Modern multiloop calculations

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Roman N. Lee
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia
E-mail: r.n.lee@inp.nsk.su

Abstract.
Loop integrals and methods of their evaluations are vital for perturbative calculations in any quantum field theory. As the order of perturbation theory increases the complexity of the relevant multiloop integrals explodes rapidly. In the present contribution I review methods of modern multiloop calculations with the emphasis on the method based on the IBP reduction and differential equations.

1. Introduction

High-precision theoretical description of Standard Model (SM) processes is of crucial importance. In particular, this precision is a necessary condition for searches of New Physics — new particles and interactions — which is expected to reveal itself as small deviations from SM predictions. From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. The complexity of the multiloop calculations grows both qualitatively and quantitatively in an explosive way with increasing the number of loops and/or scales. Therefore, in this field new methods and approaches are always in demand. Recently the multiloop calculations have experienced a great progress. This was possible due to the rise of computers, on one hand, and due to the adaptation of methods from several branches of mathematics, on the other hand. Nowadays we have a full stack of tools for perturbative calculations, but still we need more. In the present contribution I will discuss some of the available tools and methods.

2. IBP reduction

Perturbative calculations in quantum field theory start from the generation of diagrams which contribute to a specific chosen order. While the number of these diagrams rapidly grows as we go for higher orders, this step is well automatized [1, 2, 3] and is never a bottleneck in real-life calculation.

The first stage which may provide substantial difficulty is the IBP reduction [4, 5]. The standard approach to the IBP reduction is the following. Given an L-loop Feynman diagram with $E+1$ external legs, we consider a family of integrals labeled by $\boldsymbol{n} = (n_1, \ldots, n_N)$

$$j(\boldsymbol{n}) = \int d\mu_L \prod_{k=1}^{N} D_k^{-n_k}, \quad d\mu_L = \prod_{k=i}^{L} d^d l_i, \tag{1}$$

where $D_1, \ldots, D_M$ are denominators of the propagators corresponding to the internal lines of the diagram and $D_{M+1}, \ldots, D_N$ are so called irreducible numerators, chosen such that any scalar

product $l_i \cdot q_j$, involving loop momentum (l_i is a loop momentum and q_k is either loop or external momentum), can be uniquely expressed as a linear function of D_k. In particular, N is equal to the number of such scalar products,

$$N = L(L+1)/2 + L \cdot E\,. \tag{2}$$

We assumed here that $D_1, \ldots, D_M$ are linearly independent, i.e., that any nonzero linear combination of D_k depends on loop momenta. Otherwise, we have to perform first a multivariate partial fraction decomposition [6, 7, 8].

There are infinitely many integrals in this family, corresponding to different choices of multiindex $\boldsymbol{n} \in \mathbb{Z}^N$, however, there are also infinitely many relations between them. These relations come from the identities

$$\int d\mu_L \frac{\partial}{\partial l_i} \cdot q_j \prod_{k=1}^{N} D_k^{-n_k} = 0\,. \tag{3}$$

This identity corresponds to the fact that the integral of a total derivative is zero in dimensional regularization. Explicitly differentiating under the integral sign and expressing emerging scalar products via D_k, we obtain the linear combination of integrals of the form (1) with shifted indices. Introducing operators $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ acting as

$$(A_k j)(\ldots, n_k, \ldots) = n_k j(\ldots, n_k + 1, \ldots), \quad (B_k j)(\ldots, n_k, \ldots) = j(\ldots, n_k - 1, \ldots), \tag{4}$$

we can represent resulting IBP identities in the following form

$$[C_{kl}^{(ij)} A_k B_l + C_k^{(ij)} A_k + C^{(ij)}] j = 0, \tag{5}$$

where the coefficients $C_{kl}^{(ij)}$, $C_k^{(ij)}$, $C^{(ij)}$ depend on the choice of the operator $\frac{\partial}{\partial l_i} \cdot q_j$ in Eq. (3), in particular, $C^{(ij)} = \delta_{ij} d$. IBP identities can be used for expressing more complicated integrals via simpler ones. A nontrivial net result of this reduction is that any integral of a given family (1) can be expressed as linear combination of the so called *master integrals*, see Refs. [9, 10].

A standard approach to the IBP reduction is described in Ref. [11]. Schematically, this approach involves the following steps:

(i) Choose admissible ordering among integrals of the family in Eq. (1).
(ii) Generate IBP identities to some fixed depth.
(iii) Solve these identities using Gauss elimination and collect the resulting reduction rules in a database.

Obtained reduction rules can then be used for the IBP reduction. With some reservations, this is the approach implemented in most of the public reduction programs: `FIRE` [12], `Kira` [13], `Reduze 2` [14] and some private packages[1]. Recently this method has been augmented by using mappings of coefficients in the IBP identities onto finite fields $\mathbb{F}_p$ for several values of p and then using a reconstruction procedure [16, 13].

A few years ago there appeared some ideas of making IBP reduction in parametric representations. In Ref. [17] it was shown how to derive IBP identities starting from the representation [10]

$$j(\boldsymbol{n}) = \frac{\tilde{j}(\boldsymbol{n})}{\Gamma((L+1)d/2 - |\boldsymbol{n}|)} = \frac{\Gamma(d/2)}{\Gamma((L+1)d/2 - |\boldsymbol{n}|)} \int\limits_{\mathbb{R}_+^n} G^{-d/2} \prod_k \frac{dx_k x_k^{n_k - 1}}{\Gamma(n_k)}\,, \tag{6}$$

[1] The `LiteRed` program uses a more complex approach, which resembles a manual search of symbolic reduction rules, see [15].

where $G = U + F$ is the sum of two Symanzik polynomials. Namely, it was shown that the syzygy module of the ideal generated by $\boldsymbol{\nabla} G, G$ can be used for constructing IBP identities. Each syzygy

$$\boldsymbol{Q}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} G + Q(\boldsymbol{x}) G = 0 \tag{7}$$

gives rise to the IBP identity of the form similar to Eq. (5):

$$\left[\frac{d}{2} Q(\boldsymbol{A}) + \boldsymbol{Q}(\boldsymbol{A}) \cdot \boldsymbol{B} \right] \tilde{j} = 0 \tag{8}$$

Note that a similar approach for the Baikov representation [18, 19] was suggested in Ref. [20, 21]. In order to explain the benefit of the IBP identities in the parametric representation (6), we note that this representation does not require the introduction of irreducible numerators. The number of parameters is equal to the number of internal lines of the diagram, I. From graph theory we have the restriction $I \leqslant E + 1 + 3(L-1)$, where $E+1$ is the number of external legs. Therefore, the number of parameters grows *linearly* with the number of loops, while the number of scalar products, Eq. (2), grows quadratically. Thus, it is quite meaningful to expect that for higher loops the IBP reduction using the parametric representation is advantageous compared to the conventional approach based on momentum representation. We would like to mention one important practical issue related to finding the syzygy module. This is a routine procedure in computational commutative algebra which is implemented, in particular, in `Singular` CAS. However, the corresponding procedure returns redundant generating set of syzygies module, which typically includes very cumbersome elements. These elements have no practical value and should be detected and removed for the sake of efficiency.

Recently, an approach based on the intersection theory has been suggested in Refs. [22, 23]. Within this approach the integral is understood as a pairing between differential M-form and M-dimensional integration cycle:

$$\langle \phi | C] = \int\limits_C G^{-\nu} \phi \,, \quad \nu = d/2 \,. \tag{9}$$

In particular, the integral in Eq. (6) corresponds to $\phi \propto \wedge_k dx_k x_k^{n_k - 1}$. This bilinear form can be naturally understood as a pairing between (elements of) twisted de Rham cohomology and twisted homology groups since the integral depends only on the corresponding equivalence classes[2]. The twisted differential is defined as

$$d_\nu = d - \nu \, d \ln G \wedge \,. \tag{10}$$

Cho and Matsumoto in Ref. [24] introduced a pairing $\langle \phi_1 | \phi_2 \rangle$ between differential forms which is invariant under shifts

$$\phi_1 \to \phi_1 + d_\nu \chi_1, \qquad \phi_2 \to \phi_2 + d_{-\nu} \chi_2. \tag{11}$$

In other words, this is a pairing between twisted de Rham cohomologies with differential d_ν and $d_{-\nu}$. Then the idea of Refs. [22, 23] is to use the decomposition

$$\langle \phi | C] = \sum_{k=1}^{K} a_k \, \langle \phi_k | C] \,, \qquad a_k = \sum_{i=1}^{K} \left\langle \phi | \tilde{\phi}_i \right\rangle \left(I^{-1} \right)_{ik} \tag{12}$$

[2] Strictly speaking, the integration domain $\mathbb{R}^n_+$ in Eq. (6) requires analytical regularization to avoid the surface terms.

Here

$$I = \left[\left\langle \phi_i | \tilde{\phi}_j \right\rangle_{1 \leqslant i,j \leqslant K} \right] \tag{13}$$

is the so called intersection matrix, $\{\phi_1, \ldots, \phi_K\}$ and $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_K\}$ form the bases of twisted de Rham cohomologies with differential d_ν and $d_{-\nu}$, respectively[3]. This gives the decomposition of integral $\langle \phi | C] = \int_C G^{-\nu} \phi$ in terms of master integrals $\langle \phi_k | C]$. Note that the coefficients a_k in front of the master integrals are defined via $\langle \ldots \rangle$ pairing. This pairing can be simpler to calculate than the original integral $\langle \phi | C]$, Eq. (9). This is especially true for 1-forms ($M = 1$) corresponding to one-fold integrals. In Ref. [22] two recipes for finding $\langle \phi_1 | \phi_2 \rangle$ in this case have been given. One is based on expansion near zeros of polynomial G, another uses expansions near zeros of ∂G. The first recipe is difficult to generalize to the case of $n > 1$ integration variables as in this case the zeros of G form $n - 1$ dimensional varieties. The second recipe can be easily generalized to $M > 1$, however, it works only for logarithmic forms. Recent papers [25, 26] try to elaborate more effective algorithms of calculating the M-form intersection numbers, however the efficiency of this approach is under question (which one may conclude, in particular, from rather simple examples presented in those papers). Therefore, the perspectives of using the intersection theory for the reduction purpose are still quite unclear.[4]

The next stage of multiloop calculations is the evaluation of master integrals. In the next section we describe an approach based on differential equations for the master integrals.

3. Differential equations for master integrals

In order to obtain the differential equations for master integrals, we differentiate them with respect to external parameter x (mass or kinematic invariant) and reduce the result of differentiation to master integrals [28, 29]. Then we obtain the differential system of the form

$$\frac{\partial}{\partial x} \boldsymbol{j} = M \boldsymbol{j} \tag{14}$$

Here $\boldsymbol{j}$ is a column of master integrals and $M = M(\epsilon, x)$ is a matrix with elements being rational functions of $\epsilon = 2 - d/2$ and x. Note that for applications in calculation of the radiative corrections we need the solution of this system in the form of expansion in ϵ.

There has been a great progress in the automatic solution of such differential systems after Henn's paper [30], where it was observed that upon a wise choice of master integrals (canonical basis) the system acquires a very special form with the matrix in the right-hand side having the form

$$M(\epsilon, x) = \epsilon S(x). \tag{15}$$

In Ref. [31] the algorithm of finding the transformation matrix T was presented, such that the change of function $\boldsymbol{j} = T \boldsymbol{J}$ results in the differential system in ϵ-form:

$$\frac{\partial}{\partial x} \boldsymbol{J} = \epsilon S(x), \qquad S(x) = \sum_k \frac{A_k}{x - a_k} \boldsymbol{J}. \tag{16}$$

The advantage of the differential system in ϵ-form is that its general solution

$$U = \mathrm{Pexp} \left[\epsilon \int dx S(x) \right] \tag{17}$$

[3] One can choose to use the same set of differential forms $\{\phi_1, \ldots, \phi_K\}$ to label bases in both d_ν and $d_{-\nu}$ de Rham cohomologies.

[4] One might think of following the opposite direction: use conventional multiloop methods to calculate the intersection matrix and twisted Riemann relations. Namely, the conventional IBP reduction provides a rather powerful tool to construct the differential systems which in turn can be used for the calculation of the intersection matrix along the lines of Ref. [27].

can be readily expanded in ϵ-series in terms of multiple polylogarithms [32].

The algorithm of Ref. [31] consists of three stages:

(i) Reducing to global Fuchsian form. The result of this step is the system (14) with $M = \sum_k \frac{A_k(\epsilon)}{x-a_k}$.

(ii) Normalizing the eigenvalues of the matrix residues. The result is the system (14) with $M = \sum_k \frac{A_k(\epsilon)}{x-a_k}$ such that all eigenvalues of any A_k are proportional to ϵ.

(iii) Factorizing ϵ. The result is the system in ϵ-form (16).

A simplified description of the first step can be found in Ref. [33, Section E.8]. Two first stages of this algorithm involve sequences of "elementary" transformations, each of them slightly improves properties of the system.

There are three public codes which implement this algorithm: `epsilon` [34], `Fuchsia` [35], and `Libra` [36]. These programs have been already used in many physical applications. In particular, `Libra` has been used in calculations of 4-loop form factors recently finished [37]. This program has proved to be able to handle huge systems of a few hundreds of differential equations.

Note that the existence of a rational, or even algebraic, transformation matrix $T(\epsilon, x)$ which reduces the system to ϵ-form is very unexpected. In other words, given a differential system with some random rational matrix $M(\epsilon, x)$, all chances are that it can not be reduced to ϵ-form. The more remarkable is the fact that so many differential systems which emerge in multiloop calculations can be reduced to this form.

With some reservations, the reducibility of the system to ϵ-form means that its solution is expressible via multiple polylogarithms[5]. However, there are many known examples of integrals which can not be expressed via polylogarithms. Perhaps, the most famous is the two-loop sunrise integral which can be expressed via complete elliptic integrals $\mathrm{K}(x)$ and $\mathrm{E}(x)$. Therefore, it is important to have a rigorous criterion of (ir)reducibility of the differential system to ϵ-form. Such a criterion has been elaborated in Ref. [38]. It is based on the following simple proposition [38, Proposition 1]: when the system is normalized at some point (i.e., all eigenvalues of the matrix residue at this point are proportional to ϵ), any further transformation which preserves this property should be given by a transformation matrix which is regular and invertible at this point.

Then, according to Ref. [38], in order to find the rational transformation to ϵ-form or to prove that it does not exist, one should use the following refined algorithm

(i) Choose some point, say $x_0 = \infty$.

(ii) Reduce the system to Fuchsian form in all points but maybe x_0. If this is not possible, then the system is not regular. In the context of multiloop calculations this usually signals an incomplete IBP reduction.

(iii) Normalize eigenvalues of the matrix residues the system to Fuchsian form in all points but x_0. This step can be performed iff after the previous step matrix residues have the form $n+k\epsilon$ with integer n. If this is not so, the system can not be reduced to ϵ-form with rational transformations.

(iv) Now choose another point, say $x_0 = 0$ and perform two previous steps.

(v) Let $T_{\infty 0}$ be the transformation matrix obtained at the previous step. According to the proposition mentioned, this matrix can be singular only at two points, 0 and ∞. Then we

[5] If the transformation matrix and final system in ϵ-form requires algebraic extensions, the emerging iterated integrals are, in general, not expressible via polylogarithms.

can use Birkhoff-Grothendieck factorization[6] [39, 40] to decompose T as

$$T = L(z)D(z)R(z^{-1}), \tag{18}$$

where $L(z), L(z)^{-1}$ and $R(z^{-1}), R(z^{-1})^{-1}$ are polynomial in their arguments and $D(z) = z^{\mathrm{diag}(n_1,\ldots,n_m)}$. Then the reduction to ϵ-form is impossible unless D is an identity matrix. If $D = 1$, apply transformation R^{-1} to secure normalization at all points.

(vi) Factorize ϵ. If this step can not be performed, the system can not be reduced to ϵ-form.

If the system is not reducible to ϵ-form, one might ask if there exists some more general form, yet simple enough for a systematic treatment. From many examples it seems that any differential system for master integrals can be transformed to the form (14) with

$$M(\epsilon, x) = A(x) + \epsilon B(x). \tag{19}$$

Such a form would at least allow to systematically organize the ϵ expansion of non-polylogarithmic master integrals. A universal algorithm of finding such a form would be a clear advance. It worth noting that the form (19) is not specific enough as any transformation independent of ϵ preserves this form only altering $A(x)$. Perhaps, understanding how to fix this arbitrariness is necessary for the reduction algorithm to appear. In Ref. [41] it was noticed that in many cases the irreducible diagonal blocks can be reduced to $(\epsilon+1/2)$-form, which corresponds to $A(x) = B(x)/2$. However, there are examples where neither ϵ-form nor $(\epsilon + 1/2)$-form exist.

4. Conclusion

To summarize, the multiloop calculations have experienced a great progress during last ten years. In this contribution I have presented a short review of existing approaches and some ideas suggested recently. Two essential ingredients of modern methods are the IBP reduction and solution of differential equations for master integrals. Both these steps are highly automatized and are ready for treatment of very complicated families involving hundreds of master integrals. However, a systematic approach to the calculation of non-polylogarithmic multiloop integrals is yet to be developed.

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