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Colour structure of three-Reggeon cuts in QCD

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The pole Regge form of QCD amplitudes with gluon quantum numbers in cross-channels and negative signature, which underlies the BFKL approach to the theoretical description of semi-hard processes, is violated in the next-to-next-to-leading logarithmic approximation by contributions of three-Reggeon cuts. The presence of a colour quantum number and gluon Reggeization makes structure of Regge cuts in QCD different from the structure of the cuts in the old (before QCD) theory of complex angular momenta. In this talk the colour structure of three-Reggeon cuts is analysed.

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1. Introduction

One of remarkable properties of QCD is the Reggeization of all its elementary particles (quarks and gluons) in perturbation theory, which is very important for theoretical description of high energy $\sqrt{s}$ processes. The gluon Reggeization is especially important because it determines the high energy behaviour of non-decreasing with energy cross sections. In particular, it appears to be the basis of the BFKL (Balitskii-Fadin-Kuraev-Lipatov) equation, which was first derived in non-Abelian theories with spontaneously broken symmetry [1–3], and whose applicability in QCD was then shown [4]. The equation was derived using unitarity and analyticity.

The use of the unitarity requires knowledge of multiple production amplitudes. In each order of perturbation theory dominant (having the largest $\ln s$ degrees) are amplitudes with gluon quantum numbers and negative signatures in cross-channels. They determine the s-channel discontinuities of amplitudes with the same and all other possible quantum numbers. Both in the leading logarithmic approximation (LLA) and in the next-to-leading one (NLLA) the amplitudes used in the unitarity relations are determined by the Regge pole contributions and have a simple factorized form (pole Regge form). Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA. But the Regge pole contributions are not sufficient in the NNLLA. The deviations can be explained by the three-Reggeon cuts.

There is a big difference between the Reggeon cuts in QCD and in the earlier (before appearance of QCD) theory of complex angular momenta. The original reason for this difference is the different concept of Reggeons. In the previous theory, Reggeons were represented by the sum of ladder diagrams, while in QCD the primary Reggeon is a Reggeized gluon. The most important circumstance is that the gluon carries colour.

An important question is the colour structure of Reggeon cuts. This question is simple in the two-Reggeon case, because in the product of two adjoint representations there is only one representation of given dimension and parity. But it becomes quite non-trivial in the case of three-Reggeon cuts.

2. Gluon Reggeization

For elastic scattering processes $A + B \to A' + B'$ in the Regge kinematic region $s \simeq -u \to \infty$, t fixed (i.e. not growing with s), where

$$s = (p_A + p_B)^2, \quad u = (p_A - p_{B'})^2, \quad t = (p_A - p_{A'})^2 \tag{1}$$

the Reggeization means that amplitudes with the gluon quantum numbers in the t-channel and negative signature (symmetry with respect to $s \leftrightarrow u$) are written as

$$\mathcal{A}_{AB}^{A'B'} = \Gamma^c_{A'A} \left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)} \right] \Gamma^c_{B'B} \,. \tag{2}$$

Here $\Gamma^c_{P'P}$ is the particle-particle-Reggeon (PPR) vertex, c is the Reggeon colour index; $j(t) = 1 + \omega(t)$ is the gluon trajectory. In what follows, we will call the gluon trajectory not $j(t)$, but $\omega(t)$, as it is usually done.

The Reggeization means definite form not only of elastic amplitudes, but of inelastic amplitudes in the multi-Regge kinematics (MRK) as well. The MRK is the kinematics where all particles have limited (not growing with s) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with s) invariant masses of any pair of the jets. The MRK gives dominant contributions to cross sections of QCD processes at high energy $\sqrt{s}$. In the LLA only gluon production amplitudes contribute to the unitarity relations. The amplitude of n-gluon production in the MRK is presented in the form

$$\mathcal{A}_{AB}^{A'B'+n} = 2s\,\Gamma_{A'A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{G_i}(q_i,q_{i+1})\left(\frac{s_i}{s_0}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_0}\right)^{\omega(t_{n+1})}\Gamma_{B'B}^{c_{n+1}} \tag{3}$$

Here $\gamma_{c_i c_{i+1}}^{G_i}(q_i,q_{i+1})$ is the Reggeon-Reggeon-Gluon (RRG) vertex, describing production of the gluon G_i with momentum $k_i = q_i - q_{i+1}$ by Reggeons with momenta q_i and q_{i+1} ($q_1 = p_A - p_{A'},\ q_{n+1} = p_{B'} - p_B$) and colour indices c_i and c_{i+1}; $t_i = q_i^2$, $s_i = (k_{i-1}+k_i)^2,\ k_0 \equiv p_{A'},\ k_{n+1} \equiv p_{B'}$. The gluon trajectory $\omega(t)$ and the PPR vertices $\Gamma_{P'P}^{c}$ are the same as in (2); s_0 sets the energy scale, which does not't matter in the LLA.

Pole Regge form (2) of elastic amplitudes remains valid also in the NLLA, evidently with account of the one-loop corrections to $\omega(t)$ and $\Gamma_{P'P}^{c}$. It is necessary to take into account also that in the NLLA instead of each of final particles can be a two-particle jet. This is accounted by the change of the PPR vertices for the PJR vertices describing jet J production at particle-Reggeon interaction. In the amplitudes of multiple production (3), in addition to the indicated changes, production of a two-particle jet has to be taken into account. This is done by the change of the RRG vertices for the RRJ vertices. It is necessary to note here, that because of complicated analytical structure, the pole Regge form (3) is valid only for the real part of the amplitude. Fortunately, it is sufficient for derivation of the BFKL equation in the NLLA. The proof of all these statements is based on bootstrap relations that connect discontinuities in all partial s channels with derivatives over rapidities of final particles (see [5] and references therein). The bootstrap relations are quite simple in the elastic case, but they are not so simple for inelastic amplitudes. There is an infinite number of the bootstrap relations for production amplitudes in the MRK, and they provide the pole Regge form in the NLLA.

It is well known that there is no consistent theory in which all singularities in j plane (plane of complex angular momenta) are poles. Regge poles in the j plane generate Regge cuts. In QCD we know only one Regge pole – the Reggeized gluon with the trajectory $j(t) = 1 + \omega(t)$ and negative signature.

3. Two-Reggeon cuts

Two-Reggeon cuts appear in amplitudes with different from gluon t-channel quantum numbers already in the LLA. In particular, the BFKL Pomeron is a two-Reggeon cut. It appears that in amplitudes with positive signature the real parts of leading logarithmic terms cancel out, so that remaining piece is pure imagine in the LLA.

From the unitarity relation, using the pole Regge form of elastic and MRK amplitudes, we obtain that the s channel imaginary parts of elastic amplitude of the process $AB \to A'B'$ are presented in the symbolic form as

$$\Phi_{A'A} \otimes \hat{\mathcal{G}} \otimes \Phi_{B'B} \,, \tag{4}$$

where impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \to A'$ and $B \to B'$, and $\hat{\mathcal{G}}$ is the Green's function for two interacting Reggeized gluons. The impact factors are expressed through the PPR and PJR vertices and do not depend on energy $\sqrt{s}$.

The Green's function is expressed through the BFKL kernel $\hat{\mathcal{K}}$ and the rapidity $Y = \ln(s/s_0)$:

$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}} \,. \tag{5}$$

In turn, the kernel

$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r \tag{6}$$

is expressed through the gluon trajectories $\omega_{1,2}$ and so called real part $\hat{\mathcal{K}}_r$, which is given by convolutions of production vertices. In the LLA only gluons are produced, and in the NLLA it is necessary to take into account production of quark-antiquark and two-gluon jets,

$$\hat{\mathcal{K}}_r = \hat{\mathcal{K}}_G + \hat{\mathcal{K}}_{Q\bar{Q}} + \hat{\mathcal{K}}_{GG} \,. \tag{7}$$

The kernel is universal (process independent) and determines all energy dependence of scattering amplitudes. But it depends on colour state in the t-channel, that is the kernels are different for different representations of the colour group in the t-channel. Since the energy dependence is connected with leading singularities of partial amplitudes in the complex angular momentum plane, the kernel determines these singularities. Evidently, the singularities can turn out to be a pole only in exceptional cases. In particular, the BFKL Pomeron is not a pole, but a fixed branch point at

$$\omega_P = \frac{4N_c\alpha_s}{\pi}\ln 2. \tag{8}$$

It may seem that contributions other than the Regge pole should appear in the amplitude (4) also in the channel with gluon quantum numbers. It turns out, however, that this is not the case thanks to fulfilment of the bootstrap relations.

There is a big difference between the Reggeon cuts in QCD and in the earlier (before appearance of QCD) theory of complex angular momenta. The original reason for this difference is the different concept of Reggeons. In the previous theory, Reggeons were represented by the sum of ladder diagrams, while in QCD the primary Reggeon is a Reggeized gluon.

In particular, this difference makes invalid the assertion [6], [7], [8] on the fallacy of the proposal [9], [10] to form a cut by contributions of plane diagrams and the statement that only non-planar diagrams do contribute to the cut. It is clearly demonstrated by the BFKL Pomeron.

4. Violation of Regge pole factorisation

Validity of the pole Regge form (2) is proved now in all orders of perturbation theory in the coupling constant g both in the LLA and in the NLLA. But this form is violated in the NNLLA. The first observation of the violation was done [11] at consideration of the high-energy limit of the two-loop amplitudes for gg, gq and qq scattering. The discrepancy appears in non-logarithmic terms. If the pole Regge form would be correct in the NNLLA, they should satisfy the factorization condition. However, it is not the case.

Using the infrared factorization techniques, consideration of the terms responsible for breaking of the pole Regge form in amplitudes of elastic scattering in QCD was performed in [12]-[14]. In particular, the non-logarithmic terms not satisfying the factorization condition at two-loops were recovered and single-logarithmic terms at three loops violating the pole Regge form were found.

It was natural to explain the observed violation by Regge cut contributions. As it was already said, Regge cuts appear in amplitudes with positive signature already in the LLA. But in amplitudes with negative signature Regge cuts must be at least three-Reggeon ones and can appear only in the NNLLA.

5. Three-Reggeon cuts

The first explanation of the violation of the pole Regge form by the three-Reggeon cut contributions was done in [15]. The explanation is based on consideration of Feynman diagrams (I call it diagrammatic approach). In the two-loop approximation, the cut contributions comes from diagrams with three t-channel gluons, which represent the Reggeized gluon in this approximation (see Fig.(1)). For separation of pole and cut contribution, difference in their energy dependence

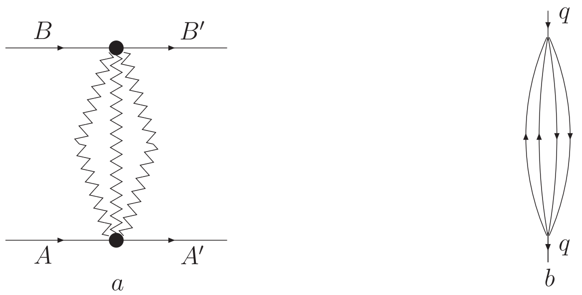

Figure 1: Schematic representation of the three-Reggeon contribution in two loops.
a – Reggeon diagram; b – its momentum part.

was used in the three-loop approximation (see Fig. (2)). Another explanation, which was used in [16], can be called Wilson line approach. It is based on representation of scattering amplitudes by Wilson lines and using the shock wave approximation. Unfortunately, the approaches used and the results given in the papers [15] and [16] are different, although both approach explain the observed violation. In particular, the Reggeon-cut mixing is introduced in the last approach, which is not needed in the first one.

The main difference between two approaches is the consideration of the colour structure of the cuts. The momentum space parts of the cut contributions are calculated in similar way. In both approaches the Reggeon propagator in the transverse momentum space is taken as $\frac{1}{\mathbf{k}_\perp^2}$ (see Fig. (1 b) and Fig. (2 b and c)). In higher approximation cut contributions are calculated supposing pair interaction between Reggeon describing by the BFKL kernel (see Fig. (2 a). In fact, such interaction was assumed many years ago in the BKP equation [17], [18] for the odderon (colourless state of three Reggeized gluons with positive signature, which differs from Pomeron by C-parity).

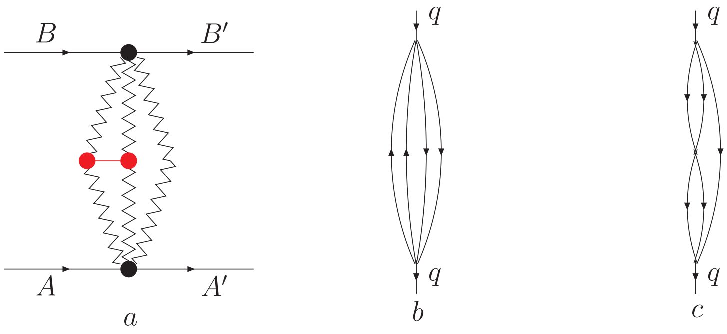

Figure 2: Schematic representation of three-Reggeon contribution in three loops.
a – Reggeon diagram; b and c – its momentum parts.

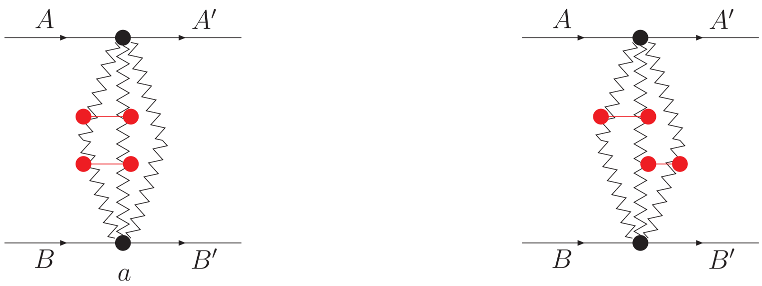

Figure 3: Schematic representation of the three-Reggeon contribution in four loops. Reggeon diagrams.

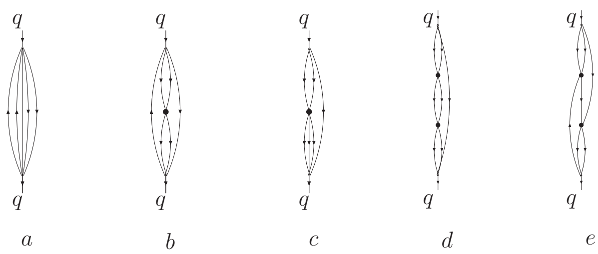

Figure 4: Schematic representation of momentum parts of the three-Reggeon contribution in four loops.

This assumption looks rather natural, although, as I know, its strict proof does not exist, as well as its check in perturbation theory.

The momentum parts of the three-Reggeon contributions are calculated in the two and three loops in [15], [19], [20] and in [16], and in four loops (see Figs. (3) and (4)) in [21], [22], [23] and in [24], [25], [26], [27] [28].

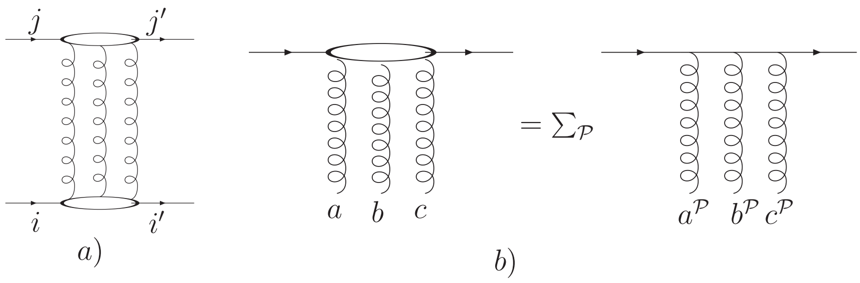

Figure 5: Three-gluon exchanges

6. Colour structure of the three-Reggeon cuts

A crucial question is the colour structure of Reggeon cuts. This question is simple in the two-Reggeon case, because in the product of two adjoint representations there is only one representation of given dimension and parity. But it becomes quite non-trivial in the case of of three-Reggeon cuts.

The difference between diagrammatic and Wilson line approaches is that in the first one colour structures of whole diagrams are considered, while in the second particle-cut interaction vertices are introduced. Note that in contrast to the Reggeon, which contributes only to amplitudes with the adjoint representation of the colour group (colour octet in QCD) in the t-channel, the cut can contribute to various representations.

Possible representations for quark-quark and quark-gluon scattering are only singlet (**1**) and octet (**8**), whereas for the gluon-gluon scattering there are also **10**, **10*** and **27**. It occurs that the difference between two approaches concern only to the adjoint representation, where the Reggeon-cut mixing is possible.

The colour structure of the two-loop three-Reggeon contribution is presented in Fig.(5).

It turns out that for the colour group representations R different from the adjoint representation the colour coefficients do not depend on σ, so that momentum dependent factors for them summed up to the eikonal amplitude. It is very important because it guarantee gauge invariance.

It is not so in the Reggeized gluon channel in the Wilson line approach.

The particle-cut interaction vertex for the octet channel introduced in [16] has the form

$$V^{(c)}_{a'a} = \mathcal{T}^{c}_{a'a} \frac{1}{3!(N_c^2-1)T_i T_j} \, \mathrm{Tr} \sum_{\sigma} \left(\mathcal{T}^{c_1^\sigma} \mathcal{T}^{c_2^\sigma} \mathcal{T}^{c_3^\sigma} \mathcal{T}^{c} \right) , \tag{9}$$

But their introduction contradicts to the limit of large number of colours and gauge invariance.

Recently, in the Wilson line approach, the contributions of the three-Reggeon cuts to the elastic scattering amplitudes have been calculated in four loops [24] To avoid contradiction with the planar $N = 4$ SYM, the authors proposed a scheme for separating the contribution of the pole and the cut in all orders of perturbation theory, based on the assertion that the diagrams for Regge cuts are non-planar. As it was already said, this statement, which comes from the old theory of complex angular momenta, is inapplicable in QCD. In addition, such a separation of the pole and cut contributions is not gauge invariant, which is unacceptable.

The question arises: is it possible to introduce particle-cut interaction vertices in the colour space without contradictions with large N_c limit and gauge invariance?

It is natural to assume that the vertex is symmetrical with respect to permutations of the Reggeon momenta. Then it should be symmetrical with respect to permutations of their colour indices. In this case for the particle A the colour part of the vertex of particle-cut interaction in the Reggeized gluon channel should have the form

$$V^A(cut)_{i\to abc} = a_A\left[\delta_{ab}\delta_{ic}+\delta_{bc}\delta_{ia}+\delta_{ac}\delta_{ib}\right] \ . + N_c b_A\left[d_{abl}d_{icl}+d_{bcl}d_{ial}+d_{acl}d_{ibl}\right] \ . \quad (10)$$

The total vertex of the interaction of three Reggeons is represented by the sum of the vertices of the interaction of the pole and the cut

$$V^A(total)_{i\to abc} = V^A(R)_{i\to abc} + V^A(cut)_{i\to abc} \ . \quad (11)$$

Due to symmetry of the $V^A(cut)$

$$V^A(cut)i \longrightarrow abc = V^A(cut)_{i\to a^\sigma b^\sigma c^\sigma} \ , \quad (12)$$

we have

$$V^A(cut)_{i\to a^\sigma b^\sigma c^\sigma} V^B(R)_{i\to abc} = V^A(cut)_{i\to abc} V^B(R)_{i\to abc} \ . \quad (13)$$

The Reggeon-cut mixing is absent in the two-loop approximation. Therefore it should be

$$V^A(cut)_{i\to abc}\left[V^B i \longrightarrow abc - V^B(cut)_{i\to abc}\right] = 0 \quad (14)$$

Direct calculation gives equations

$$(N_c^2+1)a_G(2-3a_G)+(N_c^2-4)(N_c^2-8)b_G(\frac{1}{2}-6b_G)+\frac{N_c^2-4}{2}(a_G+8b_G-24a_Gb_G) = 0 \, , \quad (15)$$

$$(N_c^2+1)a_Q(\frac{1}{4N_c}-3a_G)$$
$$+(N_c^2-4)(N_c^2-8)b_Q(\frac{1}{4N_c}-6b_Q)+\frac{N_c^2-4}{4N_c}(a_Q+2b_Q-48N_ca_Qb_Q) = 0 \, , \quad (16)$$

$$(N_c^2+1)(2a_Q+\frac{a_G}{4N_c}-6a_Ga_Q)+(N_c^2-4)(N_c^2-8)(\frac{b_Q}{2}+\frac{b_G}{4N_c}-12b_Qb_G)$$
$$+\frac{N_c^2-4}{4N_c}(2a_QN_c+a_G+4b_QN_c2b_Q-48N_c(a_Qb_G+a_Gb_Q) = 0 \, . \quad (17)$$

These equations have solution (that is not trivial), but it gives the vertex introduced in [16]. As it was already discussed, this result is not acceptable.

7. Conclusion

The pole Regge form of amplitudes with gluon quantum numbers in cross channels and negative signature, which is the basis of the BFKL equation, is violated in the NNLLA. The observed violation can be explained by the three-Reggeon cut contributions. There are two different approaches to explanation of the violation. The main difference between the approaches is the colour structure of the three-Reggeon cut contributions. In the diagrammatic approach it is determined from direct calculations of Feynman diagrams. In the Wilson line approach vertices of particle-cut interaction are introduced. But it contradicts to large N_c limit and gauge invariance. Separation of the cut contributions as contributions of non-planar diagrams seems not having serious ground. It seems that using particle -Reggeon vertices in the colour space is not possible in the case of three-Reggeon cut.

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