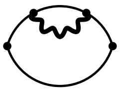
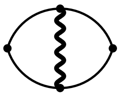
The conformal anomaly (2.6) at the Lβ_0 order is

$$\begin{aligned}
&\Delta = 4\pi\frac{b^3}{\beta_0}\delta_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,,\\
&\delta_0(b) = \frac{V_0(b)-\gamma_0(b)}{b^2} = \frac{28}{9} + 2\left(\zeta_3 + \frac{58}{27}\right)b - \frac{1}{3}\left(\frac{\pi^4}{10} - 10\zeta_3 - \frac{652}{27}\right)b^2 + \cdots
\end{aligned} \tag{3.13}$$

The first term here reproduces the $T_F n_f$ term in (2.7).

4. Next to leading β_0 order

To obtain the photon propagator with the NLβ_0 accuracy, we need the photon self-energy up to $1/\beta_0$:

$\Pi(k^2) =$  $+2$  $+$  $= \Pi_0(k^2) + \dfrac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\dfrac{1}{\beta_0^2}\right)$, (4.1)

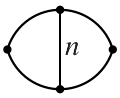
where the photon propagators in Π_1 are taken at the Lβ_0 order. The NLβ_0 contribution can be written in the form [25, 26]

$$\Pi_1(k^2) = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0(k^2)^L . \tag{4.2}$$

Using integration by parts, one can reduce it to

$$\begin{aligned} F(\varepsilon,u) = {} & \frac{2(1-2\varepsilon)^2(3-2\varepsilon)\Gamma^2(1-2\varepsilon)}{9(1-\varepsilon)(1-u)(2-u)\Gamma^2(1-\varepsilon)\Gamma^2(1+\varepsilon)} \\ & \times \left[-u \frac{2-3\varepsilon-\varepsilon^2+\varepsilon(2+\varepsilon)u-\varepsilon u^2}{\Gamma^2(1-\varepsilon)} I(1+u-2\varepsilon) \right. \\ & \left. + 2\frac{2(1+\varepsilon)(3-2\varepsilon)-(4+11\varepsilon-7\varepsilon^2)u+\varepsilon(8-3\varepsilon)u^2-\varepsilon u^3}{(1-u)(2-u)(1-u-\varepsilon)(2-u-\varepsilon)} \frac{\Gamma(1+u)\Gamma(1-u+\varepsilon)}{\Gamma(1-u-\varepsilon)\Gamma(1+u-2\varepsilon)} \right] \\ = {} & \sum_{n,m=0}^{\infty} F_{nm} \varepsilon^n u^m , \end{aligned} \tag{4.3}$$

where the integral

$I(n) =$  $= \dfrac{1}{\pi^d} \displaystyle\int \dfrac{d^d k_1 \, d^d k_2}{k_1^2 k_2^2 (k_1+p)^2 (k_2+p)^2 \left[(k_1-k_2)^2\right]^n}$

(euclidean, $p^2 = 1$) can be expressed via a ${}_3F_2$ function of unit argument [27, 28] (see the review [29] for more references). The ${}_3F_2$ function can be expanded up to any desired order using known algorithms, the coefficients are expressed via multiple ζ values; therefore, the coefficients F_{nm} can be calculated to any desired order.

The function $F(\varepsilon,u)$ simplifies in some cases. In particular [25],

$$F(\varepsilon,0) = \frac{(1+\varepsilon)(1-2\varepsilon)^2(1-\frac{2}{3}\varepsilon)^2\Gamma(1-2\varepsilon)}{(1-\varepsilon)^2(1-\frac{1}{2}\varepsilon)\Gamma(1+\varepsilon)\Gamma^3(1-\varepsilon)} , \tag{4.4}$$

so that F_{n0} contain no multiple ζ values, only ζ_n. Also [26]

$$F(0,u) = \frac{2}{3} \frac{\psi'\left(2-\frac{u}{2}\right) - \psi'\left(1+\frac{u}{2}\right) - \psi'\left(\frac{3-u}{2}\right) + \psi'\left(\frac{1+u}{2}\right)}{(1-u)(2-u)} \tag{4.5}$$

so that F_{0m} contains only ζ_{2n+1} [26]:

$$F_{0m} = -\frac{32}{3} \sum_{s=1}^{[(m+1)/2]} s\left(1-2^{-2s}\right)\left(1-2^{2s-m-2}\right)\zeta_{2s+1} + \frac{4}{3}(m+1)\left(m+(m+6)2^{-m-3}\right) . \tag{4.6}$$

The two-loop case is, of course, trivial:

$$F(\varepsilon,2\varepsilon)=\frac{2}{9\varepsilon^2}\frac{3-2\varepsilon}{1-\varepsilon}\left[2\frac{(1-2\varepsilon)^2(2-2\varepsilon+\varepsilon^2)}{(1-3\varepsilon)(2-3\varepsilon)}\frac{\Gamma(1+2\varepsilon)\Gamma^2(1-2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)}-2+\varepsilon-2\varepsilon^2\right].$$

Let's write the charge renormalization constant Z_α with the NLβ_0 accuracy as

$$Z_\alpha(b)=\frac{1}{1+b/\varepsilon}\left[1+\frac{Z_{\alpha1}(b)}{\beta_0}+\mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right],$$
$$Z_{\alpha1}(b)=\frac{Z_{\alpha11}(b)}{\varepsilon}+\frac{Z_{\alpha12}(b)}{\varepsilon^2}+\cdots,\quad Z_{\alpha1n}=\mathcal{O}(b^{n+1}). \tag{4.7}$$

In the abelian theory, $\log(1-\Pi)$ expressed (3.2) via renormalized b should be equal to $\log Z_\alpha+$ finite. Equating the coefficients of ε^{-1} in the $1/\beta_0$ terms in this relation, we see that $Z_{\alpha11}$ (4.7) is given by the coefficient of ε^{-1} in

$$-\left(1+\frac{b}{\varepsilon}\right)\Pi_1\,.$$

It is convenient to choose

$$\mu^2=D(\varepsilon)^{-1/\varepsilon}(-k^2)\to e^{-\frac{5}{3}\varepsilon}(-k^2)\,,$$

then

$$\Pi_1=3\varepsilon\sum_{L=2}^{\infty}\frac{F(\varepsilon,L\varepsilon)}{L}\left(\frac{b}{\varepsilon+b}\right)^L.$$

We expand in b and expand $F(\varepsilon,u)$ in ε and u; selecting ε^{-1} terms, we find that all coefficients but F_{n0} cancel:

$$Z_{\alpha11}=-3\sum_{n=0}^{\infty}\frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}\,. \tag{4.8}$$

The β function with NLβ_0 accuracy is

$$\beta(b)=b+\frac{\beta_1(b)}{\beta_0}+\mathcal{O}\left(\frac{1}{\beta_0^2}\right), \tag{4.9}$$

where [25, 26]

$$\beta_1(b)=-\frac{dZ_{\alpha11}(b)}{d\log b}=3\sum_{n=0}^{\infty}\frac{F_{n0}(-b)^{n+2}}{n+1}$$
$$=3b^2+\frac{11}{4}b^3-\frac{77}{36}b^4-\frac{1}{2}\left(3\zeta_3+\frac{107}{48}\right)b^5+\frac{1}{5}\left(\frac{\pi^4}{10}-11\zeta_3+\frac{251}{48}\right)b^6+\cdots \tag{4.10}$$

(the coefficients F_{n0} follow from $F(\varepsilon,0)$ (4.4)). The corresponding terms in the 5-loop QED β function [30] are reproduced. We shall need the full $Z_{\alpha1}$, not just $Z_{\alpha11}$; integrating the RG equation with the $1/\beta_0$ accuracy we obtain

$$Z_{\alpha1}(b)=-\varepsilon\int_0^b\frac{\beta_1(b)\,db}{b(\varepsilon+b)^2}=-\frac{3}{2}\frac{b^2}{\varepsilon}+\frac{1}{2}(4+F_{10}\varepsilon)\frac{b^3}{\varepsilon^2}-\frac{1}{4}\left(9+3F_{10}\varepsilon+F_{20}\varepsilon^2\right)\frac{b^4}{\varepsilon^3}+\cdots$$

At the NLβ_0 order we should expand the photon propagator $(1-\Pi_0-\Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$ (Fig. 4). The vertex function (3.7) becomes

$$V(\omega,\omega;\varphi)-V(\omega,\omega;0)=\frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{\bar{f}(\varepsilon,L\varepsilon;\varphi)}{L}\left(\frac{b}{\varepsilon+b}\right)^L \times\left[1+L\frac{Z_{\alpha 1}}{\beta_0}+\frac{3\varepsilon}{\beta_0}\sum_{L'=2}^{L-1}\frac{L-L'}{L'}F(\varepsilon,L'\varepsilon)\right]+\mathcal{O}\left(\frac{1}{\beta_0^3}\right)\,, \quad (4.11)$$

where L' is the number of loops in the Π_1 insertion, and the $1/\beta_0$ correction $Z_{\alpha 1}$ to the charge renormalization (4.7) is taken into account. We expand in b and substitute the expansions (4.3) and (3.6); in Z_1, the coefficient of ε^{-1}, all $\bar{f}_{nm}$ except $\bar{f}_{n0}$ cancel. At the NLβ_0 order the cusp anomalous dimension is determined by the same $\hat{f}_n$ coefficients as at the Lβ_0 order:

$$\Gamma(b;\varphi)=4\left[\frac{b}{\beta_0}\gamma_0(b)-\frac{b^3}{\beta_0^2}\gamma_1(b)\right](\varphi\cot\varphi-1)+\mathcal{O}\left(\frac{1}{\beta_0^3}\right)\,, \quad (4.12)$$

where

$$\gamma_1(b)=-\frac{3}{2}\left[F_{10}+2F_{01}-2\hat{f}_1\right]+\left[2F_{20}+3(F_{11}+F_{02})+3F_{01}\hat{f}_1-6\hat{f}_2\right]b - \left[\frac{3}{4}(3F_{30}+4(F_{21}+F_{12}+F_{03}))+(F_{20}+3(F_{11}+F_{02}))\hat{f}_1-\frac{3}{2}(F_{10}-2F_{01})\hat{f}_2-9\hat{f}_3\right]b^2+\cdots$$

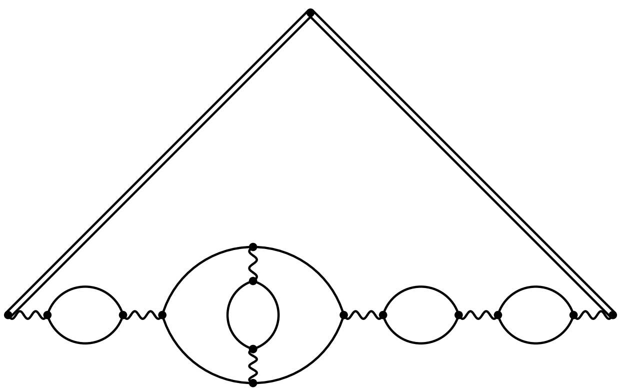


Figure 4: NLβ_0 order diagrams contain one Π_1 insertion (with any number of Π_0 insertions inside) and any number of Π_0 insertions to the left and to the right of it.

Substituting F_{nm} we obtain

$$
\begin{aligned}
&\gamma_1(b) = 12\zeta_3 - \frac{55}{4} + \left(-\frac{\pi^4}{5} + 40\zeta_3 - \frac{299}{18}\right) b\\
&+ \left(24\zeta_5 - \frac{2}{3}\pi^4 + \frac{233}{6}\zeta_3 + \frac{15211}{864}\right) b^2\\
&+ \left(-48\zeta_3^2 - \frac{2}{63}\pi^6 + 80\zeta_5 - \frac{167}{225}\pi^4 + \frac{1168}{15}\zeta_3 - \frac{971}{240}\right) b^3\\
&+ \left(36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 160\zeta_3^2 - \frac{20}{189}\pi^6 + \frac{377}{3}\zeta_5 - \frac{23}{15}\pi^4 + \frac{929}{12}\zeta_3 - \frac{8017}{1728}\right) b^4\\
&+ \left(-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 120\zeta_7 + \frac{16}{3}\pi^4\zeta_3 - \frac{2776}{21}\zeta_3^2 - \frac{914}{3969}\pi^6\right.\\
&\qquad \left. + \frac{6826}{21}\zeta_5 - \frac{1793}{1350}\pi^4 - \frac{31693}{315}\zeta_3 + \frac{79433}{4320}\right) b^5 + \cdots
\end{aligned}
\tag{4.13}
$$

This expansion can be extended to any number of loops. The first term in (4.13) agrees with the $C_F^2 T_F n_f$ term in the three-loop result [5, 6, 1]. The next term coincides with the $C_F^2 (T_F n_f)^2 \alpha_s^4$ term in Γ recently calculated in [31]. Note that the last (8-loop) term here contains F_{nm} with $n+m=6$, $n>0$, $m>0$, which contain $\zeta_{5,3}$; but they enter as the combination $F_{51}+F_{42}+F_{33}+F_{24}+F_{15}$ in which this $\zeta_{5,3}$ cancels.

Similarly, the field anomalous dimension in Landau gauge at the NLβ_0 order is

$$
\begin{aligned}
&\gamma_h(b) = -6\left[\frac{b}{\beta_0}\gamma_{h0}(b) - \frac{b^3}{\beta_0^2}\gamma_{h1}(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right),\\
&\gamma_{h1}(b) = 3\left(4\zeta_3 - \frac{17}{4}\right) + \left(-\frac{\pi^4}{5} + 36\zeta_3 - \frac{103}{9}\right) b\\
&+ \left(24\zeta_5 - \frac{3}{5}\pi^4 + \frac{59}{2}\zeta_3 + \frac{14579}{864}\right) b^2\\
&+ \left(-48\zeta_3^3 - \frac{2}{63}\pi^6 + 72\zeta_5 - \frac{44}{75}\pi^4 + \frac{3229}{45}\zeta_3 - \frac{5191}{540}\right) b^3\\
&+ \left(36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 144\zeta_3^2 - \frac{2}{21}\pi^6 + 107\zeta_5 - \frac{946}{675}\pi^4 + \frac{9601}{180}\zeta_3 + \frac{22859}{8640}\right) b^4\\
&+ \left(-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 108\zeta_7 + \frac{24}{5}\pi^4\zeta_3 - \frac{664}{7}\zeta_3^2 - \frac{272}{1323}\pi^6\right.\\
&\qquad \left. + \frac{18574}{63}\zeta_5 - \frac{119}{135}\pi^4 - \frac{6263}{63}\zeta_3 + \frac{16103}{1296}\right) b^5 + \cdots
\end{aligned}
\tag{4.14}
$$

The first term here coincides with the $C_F^2 T_F n_f$ term in the three-loop result obtained by a direct calculation [13, 14]. The last term contains the same combination of F_{nm} with $n+m=6$, so that $\zeta_{5,3}$ cancels.

The static potential at the NLβ_0 level is

$$
\begin{aligned}
V(\vec{q}\,) &= -\frac{(4\pi)^2}{\beta_0 \vec{q}^{\,2}} \varepsilon \sum_{L=1}^{\infty} g(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon+b}\right)^L \left[1 + L\frac{Z_{\alpha 1}}{\beta_0} + \frac{3\varepsilon}{\beta_0}\sum_{L'=2}^{L-1}\frac{L-L'}{L'}F(\varepsilon, L'\varepsilon)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)\\
&= -\frac{(4\pi)^2}{\vec{q}^{\,2}}\left[\frac{b}{\beta_0}V_0(b) - \frac{b^3}{\beta_0^2}V_1(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)
\end{aligned}
\tag{4.15}
$$

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where

$$V_1(b) = -\frac{3}{2}\left[F_{10}+2F_{01}+2g_{01}\right] + \frac{1}{2}\left[F_{20}-6F_{02}-6\left(F_{10}+3F_{01}\right)g_{01}-30g_{02}\right]b$$
$$-\frac{1}{4}\left[F_{30}+24F_{03}-4\left(F_{20}+12F_{02}\right)g_{01}+36\left(F_{10}+4F_{01}\right)g_{02}+312g_{03}\right]b^2+\cdots$$

contains only the same coefficients g_{0n} (3.11) as the Lβ_0 result, and only F_{n0} and F_{0m} are involved (see (4.4–4.6)). We obtain

$$\begin{aligned}
V_1(b) &= 12\zeta_3 - \frac{55}{4} + \left(78\zeta_3 - \frac{7001}{72}\right)b + \left(60\zeta_5 + \frac{723}{2}\zeta_3 - \frac{147851}{288}\right)b^2 \\
&+ \left(770\zeta_5 + \frac{\pi^4}{200} + \frac{276901}{180}\zeta_3 - \frac{70418923}{25920}\right)b^3 \\
&+ \left(1134\zeta_7 + \frac{32297}{5}\zeta_5 + \frac{41}{1800}\pi^4 + \frac{402479}{60}\zeta_3 - \frac{1249510621}{77760}\right)b^4 \\
&+ \left(21735\zeta_7 + \frac{\zeta_3^2}{7} + \frac{\pi^6}{1323} + \frac{5911849}{126}\zeta_5 + \frac{41}{720}\pi^4 + \frac{48558187}{1512}\zeta_3 - \frac{10255708489}{93312}\right)b^5 + \cdots
\end{aligned} \tag{4.16}$$

Thus we have reproduced the $C_F(T_Fn_f)^2\alpha_s^3$ and $C_F^2T_Fn_f\alpha_s^3$ terms in the two-loop potential [17], as well as the $C_F(T_Fn_f)^3\alpha_s^4$ and $C_F^2(T_Fn_f)^2\alpha_s^4$ terms in the three-loop one [18]. This expansion can be extended to any order; it contains only ζ_n because only F_{n0} and F_{0m} are present. Note the pattern of the highest weights in (4.16): 3, 3, 5, 5, 7, 7, whereas one would expect 3, 4, 5, 6, 7, 8, as in (4.13), (4.14). The conformal anomaly (2.6) at the NLβ_0 order is

$$\begin{aligned}
\Delta &= 4\pi\left[\frac{b^3}{\beta_0}\delta_0(b) - \frac{b^4}{\beta_0^2}\delta_1(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right), \\
\delta_1(b) &= \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left(36\zeta_5 + \frac{2}{3}\pi^4 + \frac{968}{3}\zeta_3 - \frac{114691}{216}\right)b \\
&+ \left(48\zeta_3^2 + \frac{2}{63}\pi^6 + 690\zeta_5 + \frac{269}{360}\pi^4 + \frac{52577}{36}\zeta_3 - \frac{14062811}{5184}\right)b^2 \\
&+ \left(1098\zeta_7 - \frac{8}{5}\pi^4\zeta_3 + 160\zeta_3^2 + \frac{20}{189}\pi^6 + \frac{95006}{15}\zeta_5 + \frac{2801}{1800}\pi^4 + \frac{198917}{30}\zeta_3 - \frac{39035933}{2430}\right)b^3 \\
&+ \left(240\zeta_3\zeta_5 + \frac{4}{225}\pi^8 + 21615\zeta_7 - \frac{16}{3}\pi^4\zeta_3 + \frac{397}{3}\zeta_3^2 + \frac{131}{567}\pi^6\right. \\
&\quad \left. + \frac{838699}{18}\zeta_5 + \frac{14959}{10800}\pi^4 + \frac{34793081}{1080}\zeta_3 - \frac{51287121209}{466560}\right)b^4 + \cdots
\end{aligned} \tag{4.17}$$

The b^3/β_0^2 term has canceled, so that the coefficient of C_F in the bracket in (2.7) is 0.

5. Conclusion

The terms with the highest powers of n_f at each order of perturbation theory ($C_F(T_Fn_f)^{L-1}\alpha_s^L$ in Γ, γ_h; $C_F(T_Fn_f)^L\alpha_s^{L+1}$ in $V(\vec{q})$) are known, and given by explicit formulas (3.8), (3.10), (3.12). The terms with the next to highest power of n_f can have abelian (C_F^2) or non-abelian (C_FC_A) color structure. The abelian terms ($C_F^2(T_F n_f)^{L-2}\alpha_s^L$ ($L \geq 3$) in Γ, γ_h; $C_F^2(T_F n_f)^{L-1}\alpha_s^{L+1}$ ($L \geq 2$) in $V(\vec{q})$) are also known to all orders in α_s, but only as algorithms which allow one to obtain (in principle) any number of terms, see (4.13), (4.14), (4.16). The simple method used here is not applicable to non-abelian terms.

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