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# Interfering resonances in a quantum billiard

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We present a method for numerically obtaining the positions, widths, and wave functions of resonance states in a two-dimensional billiard connected to a waveguide. For a rectangular billiard, we study the dynamics of three resonance poles lying separated from the other ones. As a function of increasing coupling strength between the waveguide and the billiard two of the states become trapped while the width of the third one continues to increase for all coupling strengths. This behavior of the resonance poles is reflected in the time delay function, which can be studied experimentally. [S1063-651X(98)14112-0]

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In present-day high-resolution experimental studies, the properties of individual resonance states can be investigated even when the level density is high. As an example, nuclear states have recently been identified and studied experimentally at very high excitation energy [1]. Collisional damping at such an energy is the same as at low excitation energy (ground-state domain). This experimental result, being in contradiction to the standard statistical theory of nuclear reactions, can be justified by taking into account the interferences between resonance states arising from their interaction via the continuum [2]. In atoms, strong laser field effects in the spectral lines of autoionizing states are studied theoretically as well as experimentally. The coherent coupling of autoionizing states is connected recently to level repulsion in the complex plane and the resulting trapping phenomena are discussed [3]. In these papers, the peculiarities of atomic states manipulated by a strong laser field are taken into account. The authors of the experimental paper [4] point to the numerous possible applications of such investigations.

Theoretically, the interferences between resonance states have been studied for several years, e.g., in nuclei [5], molecules [6], and atoms [3,7]. General questions are considered in, e.g., [8,9]. Many quantum systems have a band-structured spectrum with a high level density inside a band. Examples are, among others, some mesoscopic systems. Here, the interferences between individual resonance states play an important role [10] and need to be considered in detail. They determine the properties of the system as a whole.

One important result of all these investigations is the phenomenon of resonance trapping, which arises from the interference of resonance states coupled to the same decay channel. As a function of a parameter controlling the coupling to the continuum, the widths of all states increase as long as the states are isolated. At a certain critical value of the parameter where the resonance states start to overlap each other, the widths bifurcate: the width of one of the states increases further while the widths of the other ones *decrease*, see, e.g., [11]. In other words: one of the states aligns with the channel and becomes short lived while the other ones decouple from the channel and become long lived in spite of the strong coupling to the continuum. This result follows mathematically from the fact that the rank of the Hermitian part of the effective Hamilton operator is equal to the number $N$ of states while that of the non-Hermitian part is equal to the number $K$ of common open decay channels, see, e.g., [5–11]. The critical values of the coupling strength to the decay channels appear usually as avoided crossings of levels or resonances. They are connected with the existence of double or multiple poles of the $S$ matrix. Their relation to the so-called exceptional points and to quantum chaos is studied in different papers (e.g., [9] and further references therein).

The phenomenon of resonance trapping is theoretically well established but not proven directly up to now in experimental studies. In this letter we investigate the behavior of three neighboring resonances in a two-dimensional billiard connected to a single waveguide. As a function of the coupling between the resonator and the waveguide, we calculate both the position of the corresponding resonance poles and the Wigner-Smith time delay function. The time delay function, containing unique information regarding the interference between the resonance states, can be studied experimentally.

The model used is as follows. We consider a two-dimensional billiard coupled to a waveguide (for instance, a flat electromagnetic resonator). We have to solve the equation

$$(-\nabla^2+\lambda V)\Psi=E\Psi, \qquad (1)$$

where $V$ is a potential barrier between the billiard and the attached lead. We use the Dirichlet boundary condition, $\Psi=0$, on the border of the billiard and of the waveguide. The waveguide has a width $W$ and the wave function inside it has the asymptotic form

$$\Psi=[e^{ikx}-R(E)e^{-ikx}]\ u(y). \qquad (2)$$

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Here $u(y)$ is the transversal mode in the waveguide, $k$ is the wave number, and $R(E)$ is the reflection coefficient. It holds $E=k^2+(\pi/W)^2$. We choose the energy of the incoming wave so that only the first transversal mode in the lead is open, i.e.,

$$u(y)=\sin\left(\frac{\pi}{W}y\right). \tag{3}$$

Since $|R(E)|=1$, we write $R(E)=\exp[i\Theta(E)]$ with $\Theta(E)$ real. A measurable quantity derived from $R$ is the Wigner-Smith time delay function,

$$\tau_w=\frac{d\Theta}{dE}. \tag{4}$$

$\tau_w$ is the time the wave spends inside the billiard [8].

The energies and widths of the resonance states are given by the poles of the function $R(E)$ analytically continued into the lower complex plane. To find the poles we use the exterior complex scaling method [12]. The general idea is to study the system after a scaling transformation applied to the $x$ coordinate, see [12,13]: $x\to\tilde{x}=g(x)$. The function $g$ is chosen as

$$g(x)=\begin{cases} x, & x\geqslant x_0, \\ \theta f(x), & x<x_0 \end{cases} \tag{5}$$

with $f(x)$ such that $g(x)$ is three times smoothly differentiable and the inverse transformation $g^{-1}(\tilde{x})$ exists. The attached waveguide extends from $-\infty$ parallel to the $x$-axis and we choose $x_0$ to be localized inside it. The related transformation of the wave function reads

$$\Psi(x,y)\to\frac{1}{\sqrt{g'(\tilde{x})}}\tilde{\Psi}(\tilde{x},y). \tag{6}$$

Using it Eq. (1) becomes

$$\left[-\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{g'^2}\frac{\partial}{\partial\tilde{x}}\right)-\frac{\partial^2}{\partial y^2}\right]\tilde{\Psi}(\tilde{x},y)+\left(\lambda V(\tilde{x},y)+\frac{2g'g'''-5g''^2}{4g'^4}\right)\tilde{\Psi}(\tilde{x},y)=E\tilde{\Psi}(\tilde{x},y). \tag{7}$$

For a real parameter $\theta$, this equation is fully equivalent to Eq. (1) since the transformation (6) is unitary. Moreover, the two equations are fully identical for $x>x_0$. Since $x_0$ lies inside the waveguide the shape of the resonator is not changed by the transformation (6), which only rescales a part of the $x$ axis related to the waveguide. Moreover, since the waveguide is oriented parallel to the $x$ axis, the transformation does not change the boundary of the system. For $\theta$ complex, Eq. (6) ceases to be unitary and the spectral properties of Eqs. (1) and (7) are different. The continuous spectrum of Eq. (1) extends over $\langle(\pi/W)^2,\infty)$, whereas the continuous spectrum of Eq. (7) is rotated into the complex plane and is equal to

$$\cup_{n=1:\infty}\{(n\pi/W)^2+\theta^{-2}\langle 0,\infty)\}. \tag{8}$$

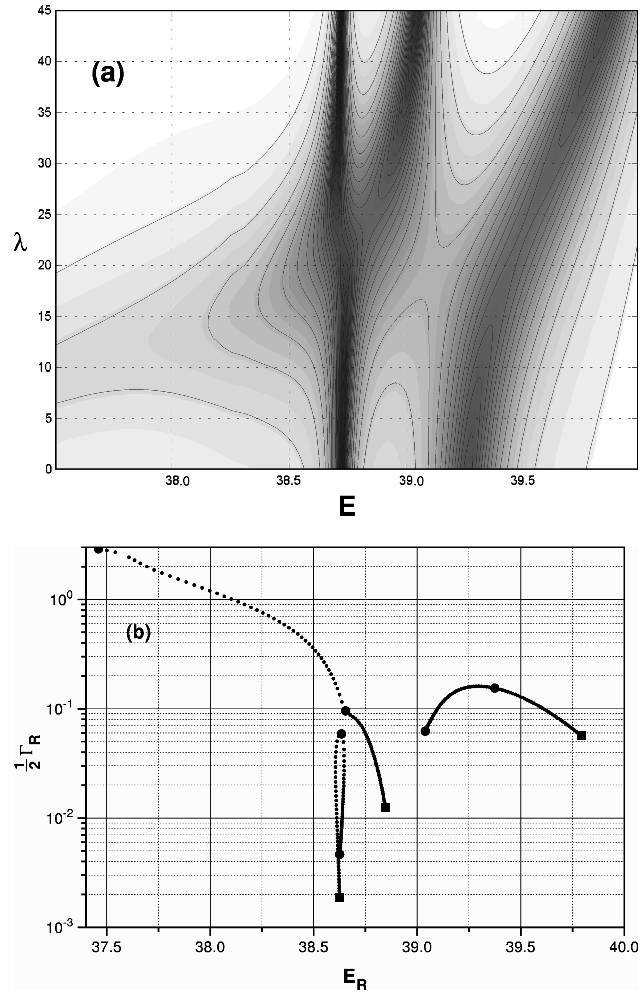


FIG. 1. Contour and surface plot of $\ln(\tau_w)$ (a) for a rectangular billiard (for parameters see text). The darker the plot is, the longer is the time delay. The motion of the corresponding resonance poles with λ (b). The positions of the resonance poles for $\lambda=44$ are denoted by squares and for $\lambda=23.5$ and $\lambda=0$ with large dots. The energies are given in units of $[x]^{-2}$.

This is a union of half-lines representing the continuous spectrum starting out from the real axis at every threshold energy $(n\pi/W)^2$ with an angle $-2\arg\theta$. The rotated continuous spectrum uncovers additive complex eigenvalues of Eq. (7), the positions of which are independent of $\theta$. These eigenvalues coincide with the resonance poles [12,13].

In the following we study the time delay $\tau_w$ and the resonance poles of a rectangular billiard of size $\Delta x\times\Delta y=2\times3.14$ connected to a single waveguide with width $W=0.6$. We choose $V$ as a rectangular potential barrier with height 1 located at $-0.3\leqslant x\leqslant 0$. By changing the parameter λ we can tune the coupling between the waveguide and the resonator. We calculate $\tau_w$ by solving Eq. (1) with the Dirichlet boundary condition $\Psi=0$, and the asymptotic boundary condition (2) imposed at $x=-13$. The resonance poles are found by the method of exterior complex scaling described above using $x_0=-2$.

In Fig. 1 we show the calculated time delay $\tau_w$ as a function of λ and energy (a) as well as the dependence of the resonance poles on λ (b). At large λ (weak coupling to the waveguide) we see three isolated resonance states. As λ de-

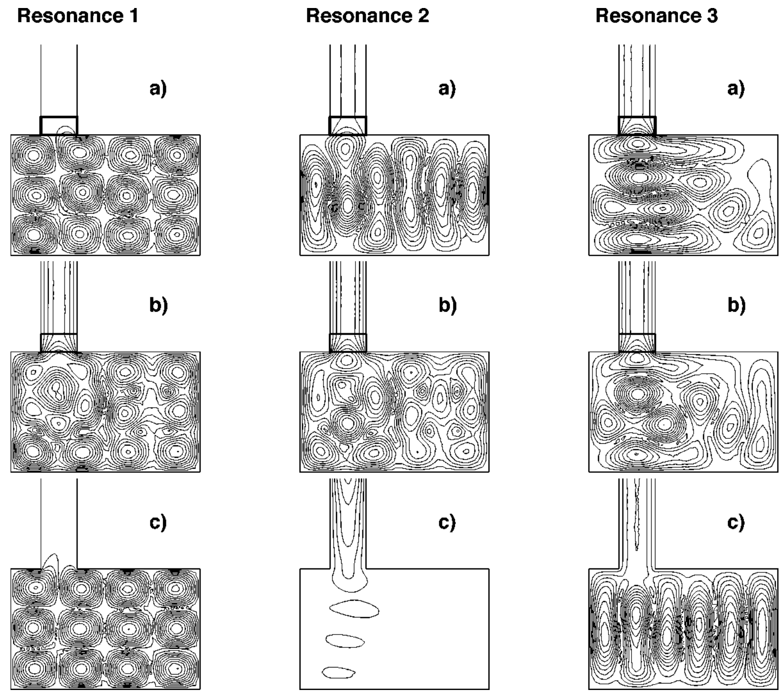


FIG. 2. The wave functions of three resonance states for $\lambda=44$ (case *a*), $\lambda=23.5$ (case *b*), and $\lambda=0$ (case *c*). The energies of the states at $\lambda=44$ are 38.6 (1), 38.8 (2), and 39.8 (3). The location of the potential barrier is marked by a rectangular in the waveguide.

creases (the coupling to the waveguide increases) the lifetimes of all three states decrease. The states attract each other in energy. As the resonances start to overlap, two of them become trapped while the third one becomes short lived. At further decreased $\lambda$ the lifetimes of the two trapped resonance states *increase*. The motion of the poles is reflected in the time delay function. The lifetime of the short-lived state is, at small $\lambda$, so short that it practically disappears when plotting the time delay. The numerical errors in the distances among the resonances are so small that the details of energy attraction and resonance trapping are stable.

Using the method of complex scaling we can also study the wave functions of the resonance states (Gamow states). The interference between the resonances states leads to a mixing of their wave functions with respect to the eigenfunctions of the resonator ($\lambda\to\infty$). We illustrate this phenomenon in Fig. 2, where the wave functions of the three resonance states are shown for $\lambda=44$, 23.5, and 0, i.e., under the condition when (a) they are isolated, (b) they are very near to one another, and (c) two of them are trapped while one is short lived.

For $\lambda\to 0$, the amplitude of the wave function related to the resonance state that finally evolves into a short-lived state is very small inside the resonator. This corresponds to a very small time delay, i.e., to a small probability of staying inside the billiard. The trapping of the two long-lived resonance states occurs in two different ways. Both wave functions correspond to (almost) pure bound states of the billiard as long as $\lambda$ is large and become mixed when their distance in the complex energy plane to the other two states becomes small. At still smaller $\lambda$ one of the resonance states demixes and approaches again its pure shape. In accordance to that, its energy for small $\lambda$ approaches that for large $\lambda$. The wave function of the other trapped resonance state remains, however, mixed. Accordingly, its position at small $\lambda$ differs from that for large $\lambda$ and its wave function is mixed with that of the short-lived resonance state.

The analysis presented in this letter shows that open billiards provide an excellent possibility to study the dynamics of resonance poles in detail. A realization can be achieved by means of flat microwave resonators connected to a waveguide where the coupling strength to the channel can be varied by hand. Such systems allow us therefore to investigate directly the formation of different time scales (resonance trapping) by tracing the corresponding time delay function. In particular, it is possible to show and to understand the contraintuitive result that the lifetimes of certain resonance states increase with increasing coupling to the continuum and that the wave functions of these long-lived states may be pure in relation to the bound states. The results of these investigations will help in analyzing high-resolution experimental data in various fields of physics.

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[1] A. Bracco *et al.*, Phys. Rev. Lett. **74**, 3748 (1995); Nuovo Cimento (to be published); M. Mattiuzzi *et al.*, Phys. Lett. B **364**, 13 (1995); Nucl. Phys. A **612**, 262 (1997).

[2] E. Persson and I. Rotter (unpublished).

[3] N. J. Kylstra and C. J. Joachain, Phys. Rev. A **57**, 412 (1998); A. I. Magunov, M. Müller, I. Rotter, and S. I. Strakhova, in 7th International Workshop on Laser Physics, Berlin, July 6–10, 1998 [Laser Phys. (to be published)].

[4] N. E. Karapanagioti *et al.*, Phys. Rev. Lett. **74**, 2431 (1995); Phys. Rev. A **53**, 2587 (1996).
[5] I. Rotter, Rep. Prog. Phys. **54**, 635 (1991); V. V. Sokolov and V. G. Zelevinsky, Ann. Phys. (N.Y.) **216**, 323 (1992); V. V. Sokolov, I. Rotter, D. V. Savin, and M. Müller, Phys. Rev. C **56**, 1031 (1997); **56**, 1044 (1997), and references therein.
[6] M. Desouter-Lecomte, J. Liévin, and V. Brems, J. Chem. Phys. **103**, 4524 (1995).
[7] V. V. Flambaum, A. A. Gribakina, and G. F. Gribakin, Phys. Rev. A **54**, 2066 (1966).
[8] Y. V. Fyodorov and H. J. Sommers, J. Math. Phys. **38**, 1918 (1997).
[9] W. D. Heiss, M. Müller, and I. Rotter, Phys. Rev. E **58**, 2894 (1998).
[10] W. Iskra, I. Rotter, and F. M. Dittes, Phys. Rev. C **47**, 1086 (1993).
[11] M. Müller, F.-M. Dittes, W. Iskra, and I. Rotter, Phys. Rev. E **52**, 5961 (1995).
[12] B. Simon, Int. J. Quantum Chem. **14**, 529 (1978).
[13] A. Jensen, J. Math. Anal. Appl. **59**, 505 (1977).