Phys. Lett. B 848 (2024) 138408

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

Letter

Total Born cross section of e^+e^--pair production by an electron in the Coulomb field of a nucleus

Roman N. Lee [a,*], Alexey A. Lyubyakin [b], Vladimir A. Smirnov [c,d]

[a] Budker Institute of Nuclear Physics, 630090, Novosibirsk, Russia
[b] Novosibirsk State University, 630090, Novosibirsk, Russia
[c] Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119992 Moscow, Russia
[d] Moscow Center for Fundamental and Applied Mathematics, 119992 Moscow, Russia

A R T I C L E I N F O

Editor: A. Ringwald

A B S T R A C T

We calculate the total Born cross section of the e^+e^--pair production by an electron in the field of a nucleus (trident process) using the modern multiloop methods. For general energies we obtain the cross section in terms of converging power series. The threshold asymptotics and the high-energy asymptotics are obtained analytically. In particular, we obtain an additional contribution to the Racah formula due to the identity of the final electrons. Besides, our result for the leading term of the high-energy asymptotics reveals a typo in an old Racah paper.

1. Introduction

The process of the e^+e^--pair production by an electron in the field of a nucleus $e^-Z \to e^-Ze^+e^-$ (trident process) is one of the basic processes of interaction of high-energy electrons with matter. Theoretical and experimental studies of this process have a long history [1–6].

First theoretical results for the total Born cross section in the leading logarithmic approximation were obtained by Landau and Lifshitz [2] and, independently, by Bhabha [3]. A little bit later the paper by Racah [1] appeared, where the total Born cross section of this process was obtained up to power corrections. The Racah result reads

$$\sigma_{e^-Z\to e^-Ze^+e^-} = \frac{\alpha^2(Z\alpha)^2}{\pi m^2}\left(\frac{28}{27}L^3 - \frac{178}{27}L^2 + \left(\frac{191}{81}+\frac{\pi^2}{27}\right)L + \frac{683\pi^2}{162} - \frac{3781}{486} - \frac{4\pi^2\ln 2}{9} + \frac{79\zeta_3}{9}\right) + \mathcal{O}(1/\gamma). \quad (1)$$

Here $L = \ln(2\gamma)$, m and γ are the electron mass and its relativistic factor in the nucleus rest frame. Note that the above formula does not take into account the identity of the two final electrons. Meanwhile, this account is expected to modify the coefficients in front of L^1 and L^0 terms.

In the present paper we calculate the total Born cross section of the trident process. We obtain the exact result in terms of convergent power series with analytic coefficients, which allows us to determine analytically both the threshold asymptotics and the high-energy asymptotics of the cross section. In particular, we find that the contribution due to the identity of the final electrons indeed modifies the $\propto L^1$ and $\propto L^0$ terms. We also uncover a typo in the Racah formula which results to the incorrect coefficient $\frac{79}{9}$ in front of ζ_3 (the correct one is $\frac{185}{18}$).

2. Details of calculation

The total cross section of the trident process is determined by the three-loop cut diagrams depicted in Fig. 1. These diagrams fall into three groups:

- Diagrams containing a cut fermion loop with four photon lines attached.
- Diagrams containing a cut fermion loop with two photon lines attached.
- Diagrams without fermion loop.

It is natural to call the corresponding contributions C-even, C-odd, and “twisted” contributions, respectively. The latter contribution appears due to the particle identity. We will denote these contributions by σ_E, σ_O, and σ_T, so that

$$\sigma_{e^-Z\to e^-Ze^-e^+} = \sigma_E + \sigma_O + \sigma_T\,. \quad (2)$$

Let p_1, $p_{2,3}$, and p_4 denote the momenta of the initial electron, of two final electrons and of the final positron, respectively. Then the cut gray

* Corresponding author.
E-mail addresses: r.n.lee@inp.nsk.su (R.N. Lee), lyubyakin@inp.nsk.su (A.A. Lyubyakin), smirnov@theory.sinp.msu.ru (V.A. Smirnov).

https://doi.org/10.1016/j.physletb.2023.138408
Received 11 September 2023; Received in revised form 8 December 2023; Accepted 15 December 2023
Available online 19 December 2023

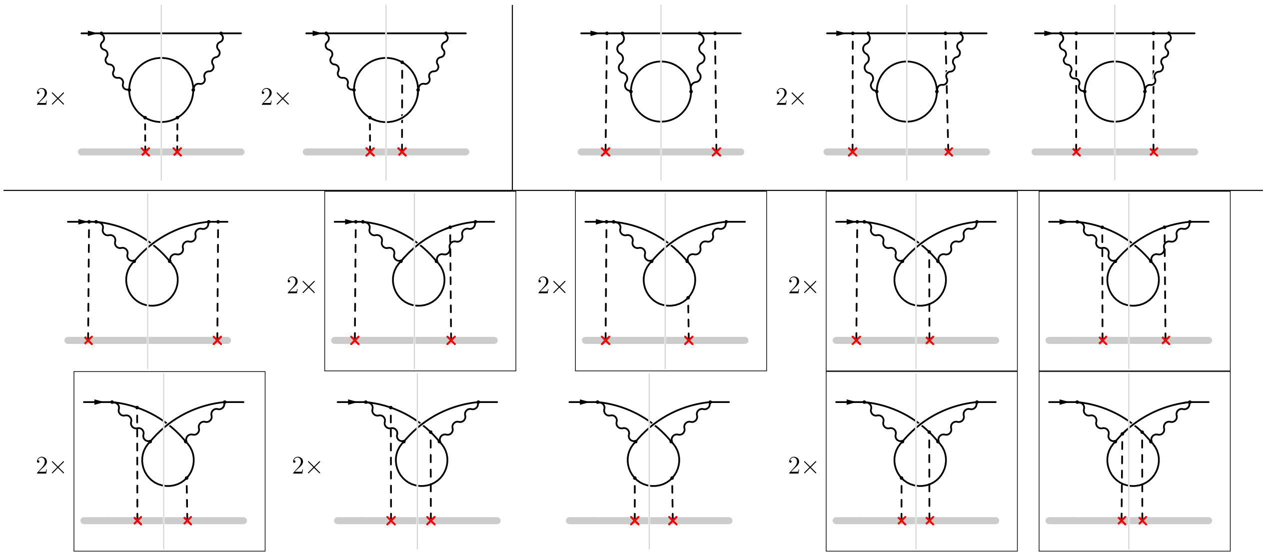
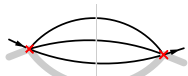


Fig. 1. Diagrams corresponding to the total Born cross section $Ze+e^- \to Ze+e^-+e^-+e^+$. The two upper left diagrams correspond to the C-even contribution, the three upper right diagrams correspond to the C-odd contribution, and the remaining ten diagrams correspond to the "twisted" contribution. Each of the 7 framed diagrams corresponds to a specific `LiteRed` basis.

line corresponds to the delta-function $\delta(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_3)$, expressing the energy conservation (here $\varepsilon_i = p_i^0 = \sqrt{\boldsymbol{p}_i^2+m^2}$).

We perform Dirac algebra using `FORM` [7] and express the three contributions in terms of scalar three-loop integrals with cut denominators. Each of the three-loop scalar Feynman integrals appearing in the calculation falls into one of the seven families of integrals corresponding to the framed diagrams in Fig. 1. We perform an IBP reduction using `LiteRed` and `FIRE` [8,9] and identify 74 unique master integrals. We introduce the dimensionless parameter $x = m^2/\varepsilon_1^2 = 1/\gamma^2$ and derive a differential equation with respect to x.

Using `Libra` [10] and criterion of Ref. [11], we have checked that the differential system is irreducible to ϵ-form in a sole 3×3 block, corresponding to the three-particle phase-space integral in the static field,

$j_1 =$  $$= \int \frac{d\boldsymbol{p}_2}{2\varepsilon_2}\frac{d\boldsymbol{p}_3}{2\varepsilon_3}\frac{d\boldsymbol{p}_4}{2\varepsilon_4}\delta(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4), \qquad (3)$$

and its first and second derivatives with respect to x. We pass to normalized Fuchsian form and fix the boundary conditions in the asymptotics $x \to 1/9$, corresponding to the threshold ($\varepsilon_1 = 3m$). Note that the physical region corresponds to $x \in (0, 1/9)$. The only non-zero boundary constant appears to be the coefficient in the leading asymptotics of j_1, Eq. (3), at the threshold. Then we construct an ϵ-regular basis [12] following the approach described in Ref. [13]. Once we find the ϵ-regular basis we can safely put $\epsilon = 0$ in the differential equations and boundary conditions for it. This is exactly the rationale behind passing to an ϵ-regular basis. When searching for the ϵ-regular basis, it is convenient to pass to the new variable z,

$$x = \left(\frac{1-z^2}{1+z^2}\right)^2, \quad z = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}, \quad \frac{1}{\sqrt{2}} < z < 1. \qquad (4)$$

Then the matrix on the right-hand side of the resulting system at $\epsilon = 0$ has many zeros as demonstrated in Fig. 2. In particular, apart from the leftmost upper 3×3 block, all the diagonal elements are zero. The non-zero matrix elements are rational functions of z, therefore, we can express all elements of the regular basis as onefold integrals of polylogarithms and the integral j_1.

However, the resulting expressions appear to be quite complicated and we choose instead to use the Frobenius method. In order to apply

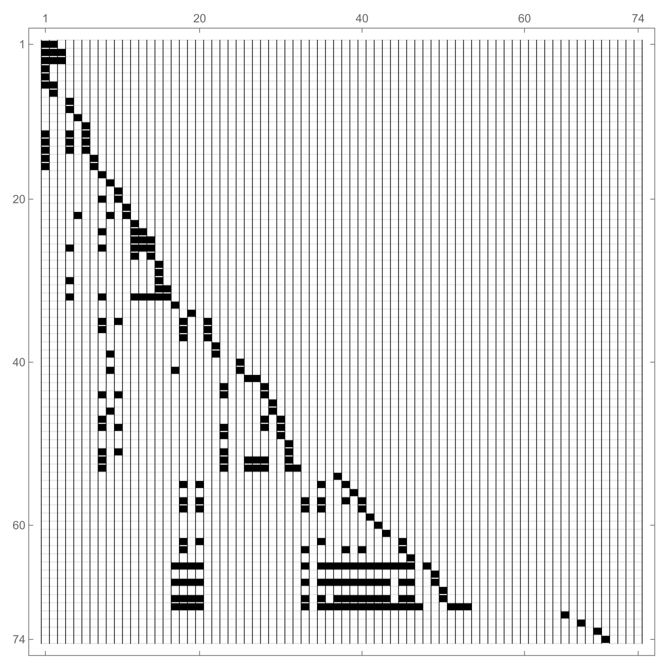


Fig. 2. Plot of the matrix on the right-hand side of the differential system for the regular basis. Black squares denote nonzero matrix elements.

this method, we pass to the variable τ which varies from 0 to ∞ in the physical region:

$$x = \left(\frac{1-\tau}{3+\tau}\right)^2, \quad \tau = \frac{1-3\sqrt{x}}{1+\sqrt{x}}, \quad 0 < \tau < 1. \qquad (5)$$

The point $\tau = 0$ corresponds to the threshold, while $\tau = 1$ corresponds to the high-energy limit.

Then we construct the generalized power series for the evolution operator $U(\tau, \theta)$ around $\theta = 0$ and $\theta = 1$:

$$U(\tau, \underline{0}) = \sum_{\nu\in\{0,\frac{3}{2}\}}\sum_{n=0}^{N_0-1} U_{n+\nu}^{(0)}\tau^{n+\nu} + \mathcal{O}(\tau^{N_0+\nu}), \qquad (6)$$

$$U(\tau,\underline{1}) = \sum_{n=0}^{N_1-1}\sum_{k=0}^{7} U^{(1)}_{n,k}\overline{\tau}^n \ln^k \overline{\tau} + \mathcal{O}(\overline{\tau}^{N_1}), \tag{7}$$

where $\overline{\tau} = 1-\tau$ and the coefficients $U^{(0)}_{n+\nu}$ and $U^{(1)}_{n,k}$ satisfy a finite-order recurrence relation automatically derived using Libra. Then the specific solution reads

$$\boldsymbol{J}(\tau) = U(\tau,\underline{0})\boldsymbol{C}_0 = U(\tau,\underline{1})\boldsymbol{C}_1\,, \tag{8}$$

where $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ are the two columns of boundary constants. Note that $\boldsymbol{C}_0$ is expressed via the threshold asymptotics of the phase-space integrals j_1, which was calculated analytically. Then we can find $\boldsymbol{C}_1$ numerically by matching the two expansions at some point $\tau_0 \in (0,1)$:

$$\boldsymbol{C}_1 = U^{-1}(\tau_0,\underline{1})U(\tau_0,0)\boldsymbol{C}_0. \tag{9}$$

Our goal was to obtain high-precision values for $\boldsymbol{C}_1$ in order to recognize their analytic form using PSLQ algorithm [14]. In order to evaluate $U(\tau_0,\underline{1})$ and $U(\tau_0,\underline{0})$ with comparable precision, we have to choose N_1 and N_0 so that

$$\tau_0^{N_0} \approx (1-\tau_0)^{N_1}\,. \tag{10}$$

Since the expansion of $U(\tau,\underline{1})$ appears to be much more computationally expensive than that of $U(\tau,\underline{0})$, we choose $\tau_0 = 9/10$, $N_0 = 6700$, $N_1 = 307$. This choice gives us about 300 digits of $\boldsymbol{C}_1$ within about 10 minutes of computation time for each $U(\tau_0,\underline{1})$ and $U(\tau_0,\underline{0})$. Then we use PSLQ to obtain the analytic form of $\boldsymbol{C}_1$ in terms of alternating multiple zeta values.

3. Results and conclusion

The results obtained within the described approach are the following:

- Threshold asymptotics with analytic coefficients to arbitrary order in the parameter $\tau = \frac{\epsilon_1-3m}{\epsilon_1+m} = \frac{\gamma-3}{\gamma+1}$.
- High-energy asymptotics with analytic coefficients to arbitrary order in the parameter $m/\epsilon_1 = 1/\gamma$.
- High-precision numerical results for arbitrary energies.

The threshold asymptotics of σ_E, σ_O, and σ_T read

$$\sigma_E = \frac{\alpha^2(Z\alpha)^2}{m^2}\left[\frac{304\tau^{9/2}}{945} - \frac{64\tau^{11/2}}{10395} + \frac{15040\tau^{13/2}}{27027} - \frac{9472\tau^{15/2}}{405405} + \mathcal{O}\left(\tau^{17/2}\right)\right], \tag{11}$$

$$\sigma_O = \frac{\alpha^2(Z\alpha)^2}{m^2}\left[\frac{8\tau^{7/2}}{35} - \frac{232\tau^{9/2}}{945} + \frac{512\tau^{11/2}}{1485} - \frac{1568\tau^{13/2}}{6435} + \frac{11528\tau^{15/2}}{61425} + \mathcal{O}\left(\tau^{17/2}\right)\right], \tag{12}$$

$$\sigma_T = \frac{\alpha^2(Z\alpha)^2}{m^2}\left[-\frac{4\tau^{7/2}}{21} + \frac{76\tau^{9/2}}{135} - \frac{1856\tau^{11/2}}{1155} + \frac{395632\tau^{13/2}}{135135} - \frac{121276\tau^{15/2}}{25025} + \mathcal{O}\left(\tau^{17/2}\right)\right]. \tag{13}$$

The high-energy asymptotics read

$$\sigma_E = \frac{\alpha^2(Z\alpha)^2}{\pi m^2}\left\{\underline{\frac{28}{27}L^3 - \frac{178}{27}L^2 + \left(\frac{430}{27} - \frac{25\pi^2}{18}\right)L + \frac{68\zeta_3}{3} + \frac{13}{9}\pi^2\ln 2 + \frac{877\pi^2}{324} - \frac{512}{27}} + \frac{1}{\gamma}\left[-\frac{27\pi^2}{2}L + 36\pi^2\ln 2 - \frac{\pi^2}{4}\right] + \frac{1}{\gamma^2}\left[\frac{L^5}{3} - \frac{5L^4}{6} + \left(\frac{179}{27} - \frac{5\pi^2}{9}\right)L^3 + \left(4\zeta_3 - \frac{655}{54} + \frac{7\pi^2}{6}\right)L^2 + \left(\frac{5\pi^4}{18} - 16\zeta_3 - \frac{25\pi^2}{2} - \frac{3907}{54}\right)L - \frac{47\zeta_5}{2} - \frac{11\pi^2\zeta_3}{6} + \frac{476\zeta_3}{3} + \frac{\pi^4}{24} - \frac{286\pi^2}{81} - \frac{15161}{108} + \frac{131}{9}\pi^2\ln 2\right] + \mathcal{O}\left(\frac{1}{\gamma^3}\right)\right\}, \tag{14}$$

$$\sigma_O = \frac{\alpha^2(Z\alpha)^2}{\pi m^2}\left\{\underline{\left(\frac{77\pi^2}{54} - \frac{1099}{81}\right)L - \frac{223\zeta_3}{18} + \frac{163\pi^2}{108} + \frac{5435}{486} - \frac{17}{9}\pi^2\ln 2} + \frac{1}{\gamma}\frac{3\pi^2}{4} + \frac{1}{\gamma^2}\left[-\frac{7L^4}{18} + \frac{11L^3}{9} + \left(\frac{5\pi^2}{18} - \frac{415}{54}\right)L^2 + \left(\frac{901}{162} - \frac{\pi^2}{18}\right)L - \frac{13\zeta_3}{2} - \frac{17\pi^4}{360} + \frac{143\pi^2}{324} - \frac{3935}{972} - \frac{2}{9}\pi^2\ln 2\right] + \mathcal{O}\left(\frac{1}{\gamma^3}\right)\right\}, \tag{15}$$

$$\sigma_T = \frac{\alpha^2(Z\alpha)^2}{\pi m^2}\left\{\underline{\left(-\frac{748\zeta_3}{105} - \frac{2729}{630} + \frac{13591\pi^2}{4725} - \frac{16}{7}\pi^2\ln 2\right)L} + \frac{93\zeta_5}{8} - \frac{7\pi^2\zeta_3}{8} - \frac{2048\mathrm{Li}_4\left(\frac{1}{2}\right)}{35} + \frac{101\pi^4}{105} - \frac{6051\zeta_3}{175} + \frac{3007\pi^2}{4725} - \frac{5282}{1575} - \frac{256\ln^4 2}{105} + \frac{496}{105}\pi^2\ln^2 2 - \frac{1242}{175}\pi^2\ln 2 + \frac{1}{\gamma}\frac{27\pi^2}{8} + \frac{1}{\gamma^2}\left[\frac{L^5}{15} - \frac{5L^4}{4} + \frac{5L^3}{18} + \left(-2\zeta_3 - \frac{91}{4} + \frac{11\pi^2}{36}\right)L^2 + \left(\frac{88\zeta_3}{21} - \frac{493}{126} + \frac{6707\pi^2}{3780} + \frac{\pi^4}{15} - \frac{166}{105}\pi^2\ln 2\right)L - \frac{77\zeta_5}{4} + \frac{11\pi^2\zeta_3}{12} - \frac{1144\mathrm{Li}_4\left(\frac{1}{2}\right)}{35} + \frac{9809\pi^4}{15120} + \frac{344\pi^2}{105}\ln^2 2 - \frac{143\ln^4 2}{105} + \frac{257\zeta_3}{140} - \frac{434\pi^2}{45}\ln 2 + \frac{2836\pi^2}{4725} - \frac{746}{21}\right] + \mathcal{O}\left(\frac{1}{\gamma^3}\right)\right\}. \tag{16}$$

Finally, the functions σ_E, σ_O, σ_T for arbitrary $\tau = \frac{\gamma-3}{\gamma+1}$ are presented in Fig. 3. The solid graphs were obtained using deep Frobenius expansions near $\tau = 0$ and $\tau = 1$.

We note that the leading terms underlined in Eqs. (14) and (15) were also considered by Racah in Ref. [1]. While we find agreement of Eq. (14) with the sum of Eqs. (59) and (66) of Ref. [1], the underlined terms in Eq. (15) slightly differ from Eq. (70) of Ref. [1]. Namely, the coefficient in front of ζ_3 is $-\frac{125}{9}$ in Ref. [1], which is to be compared with $-\frac{223}{18}$ in Eq. (15). Fortunately, it is easy to identify the place where this typo appeared: simply integrating Eq. (69) of Ref. [1] over dt/t we recover our result. It turns out that the logarithmically amplified term in the leading asymptotics of the contribution σ_T, underlined in Eq. (16) can also be found in old papers. Namely, Kuraev, Lipatov and Strikman in Ref. [15] considered the effect of electron identity in the process $e^+e^- \to e^+e^+e^-e^-$. After taking into account the combinatorial factor 2 due to the appearance of two pairs of identical particles in this process instead of one pair in $e^-Z \to e^-Ze^-e^+$ we find agreement with Ref. [15].

Finally, we note that the leading high-energy asymptotics of the cross sections suffice only for rather high energies. In particular, they provide 5% accuracy at $\epsilon_1 \gtrsim 200m$, to be compared with $\epsilon_1 \gtrsim 50m$ and $\epsilon_1 \gtrsim 17m$ when one takes into account the $1/\gamma$ and $1/\gamma^2$ corrections, respectively.

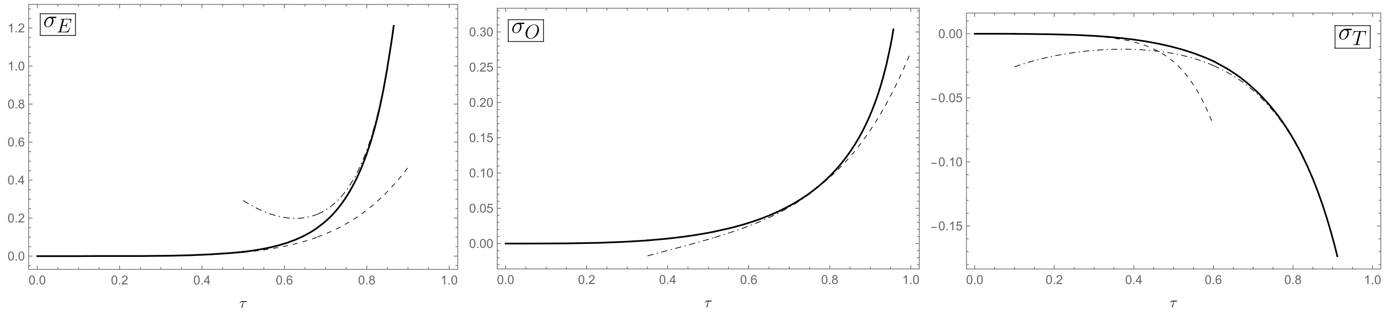


Fig. 3. Cross sections σ_E, σ_O, σ_T as functions of τ. Dashed and dash-dotted curves correspond to the threshold, Eqs. (11)-(13) and high-energy, Eqs. (14)-(16), asymptotics, respectively.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are grateful to V. Fadin for useful discussions and interest to the work. The work of R.L. and A.L. was supported by the Russian Science Foundation, grant number 20-12-00205. The work of V.S. was supported by the Ministry of Education and Science of the Russian Federation as part of the program of the Moscow Center for Fundamental and Applied Mathematics under Agreement No. 075-15-2022-284.

References

[1] G. Racah, Sulla nascita di coppie per urti di particelle elettrizzate, Il Nuovo Cimento 14 (1937) 93–113, https://doi.org/10.1007/BF02960616.

[2] L.D. Landau, E.M. Lifschitz, On the production of electrons and positrons by a collision of two particles, Phys. Z. Sowjetunion 6 (1934) 244, https://doi.org/10.1016/B978-0-08-010586-4.50021-3.

[3] H.J. Bhabha, The creation of electron pairs by fast charged particles, Proc. R. Soc. Lond. A 152 (1935) 559, https://doi.org/10.1098/rspa.1935.0208.

[4] M. Camac, Pair production by electrons, Phys. Rev. 88 (1952) 745–750, https://doi.org/10.1103/PhysRev.88.745, https://link.aps.org/doi/10.1103/PhysRev.88.745.

[5] F. Brisbout, C. Dahanatake, A. Englee, D. Perkins, On the trident cross-section between 1 and 10 gev, Il Nuovo Cimento (1955-1965) 4 (1956) 1496–1503.

[6] F.J. Loeffler, Production of electron pairs by 550-mev electrons, Phys. Rev. 108 (1957) 1058–1062, https://doi.org/10.1103/PhysRev.108.1058, https://link.aps.org/doi/10.1103/PhysRev.108.1058.

[7] B. Ruijl, T. Ueda, J. Vermaseren, FORM version 4.2, arXiv:1707.06453, 7.2017.

[8] R.N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals 523 (2014) 012059, https://doi.org/10.1088/1742-6596/523/1/012059, arXiv:1310.1145.

[9] A. Smirnov, V. Smirnov, FIRE4, LiteRed and accompanying tools to solve integration by parts relations, arXiv:1302.5885, 2013.

[10] R.N. Lee, Libra: a package for transformation of differential systems for multiloop integrals, Comput. Phys. Commun. 267 (2021) 108058, https://doi.org/10.1016/j.cpc.2021.108058, arXiv:2012.00279.

[11] R.N. Lee, A.A. Pomeransky, Normalized Fuchsian form on Riemann sphere and differential equations for multiloop integrals, arXiv:1707.07856, 2017.

[12] R.N. Lee, A.I. Onishchenko, ϵ-regular basis for non-polylogarithmic multiloop integrals and total cross section of the process $e^+e^- \to 2(Q\bar{Q})$, J. High Energy Phys. 12 (2019) 084, https://doi.org/10.1007/JHEP12(2019)084, arXiv:1909.07710.

[13] P.A. Krachkov, R.N. Lee, Two-loop corrections to lamb shift and hyperfine splitting in hydrogen via multi-loop methods, J. High Energy Phys. 07 (2023) 211, https://doi.org/10.1007/JHEP07(2023)211, arXiv:2306.13369.

[14] H.R.P. Ferguson, D.H. Bailey, A polynomial time, numerically stable integer relation algorithm, Tech. Rep., 1991.

[15] E.A. Kuraev, L. Lipatov, M. Strikman, Identity of electrons in the final state in pair production on colliding e+ e–beams, Yad. Fiz. 18 (6) (1973) 1270–1282.