Channel Capacity Calculation at Large SNR and Small Dispersion within Path-Integral Approach

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We develop the path-integral approach [1–4] to the calculation of the information theory quantities (conditional probability density function $P[Y|X]$, output signal correlators, mutual information $I_{P[X]}$, optimal input signal distribution $P_X^{opt}[X]$, channel capacity C, and so on) for the communication channel modelled by the nonlinear Shrödinger equation (NLSE) with additive white Gaussian noise (AWGN). Now we consider the perturbation theory in this NLSE model for small dispersion parameter β and find the first and the second corrections to the conditional probability density function $P[Y|X]$, to the optimal input signal distribution $P^{opt}[X]$, and to the channel capacity C at large signal-to-noise ratio (SNR).

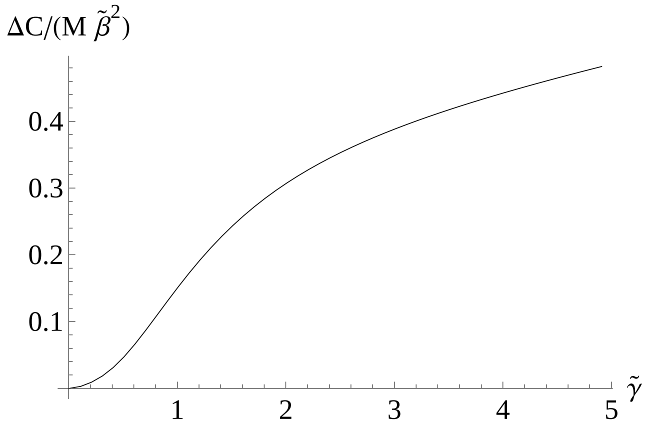

Figure 1: The correction $\Delta C/(M\widetilde{\beta}^2)$ to the channel capacity as a function of $\widetilde{\gamma} = \frac{\gamma L P_{ave}}{\sqrt{3}}$ for low powers

We consider the optical fiber channel modelled by the NLSE with AWGN:

$$\partial_z \psi + i\beta \partial_t^2 \psi - i\gamma |\psi|^2 \psi = \eta(z,t)\,, \tag{1}$$

where β is the dispersion coefficient, γ is the Kerr nonlinearity coefficient, $\eta(z,t)$ is an additive complex white noise with zero mean $\langle \eta(z,t)\rangle_\eta = 0$ and with correlation function

$$\langle \eta(z,t)\bar{\eta}(z',t')\rangle_\eta = Q\delta(z-z')\delta(t-t')\,, \tag{2}$$

where bar means complex conjugation, and Q is a power of the white Gaussian noise $\eta(z,t)$ per unit length and per unit frequency. The conditions for a complex signal $\psi(z,t)$ are as follows: $\psi(z=0,t) = X(t)$ and $\psi(z=L,t) = Y(t)$, where $X(t)$ is the input signal, and $Y(t)$ is the output signal.

We apply the Feynman path-integral approach for this model introduced in [3]. We use the following representation of the conditional probability density function $P[Y|X]$ which is convenient for the large SNR case:

$$P[Y|X] = \int\limits_{\psi(0,t)=X(t)}^{\psi(L,t)=Y(t)} \mathcal{D}\psi\, \exp\left\{-\frac{S[\psi]}{Q}\right\} = \Lambda \exp\left[-\frac{S[\Psi_{cl}]}{Q}\right], \tag{3}$$

$$\Lambda = \int_{\phi(z=0)=0}^{\phi(z=L)=0} \mathcal{D}\phi \exp\Big[-\frac{1}{Q}\Big\{S[\Psi_{cl}+\phi] - S[\Psi_{cl}]\Big\}\Big]\,, \tag{4}$$

where the action $S[\psi]$ has the form which corresponds to the noise statistics (2):

$$S[\psi] = \int_0^L dz \int_T dt\, \left|\partial_z \psi + i\beta\partial_t^2\psi - i\gamma\psi|\psi|^2\right|^2\,, \tag{5}$$

and in the leading order in 1/SNR the action difference in (4) reads (in discrete time form) [4]:

$$S[\Psi_{cl}+\phi]-S[\Psi_{cl}]\approx\delta_t\sum_{k=0}^{M-1}\int_0^L dz\,\mathcal{L}_{eff}[\phi(z,t_k)], \tag{6}$$

$$\mathcal{L}_{eff}[\phi(z,t_k)]=\left|\partial_z\phi(z,t_k)+i\beta\partial_t^2\phi(z,t_k)-i\gamma\left(2\phi(z,t_k)|\Phi(z,t_k)|^2+\bar{\phi}(z,t_k)\Phi^2(z,t_k)\right)\right|^2. \tag{7}$$

In (3) $\Psi_{cl}(z,t)$ is the solution of the equation of motion $\delta S[\Psi_{cl}]=0$ with the boundary conditions: $\Psi_{cl}(z=0,t)=X(t)$, $\Psi_{cl}(z=L,t)=Y(t)$, see [3, 4]. In (6) M is the total mesh points for the time interval T in which the input signal $X(t)$ is periodically embedded. It means that the whole time interval T is assumed to be divided into M mesh points (independent time channels) with grid spacing $\delta_t=T/M=2\pi/W$, where W is the frequency domain (bandwidth) of an input complex T-periodic signal $X(t_j)$, and a time moment (in one period) $t_j=-T/2+j\delta_t$, $j=0,1,\ldots,M-1$. In (7) $\Phi(z,t)$ is the solution of NLSE (1) with zero noise and with one condition $\Phi(z=0,t)=X(t)$. We have shown [4] that $\Psi_{cl}=\Phi(z,t)+\varkappa(z,t)$ can be approximated by $\Phi(z,t)$ with the small remainder $\varkappa(z,t)\sim\sqrt{Q}$. We have used it in (7).

The general representation for the mutual information $I_{P[X]}$ (in nat units) in the leading order in 1/SNR reads [4]:

$$I_{P[X]}=M\log\left[P_{ave}/P_{noise}\right]+\left\langle\log\left[\frac{\Lambda}{\Lambda_{QL}}\right]\right\rangle_{P[X]}, \tag{8}$$

where $\Lambda_{QL}=(\delta_t/(\pi QL))^M$, and for any $F[X]$ the quantity $\left\langle F[X]\right\rangle_{P[X]}=\int\mathcal{D}X P[X]F[X]$ is the averaging over the input signal PDF $P[X]$. In the first logarithm in (8), i.e. in Shannon's term $\log\left[P_{ave}/P_{noise}\right]$, $P_{noise}=QL/\delta_t=QLW/(2\pi)$, is the noise power in the bandwidth W and length L, and P_{ave} is the average power of the input signal:

$$P_{ave}=\lim_{T\to\infty}\int\mathcal{D}XP[X(t)]\frac{1}{T}\int_T|X(t)|^2dt=\lim_{T\to\infty}\int\prod_{j=0}^{M-1}dReX(t_j)dImX(t_j)P[X(t)]\frac{1}{M}\sum_{j=0}^{M-1}|X(t_j)|^2. \tag{9}$$

For an input complex signal $X(t)$ we use the notations $\rho(t_j)\equiv\rho_j=|X(t_j)|$ and $\phi_0(t_j)\equiv\phi_{0,j}=\arg(X(t_j))$.

To calculate the channel capacity $C=\max_{P[X]}I_{P[X]}$ we should find the normalization factor $\Lambda=\Lambda^{(0)}\left\{1+\frac{\Lambda^{(1)}}{\Lambda^{(0)}}+\frac{\Lambda^{(2)}}{\Lambda^{(0)}}+\mathcal{O}(\widetilde{\beta}^3)\right\}$, see (4), and perform the maximization procedure over the input signal distribution $P[X]$ ($P[X]=P^{opt}[X]$ for this maximum). Here $\Lambda^{(0)}$ is the normalization factor for $\beta=0$ [1]:

$$\Lambda^{(0)}=\prod_{j=0}^{M-1}\frac{\delta_t}{\pi QL\sqrt{1+\mu_j^2/3}}, \tag{10}$$

where in (10) we have used the dimensionless parameter $\mu(t_j)=\mu_j=\gamma L\rho^2(t_j)$ characterizing the impact of nonlinearity in the phase evolution for the NLSE (1) with $\beta=0$. For the nondispersive case the maximization procedure in question was described in [1] and the optimal input signal distribution $P_X^{opt(\beta=0)}[X]$ was obtained in the intermediate power regime $\widetilde{\gamma}_{noise}\ll\widetilde{\gamma}\ll(\widetilde{\gamma}_{noise})^{-1}$. Here $\widetilde{\gamma}$ is the dimensionless nonlinearity parameter $\widetilde{\gamma}=\gamma LP_{ave}/\sqrt{3}$, $\widetilde{\gamma}_{noise}=\gamma L^2QW/(2\pi\sqrt{3})$, so that $\mathrm{SNR}=\widetilde{\gamma}/\widetilde{\gamma}_{noise}$.

In the first order in dimensionless dispersion parameter $\widetilde{\beta}=\beta LW^2/(2\pi)^2=\beta L/\delta t^2$ we obtained the following result

$$\frac{\Lambda^{(1)}}{\Lambda^{(0)}}=\sum_{j=0}^{M-1}\left\{\frac{4\widetilde{\beta}\mu^3}{15\left(\mu^2+3\right)}-\frac{\beta L\mu^2}{105\left(\mu^2+3\right)\rho^2}\left\{14\left(2\mu^2+15\right)\rho\dot{\rho}\dot{\phi}_0+\mu\left(16\mu^2+189\right)\dot{\rho}^2+\right.\right. \tag{11}$$
$$\left.\left.7\rho\left(\rho\left(2\mu\dot{\phi}_0{}^2+15\ddot{\phi}_0\right)+7\mu\ddot{\rho}\right)\right\}\right|_{t=t_j}.$$

It is easy to see that the correction to the channel capacity linear in $\widetilde{\beta}$ vanishes, since $\left\langle \Lambda^{(1)}/\Lambda^{(0)} \right\rangle_{P_X^{opt(\beta=0)}[X]} = 0$. To calculate the first non-vanishing correction ΔC to the channel capacity in small dimensionless dispersion parameter $\widetilde{\beta}$ one should find the second correction $\Lambda^{(2)}$ to the normalization factor (4), and:

$$\Delta C = \langle \frac{\Lambda^{(2)}}{\Lambda^{(0)}} \rangle_{P_X^{opt(\beta=0)}[X]} = \int \mathcal{D}X P_X^{opt(\beta=0)}[X] \, \frac{\Lambda^{(2)}}{\Lambda^{(0)}}. \tag{12}$$

We found this correction ΔC and demonstrated that it is positive, $\Delta C > 0$, therefore increasing the earlier calculated capacity for a nondispersive nonlinear optical fiber channel [1]. In figure 1 we present the ratio $\Delta C/(M\widetilde{\beta}^2)$ as the function of $\widetilde{\gamma}$ for low powers ($\widetilde{\gamma}_{noise} < \widetilde{\gamma} < 5$). For typical fiber optical links [5] one has: $L = 1000\,\mathrm{km}$, $\gamma = 1.31(\mathrm{Wkm})^{-1}$, $W = 100\,\mathrm{GHz}$, $P_{noise} = QLW/(2\pi) = 5.3 \times 10^{-4}\mathrm{mW}$. For these parameters one has $\widetilde{\gamma}_{noise} \approx 4 \times 10^{-7}$, and our results are reliable if $\widetilde{\beta} \ll 1$, i.e., $\beta \ll 4\,\mathrm{ps}^2/\mathrm{km}$.

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