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Periodical properties of the ray transfer matrix and generation of sideband modes in a stable laser resonator

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The paper describes a subclass of stable laser cavities, periodic stable laser cavities, in which perturbations consisting of deviations of the mode axis from the ideal direction are of a strictly periodic oscillatory nature. In such resonators, in addition to unperturbed longitudinal–transverse spatial modes with an ideal direction of the optical axis, additional modes can appear at sideband frequencies, associated with the resonant buildup of perturbation oscillations. These modes have approximately the same spatial structure as those of the unperturbed fundamental modes, and their frequency detuning from the frequencies of the fundamental modes is governed by the resonator geometry and the periodicity parameter, i.e., the number of passes in the resonator in one period of perturbation oscillations. For many repetitively pulsed laser systems emitting comb spectrum structures, such as free electron lasers, modern frequency standards using femtosecond lasers, and various comb spectrometers, it is desirable to avoid such periodic stable cavities in order to preserve the purity of the comb spectrum used in them. This may also be important for CW lasers with extreme radiation monochromaticity. In some repetitively pulsed lasers, on the contrary, it may be desirable to use such periodic stable laser cavities for a more complete frequency filling and higher quasi-continuity of their emission spectra.

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The well-known and widely used theory of open laser cavities, developed by many authors and summarized in Kogelnik and Li [1], considers separately the matrix theory of propagation of paraxial optical rays in the geometrical optics approximation and the diffraction theory of Gaussian beams, which determine the real spatial structure of laser modes. Within the framework of the matrix theory, the concept of stability of a laser resonator is formulated for certain interrelations of its geometric parameters, when the laser ray oscillating between mirrors does not go beyond a certain radial size. Further, it is assumed that it is in such stable geometries that stable configurations of the intracavity field (resonator modes) are formed, although the mode shape itself can be found only by wave optics methods. Analysis of the dynamics of the laser mode axis in a laser cavity in the geometrical optics approximation is widely used in practice, for example, in lasers based on dielectric microcavities [2] and multipass lasers [3].

In this paper, the matrix theory of ray transfer is applied to analysis of the behavior of the mode axis perturbation due to the asymmetry of the amplifying medium—a tilted electron beam. Thus, we consider here the mode axis as an analog of the ray in the matrix theory.

For simplicity, we will consider open axially symmetric laser cavities of length L with identical mirrors with radius of curvature R, in which the optical mode axis is characterized by position r and angular inclination r'. According to the matrix theory of beam propagation in a stable laser resonator [1], the beam parameters after n passes in the resonator will be determined as

$$\begin{pmatrix} r_n \\ r'_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix}, \quad (1)$$

where $A = 1$, $B = L$, $C = -2/R$, $D = 1 - 2L/R$, $A_n = [A \sin n\Theta - \sin(n-1)\Theta]/\sin\Theta$, $B_n = B \sin n\Theta/\sin\Theta$, $C_n = C \sin n\Theta/\sin\Theta$, $D_n = [D \sin n\Theta - \sin(n-1)\Theta]/\sin\Theta$, $\Theta = \arccos(A+D)/2$, and r_0 and r'_0 are the initial parameters of the mode axis. As in Kogelnik and Li [1], the paraxial approximation $r_n \ll L$ is used here.

For simplicity of further consideration, we will choose the initial parameters of the mode axis as shown in Fig. 1, which correspond to the most frequent perturbation of the mode axis in free electron lasers and optically pumped lasers when the mode axis passes obliquely through the center of the resonator. Obviously, such initial perturbation parameters of the axis are in the relation $r'_0 = -2r_0/L$. In the absence of perturbation, when the mode axis coincides with the geometric axis of the resonator (the line passing through the centers of the mirrors), $r_0 = r'_0 = 0$.

The elements of the finite ray transfer matrix A_n, B_n, C_n, D_n, written in Eq. (1) in the form of bounded trigonometric functions, correspond to stable laser cavities, the geometry of which must obey the inequality $-1 < \cos\Theta = 1 - L/R < 1$ [1].

Note that, in addition to the property of boundedness, the matrix of a stable laser resonator with certain geometric parameters has another interesting feature, the property of periodicity. By this, we understand the exact periodic repetition of the initial parameters of the mode axis r_0 and r'_0 after some integer number m of passes through the resonator. We will call m the periodicity parameter. Let us consider stable resonators with parameters at which $\Theta = \arccos(1 - L/R) = (k/m)\pi$, where k is a positive

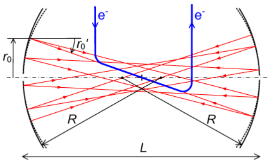

Fig. 1. Optical scheme of the periodic resonator ($k = 7$, $m = 9$): the direction of the first light passage coincides with the direction of the inclined amplifying electron beam; the dashed lines are the spheres of radius $L/2$ from the center of the resonator.

integer, including zero (0, 1, 2, 3, . . .), and m is a natural number (1, 2, 3, . . .). It is assumed that the numbers k and m are mutually prime numbers, i.e., do not contain the same integer factors other than 1, which must cancel. Then, from Eq. (1),

$$r_m = [\sin(\Theta - k\pi)/\sin\Theta]r_0 = (-1)^k r_0;$$
$$r'_m = [\sin(\Theta - k\pi)/\sin\Theta]r'_0 = (-1)^k r'_0. \quad (2)$$

Thus, the initial parameters of the mode axis will occur after m passes in the cavity for an even k and after $2m$ passes in the cavity for an odd k. We call such stable cavities periodic stable laser cavities.

In addition to the periodicity condition [Eq. (2)], it will be essential for further analysis to consider the behavior of the perturbation in the intermediate passages in the resonator between the periods, especially for $m \gg 1$. For definiteness, we consider a periodic stable resonator with parameters ($k = 7$, $m = 9$), which happens to be very similar to the resonator of the Novosibirsk free electron laser (NovoFEL), and the typical perturbation of its optical axis, associated with the inclination of the emitting electron beam relative to the optical axis of the laser resonator (Fig. 1). In solid-state lasers with optical pumping, a similar perturbation occurs when the axis of the pump laser is tilted to the axis of the optical cavity of the pumped laser. The path of rays in the resonator, in particular, their periodicity, depends only on the ratio L/R and does not depend on their absolute values, as well as on the value of r_0. For this reason, Fig. 1 shows, for clarity, a resonator similar to the NovoFEL resonator, but with greatly enlarged transverse dimensions of the resonator and much higher perturbations of the mode axis.

In a typical NovoFEL regime, electron pulses follow with intervals equal to the time for the light pulse to make a round trip in the resonator. Therefore, single-pass electron pulses are accompanied by an intracavity light pulse in even passes of the resonator ($n = 0, 2, 4, 6, \ldots$). Further, we will be interested in the time evolution of perturbations arising at each even pass. Figure 2(a) shows the change in the parameters of the mode axis after one zero perturbation ($n = 0$). It can be seen that the pattern of both perturbation parameters is strictly periodic after 18 passes in the resonator (nine round trips in the resonator), as follows from Eq. (2). However, NovoFEL electron pulses follow continuously with a period of two passes. If we assume that all electron pulses are identical, then the stationary regime will feature average smoothing of all perturbations, and the mode axis will probably only slightly tilt toward the electron beam. There will be no oscillation of the mode axis. In this sense, the situation is absolutely similar to appearance of lasing in a free electron laser (FEL). When the electron beam is uniform in density, its radiation power is zero. Laser radiation arises due to random statistical fluctuations in the beam density.

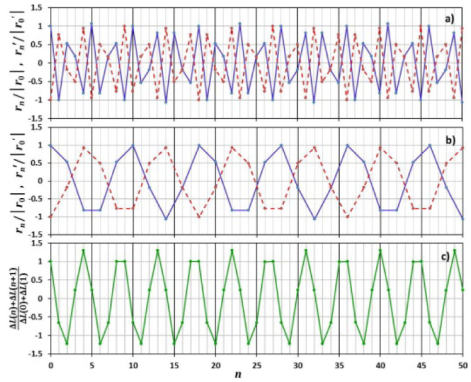

Fig. 2. (a) Normalized perturbations of the mode axis as a function of pass number (solid line, position; dashed line, angular inclination). (b) The same as (a), except for even passes only. (c) Lengthening of the round trip in the resonator as a function of the pass number for the periodic resonator ($k = 7$, $m = 9$).

Similarly, at growing perturbations, we must consider fluctuations in the total charge (current) in individual electron pulses, especially because they are quite real for both technical and statistical reasons. Then it makes sense to consider how effectively the perturbation from the zero, most powerful, electronic pulse ($n = 0$) will be maintained by subsequent electron pulses. Obviously, this efficiency will depend on how close the mode axis parameters in subsequent even passes of the cavity are to these parameters for the zero pass (all electron pulses of NovoFEL have the same axis). The change in the parameters of the mode axis in even passes is shown in Fig. 2(b). Despite the same rigorously mathematical repetition period of 18 passes, the picture is substantially different from the original picture in Fig. 2(a). Firstly, it has acquired a cosine form, which is optimal for the excitation of oscillations. Secondly, it has a new approximate physical period of nine passes. Thus, a random fluctuation of the electron beam in the initial zero pass will be swung in amplitude by almost all subsequent NovoFEL pulses, with maximum efficiency in each $(18 \times p)$th ($p = 0, 1, 2, \ldots$) pass, almost the same efficiency in half the period, and minimum efficiency in passages with opposite signs of disturbances [Fig. 2(b)]. Note that this periodicity, like the classical periodicity in a laser cavity, is fundamentally important. For resonators that are non-periodic in terms of perturbation, after a certain number of passes in the resonator, the phase of the perturbation will change to the opposite, owing to the drift, and excitation of perturbations will be impossible in the stationary regime. Laser generation in NovoFEL grows from spontaneous emission to a steady-state level in a time of about several tens of round trips in the cavity by the intracavity light pulse. The developed steady-state perturbation of the mode will grow during several hundred round trips and will consist in oscillation of the mode axis tilt relative to the direction of the laser resonator axis with a period of nine passes. Figure 2(b) also suggests a general remark for all periodic resonators: the greatest hazard in terms of oscillation buildup will be constituted by resonators with the smallest number m, e.g., a confocal resonator located on the stability boundary with parameters ($k = 1$, $m = 4$), often used in comb systems of frequency standards [4].

Let us now consider how oscillation of the axis of the main laser modes generates additional modes to the main modes at sideband frequencies. The ideal NovoFEL super-mode is a comb structure, i.e., a set of equidistant, very narrow, modes with different longitudinal indices within a gain contour 6–10 GHz wide, whose center frequencies differ by $\Delta\nu = c/2L = 5.6$ MHz, i.e., the frequency of the round trip of a light pulse inside the optical resonator [5]. It is assumed that, in the case of a well-tuned electron beam, the mode axis of length L_0 does not change and coincides with the resonator axis of length L (Fig. 1). In a regime with significant perturbation of the optical axis of the mode, the optical axis undergoes periodic oscillations in the coordinate and angle. In this case, the effective length of the resonator, which is determined by the reflection points on the mirrors, and the effective-length-specific frequencies of the comb structure, will also change periodically. The length of the mode axis between the reflection points on the mirrors n and $n+1$ is $L(n) = (r_n - r_{n+1})/\sin[arctg(r'_n)] - (r_n^2 + r_{n+1}^2)/2R$. Each mode of the comb structure must satisfy the main resonance condition, i.e., the integer number of its wavelengths must fit into the round trip in the resonator $L(n) + L(n+1)$. In this case, when the change in the round-trip length of the resonator, $\Delta L(n) = L(n) + L(n+1) - 2L_0$, grows, the distance between the frequencies of the comb structure decreases, and when $\Delta L(n)$ becomes as large as the wavelength, the frequency of the ith comb mode is shifted to the place of the adjacent $(i-1)$th comb mode in the unperturbed comb structure. Figure 2(c) shows the normalized change in the round-trip length of the resonator $\Delta L(n)$ for the parameters of the resonator ($k = 7$, $m = 9$), which does not depend on the magnitude of the initial perturbation r_0, but depends only on the geometry of the resonator. It can be seen that this function has an approximately cosine form and is periodic with a mathematical (exact) period of nine passes, but has a smaller physical (approximate) period of 4.5 passes. It can also be seen that each positive disturbance-"feeding" half-wave [Fig. 2(b)] with a period of nine passes coincides with one period of 4.5 passes of the resonator length oscillations in Fig. 2(c). This means that, without taking into account the feeding of the perturbation by electron pulses, we shall observe a frequency modulation of the modes of the comb structure with a period of 4.5 passes. However, given this fact, as well as a slightly larger value of modulation with a mathematical period of nine passes [Fig. 2 (c)], we should assume that, in many cases, we will have a frequency modulation with a period of nine passes. It will be subsequently shown that both of these cases have been observed at NovoFEL. Thus, in the assumption of the cosine form of the perturbation in Fig. 2(c), the temporal form of the field of each individual ith mode in the comb structure can be written as

$$E_i(t) \sim \cos[(\omega_{0i} - \beta\Omega \times \cos\ \Omega t)t] = \sum_p J_p(\beta) \times \cos[(\omega_{0i} + p\Omega)t], \tag{3}$$

where β is the frequency modulation index, Ω is the frequency modulation frequency, equal in our case to $\Omega = 2\pi(c/2L)/G$, $G = 9/2 = 4.5$ for a modulation of the round-trip length in the resonator with a period of nine passes, and $G = 4.5/2 = 2.25$ for this modulation with a period of 4.5 passes. In addition, J_p is the Bessel function of order p, and p is an integer $(\ldots, -2, -1, 0, 1, 2, \ldots)$. Taking into account the fact that $J_{-p}(\beta) = (-1)^p J_p(\beta)$, we obtain that the laser radiation intensity spectrum will be a periodic structure in which each unit-power line of the comb structure is surrounded from both sides by symmetric sideband modes (more precisely, by the sideband frequencies of the same comb laser line) with relative intensities $J_p(\beta)^2/J_0(\beta)^2$. Thus, the positions of the sideband modes are governed by the modulation frequency Ω, and their intensity by the frequency modulation index β. Considering the form of the squares of the Bessel functions in dependence on the value of β, we conclude that the magnitude of the sideband modes can be either less or greater than that of the central comb laser line. In particular $J_1(1.45)^2 = J_0(1.45)^2$. In this case, $J_2(1.45)^2/J_0(1.45)^2 = 0.16$, and the angular frequency deviation required at $G = 4.5$ is $\beta\Omega = 0.32 \times 2\pi(c/2L) = 11.3$ MHz. Such frequency deviation corresponds to the maximum amplitude of the resonator extension $\Delta L_{\max} = 0.32\ \lambda$, where λ is the radiation wavelength.

The repetitively pulsed emission of the terahertz NovoFEL in a typical regime is a continuous sequence of 100-ps pulses with repetition rate of 5.6 MHz, emitted by a single intracavity light pulse [6]. Its optical resonator for wavelengths shorter than 200 µm is of the open type [7,8] and consists of two spherical mirrors of 190 mm in diameter and curvature radius of 15 m, located at a distance $L = 26.589$ m. The mirrors have small round holes in the center for radiation output and alignment. The length of the resonator was chosen for the reason of synchronization with the frequency of the RF resonators of the linear accelerator and turned out to be very close to the length of the periodic resonator $(k = 7,\ \ m = 9)$: $L_{\text{per}} = R[1 - \cos(k\pi/m)] = 26.491$ m. Therefore, high-power sideband modes can be excited in the NovoFEL cavity if the electron beam is not perfectly aligned.

Note that, from a physical point of view, the generation of sideband modes is universal in nature and occurs when some lower-frequency perturbation oscillation is superimposed on the main oscillatory motion of electrons or the radiation field. In the FEL, this effect is much more pronounced than that in quantum lasers, owing to the much larger number of degrees of freedom of the radiating system, i.e., the electron beam. Thus, in the NovoFEL resonance regime, when the repetition frequencies of the electron and light pulses inside the optical resonator coincide exactly, one or both of the instabilities (trapped electron instability and modulation instability) are always observed. In the regime with a pulse duration of ∼200 ps, NovoFEL light pulses are split into 6–7 or 2 parts, which are incoherent with each other, and sideband modes appear in the emission spectrum at frequencies of 30 and 6 GHz [9]. These instabilities and associated sideband modes were completely suppressed by the introduction of a stabilizing factor, i.e., a negative detuning of the electron pulse repetition rate Δf from the intracavity light pulse repetition frequency f [9].

This paper describes another instability with frequencies lower by three orders of magnitude, whose sideband modes are visible only in the hyperfine mode structure of the NovoFEL emission spectrum. Although the super-mode comb structure of the spectrum has been known since the first works on the FEL theory [10], its practical measurement became possible only after the creation of a special device, the ultralong-resonance Fabry–Pérot interferometer, described in detail elsewhere [5]. Measurement of the mode emission spectrum of NovoFEL with this device sometimes gave completely unexpected results (Fig. 3). For example, along with generation of one super-mode in the case of the ideal axial injection of the electron beam (Curve 1 in Fig. 3), in the case of its non-ideal inclined injection, four more modes (Curve 4 in Fig. 3) were observed with approximately the same radiation power [9]. For lack of a better explanation, in Kubarev *et al.* [9] these additional modes were associated by the author of this paper with the transverse laser cavity modes excited by an inclined electron beam. However,

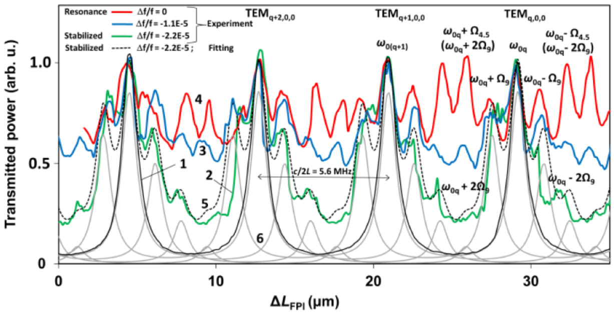

Fig. 3. Mode structure of NovoFEL radiation at a wavelength of 164 μm (normalized transmitted power as a function of change in the length of the resonance Fabry–Pérot interferometer): 1, a single super-mode without any sideband modes at an ideal axial electron beam injection; 2–4, a single super-mode with sideband frequencies at an inclined injection of the electron beam for different degrees of radiation stabilization; 5, a fitting of experimental stabilized regime (2) using modes (6, gray lines) of the instrumental width of the interferometer.

despite formally suitable frequencies, these modes contradicted the fundamentals of laser physics, according to which the powers of laser modes are inversely related to their losses, which grow rapidly with increasing transverse indices. Moreover, gain and loss measurements showed that the generation of these higher transverse modes is simply impossible with their classical transverse spatial distribution. The experimental losses in the multi-mode and single-mode regimes differed weakly and were close to the calculated losses of the fundamental mode. Higher laser modes have much higher losses and, if they were generated, a large increase in the experimental losses would have been observed when switching from a single-mode regime to a multi-mode one. In addition, a large number of transverse laser modes is not characteristic of a homogeneous FEL amplifying medium, in which the main generating mode must suppress the others, owing to mode competition.

All these contradictions are removed with the theory of excitation of sideband modes in periodic laser cavities, presented in this paper. Thus, the mode spectra with inclined injection in Fig. 3 also correspond to the single TEM$_{q00}$ super-mode, which, oscillating inside the optical cavity, generates, in the stabilized regime (Curve 2 in Fig. 3), additional sideband modes at the reciprocal frequency of nine cavity passes ($\Omega_9/2\pi$ = 1.24 MHz). In the resonance regime (Curve 4 in Fig. 3), when the perturbation of the electron beam and mode is maximal, it is possible to generate sideband modes at two characteristic reciprocal frequencies of 9 and 4.5 ($\Omega_{4.5}/2\pi$ = 2.49 MHz) passes in the cavity [Fig. 2(c)]. Moreover, the power of the sideband mode $\Omega_{4.5}$ in the far right period of the interferogram almost equaled the power of the main mode. The second harmonic $2\Omega_9$ coincides with this sideband mode, but its share does not exceed 16%. The violation of the strict periodicity of Interferograms 2–4 with inclined injection of the electron beam is associated with the rather long measurement time (∼5–10 min [5]) and the drift of the amplitude parameters of the sideband modes. Similar smaller sideband modes (like Curve 2 in Fig. 3) have also been observed at many other NovoFEL generation wavelengths. Owing to the mode competition (visible in Fig. 3), the coherence of the NovoFEL radiation decreased, which worsened the quality of the experiments carried out on it in the field of ultrafast heterodyne time-domain spectroscopy.

With the given length of the NovoFEL resonator L = 26.589 m, the observed resonance (k = 7, m = 9) corresponds to R = 15.056 m. The nearest strong resonances, (k = 4, m = 5) and (k = 10, m = 13), will be at R = 14.698 m and R = 15.207 m, respectively. Therefore, the excitation of sideband modes will be minimal for R = 15.131 m, which is intermediate between the two weakest resonances. In this case, dephasing of the perturbation periodicity [a phase shift of π of the function $\Delta L(n)$] occurs in the 35th round trip in the resonator.

The real relative width of the lines of the NovoFEL comb structure (Fig. 3) is very small. It was accurately measured in the time-domain regime of the resonance interferometer and found to be $\delta\nu/\nu = 2.2 \times 10^{-8}$ [5]. Therefore, separating one such line from the NovoFEL radiation by using three resonance interferometers, one can obtain an ultramonochromatic tunable CW source for THz spectroscopy with a power many orders of magnitude higher than that of other alternative sources [11]. Another example of possible application of NovoFEL in the field of ultrahigh-resolution spectroscopy is the use of the NovoFEL comb structure in coherent comb spectroscopy, similar to that successfully tested by Tammaro *et al.* [12]. In this case, sideband modes will even be useful, because they increase the spectral resolution but, of course, this must be under the condition of good stability of these modes.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the author upon reasonable request.

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