Reconnection Rate for the Inhomogeneous Resistivity Petschek Model

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The reconnection rate for the canonical simplest case of steady-state two-dimensional symmetric reconnection in an incompressible plasma is found by matching of an outer Petschek solution and an internal diffusion region solution. The reconnection rate obtained naturally incorporates both Sweet-Parker and Petschek regimes, while the latter is possible only for a strongly localized resistivity.

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Magnetic reconnection is an energy conversion process which occurs in astrophysical, solar, space, and laboratory plasmas [1,2]. First attempts to explain the fast energy release in solar flares based on pure resistive magnetic field dissipation [3,4] showed that the energy conversion rate is estimated as $1/\sqrt{Re_m}$, where $Re_m = V_A L/\eta$ is the global Reynolds number, L is the half length of the reconnection layer, V_A is the Alfvénic velocity, and η denotes the resistivity. For typical conditions in the solar corona, the Sweet-Parker rate turns out to be orders of magnitude too small when compared with experimental data.

In 1964, Petschek [5] pointed out that in a highly conducting plasma dissipation needs to be present only within a small region known as the diffusion region, and energy conversion occurs primarily across nonlinear waves, or shocks. This gives another estimation of the maximum reconnection rate $1/\ln Re_m$ which is much more favorable for energy conversion.

Unfortunately, it is still unclear which conditions make Petschek-type reconnection possible and which are responsible for the Sweet-Parker regime. Numerical simulations [6,7] were not able to reproduce solutions of Petschek type but rather were in favor of Sweet-Parker–type solutions unless the resistivity was localized in a small region [7–9]. In laboratory experiments, one also seems to observe the Sweet-Parker regime of reconnection [10,11].

From the mathematical point of view, the problem of reconnection rate is connected with the matching of a solution for the diffusion region where dissipation is important, and a solution for the convective zone where the ideal MHD equations can be used. But, until now, this question is still not resolved even for the canonical simplest case of steady-state, two-dimensional symmetric reconnection in an incompressible plasma.

It is the aim of this paper to present a matching procedure for the canonical reconnection problem. The reconnection rate obtained from the matching turns out to incorporate naturally both Petschek and Sweet-Parker regimes as limiting cases.

Petschek solution.—We consider the simplest theoretical system consisting of a two-dimensional current sheet which separates two uniform and identical plasmas with oppositely oriented magnetic fields $\pm \mathbf{B}_0$. Petschek [5] pointed out that the diffusion region can be considerably smaller than the whole size of the reconnection layer and that the outer region contains two pairs of standing slow shocks. These shocks deflect and accelerate the incoming plasma from the inflow region into two exit jets wedged between the shocks (Fig. 1). This jet area between the shocks with accelerated plasma is traditionally called the outflow region.

In the dimensionless form, the Petschek solution can be presented as follows [5,12]:

Inflow region:

$$v_x = 0; \qquad v_y = -\varepsilon, \tag{1}$$

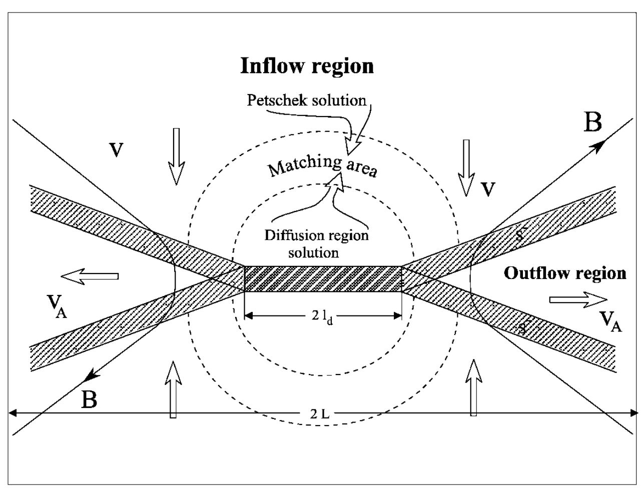

FIG. 1. Scheme of matching of the outer Petschek solution and diffusion region solution.

$$B_x = 1 - \frac{4\varepsilon}{\pi} \ln \frac{1}{\sqrt{x^2 + y^2}}; \qquad B_y = \frac{4\varepsilon}{\pi} \arctan \frac{x}{y}. \tag{2}$$

Outflow region:

$$v_x = 1; \qquad v_y = 0; \qquad B_x = 0; \qquad B_y = \varepsilon. \tag{3}$$

The equation of shock in the first quadrant is $y = \varepsilon x$.

Here x, y are directed along the current sheet and in the perpendicular direction, respectively. We normalized the magnetic field to B_0, the length to L, the plasma velocity to Alfvénic velocity V_A, and the electric field E to the Alfvénic electric field $E_A = V_A B_0$.

The reconnection rate $\varepsilon = E/E_A \ll 1$ is supposed to be a small parameter of the problem.

Expressions (1)–(3) are the asymptotic (with respect to ε) steady-state solution of the ideal MHD equations and the Rankine-Hugoniot shock relations. Petschek did not obtain a solution in the diffusion region; instead he estimated a maximum reconnection rate as $1/\ln Re_m$ using some simple physical suggestions. Generally speaking, this implies that the Petschek model gives any reconnection rate from the Sweet-Parker value $1/\sqrt{Re_m}$ up to $1/\ln Re_m$, and it is still unclear whether Petschek reconnection faster than Sweet-Parker reconnection is possible. The problem can be solved by matching of a solution for the diffusion region and the Petschek solution [(1)–(3)].

Diffusion region scaling.—We renormalize the MHD equations to the new scales B_0', V_A', $E_A' = B_0' V_A'$, where all quantities are supposed to be taken at the upper boundary of the diffusion region, and $x' = x/l_d$, where l_d is the half length of the diffusion region. We have to use the dissipative MHD equations for the diffusion region with the Reynolds number

$$Re_m' = \frac{V_A' l_d}{\eta}, \tag{4}$$

and the electric field $E = \varepsilon'$.

The scaling for the diffusion region is similar to that of the Prandtl viscous layer [13]:

$$\begin{aligned} x', B_x', v_x', P' &\sim O(1), \\ y', B_y', v_y', \varepsilon' &\sim 1/\sqrt{Re_m'}, \end{aligned} \tag{5}$$

Consequently, the new boundary layer variables are the following:

$$\begin{aligned} &\tilde{x} = x', \qquad \tilde{B}_x = B_x', \qquad \tilde{v}_x = v_x', \qquad \tilde{P} = P', \\ &\tilde{y} = y'\sqrt{Re_m'}, \qquad \tilde{B}_y = B_y'\sqrt{Re_m'}, \\ &\tilde{v}_y = v_y'\sqrt{Re_m'}, \qquad \tilde{\varepsilon} = \varepsilon'\sqrt{Re_m'}. \end{aligned} \tag{6}$$

The diffusion region Reynolds number is supposed to be $Re_m' \gg 1$, and therefore in the zero order with respect to the parameter $1/\sqrt{Re_m'}$ the boundary layer equations turn out to be

$$\frac{\partial \tilde{v}_x}{\partial t} + \tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} - \tilde{B}_x \frac{\partial \tilde{B}_x}{\partial \tilde{x}} - \tilde{B}_y \frac{\partial \tilde{B}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{P}(\tilde{x})}{\partial \tilde{x}}, \tag{7}$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \mathrm{curl}(\tilde{\mathbf{v}} \times \tilde{\mathbf{B}}) - \mathrm{curl}[\eta(\tilde{x}, \tilde{y})\, \mathrm{curl} \tilde{\mathbf{B}}], \tag{8}$$

$$\mathrm{div} \tilde{\mathbf{B}} = 0, \qquad \mathrm{div} \tilde{\mathbf{v}} = 0, \tag{9}$$

where $\tilde{\eta}(\tilde{x}, \tilde{y})$ is the normalized resistivity of the plasma with maximum value 1.

Unfortunately, the appropriate exact solutions of the boundary layer Eqs. (7)–(9) are unknown even in a steady-state case; therefore we have to solve the problem numerically. The main difficulty is that the internal reconnection rate $\tilde{\varepsilon}$ is unknown in advance and has to be determined for given resistivity $\tilde{\eta}(\tilde{x}, \tilde{y})$, given total pressure $\tilde{P}(\tilde{x})$, and $\tilde{B}_x(\tilde{x})$ given at the upper boundary of the diffusion region. In addition, the solution must have the Petschek-type asymptotic behavior [(1)–(3)] outside of the diffusion region.

Starting with an initial MHD configuration under fixed boundary conditions, we look for the convergence of the time-dependent solutions to a steady state.

As initial configuration, we choose an X-type flow and the following magnetic field: $\tilde{v}_x = \tilde{x}$, $\tilde{v}_y = -\tilde{y}$, $\tilde{B}_x = \tilde{y}$, $\tilde{B}_y = \tilde{x}$. The distribution of the resistivity is traditional [7,9]:

$$\eta(\tilde{x}, \tilde{y}) = d e^{(-s_x \tilde{x}^2 - s_y \tilde{y}^2)} + f, \tag{10}$$

with $d + f = 1$.

The problem considered here consists essentially of two coupled physical processes: diffusion and wave propagation. To model these processes, a nonsteady two-step numerical scheme has been used. At first, convectional terms were calculated using the Godunov characteristic method, and then the elliptical part was treated implicitly. Calculations were carried out on a rectangular uniform grid 145×100 in the first quadrant with the following boundary conditions. Lower boundary: symmetry conditions $\partial \tilde{v}_x/\partial \tilde{y} = 0$, $\tilde{v}_y = 0$, and $\tilde{B}_x = 0$; induction equation (8) has been used to compute the $\tilde{B}_y$ component at the x axis. Left boundary: symmetry conditions $\tilde{v}_x = 0$, $\partial \tilde{v}_y/\partial \tilde{x} = 0$, $\partial \tilde{B}_x/\partial \tilde{x} = 0$, and $\tilde{B}_y = 0$. Right boundary: free conditions $\partial \tilde{v}_x/\partial \tilde{x} = 0$, $\partial \tilde{v}_y/\partial \tilde{x} = 0$, $\partial \tilde{B}_x/\partial \tilde{x} = 0$, and $\partial \tilde{B}_y/\partial \tilde{x} = 0$. Upper (inflow) boundary: $\tilde{v}_x = 0$ and $\tilde{B}_x = 1$.

Note that this implies that we do not prescribe the incoming velocity, and, hence, the reconnection rate: the system itself has to determine how fast it wants to reconnect.

The total pressure can be fixed to 1 in the zero-order approximation: $\tilde{P} = 1$.

Let us discuss the result of our simulations. For the case of localized resistivity where we chose $d = 0.95$, $f = 0.05$, and $s_x = s_y = 1$ in Eq. (10), the system reaches the Petschek steady state (see Fig. 2) with a clear asymptotic behavior, a pronounced slow shock, and the reconnection rate turns out to be $\tilde{\varepsilon} \sim 0.7$.

On the other hand, for the case of a homogeneous resistivity $d = 0$, $f = 1$, the system reaches the Sweet-Parker

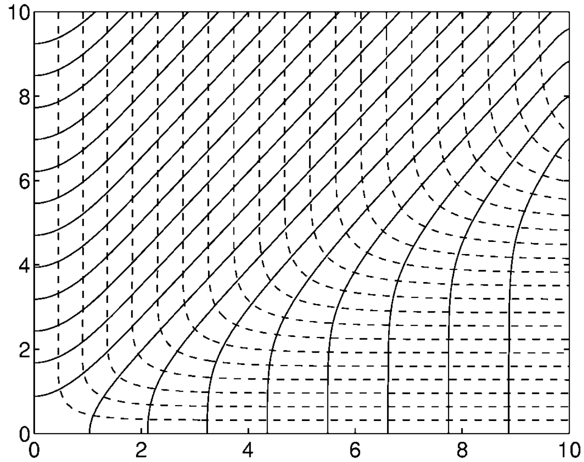

FIG. 2. Computer plots of reconnecting magnetic field lines (solid line) and stream lines (dashed line) in the Petschek regime.

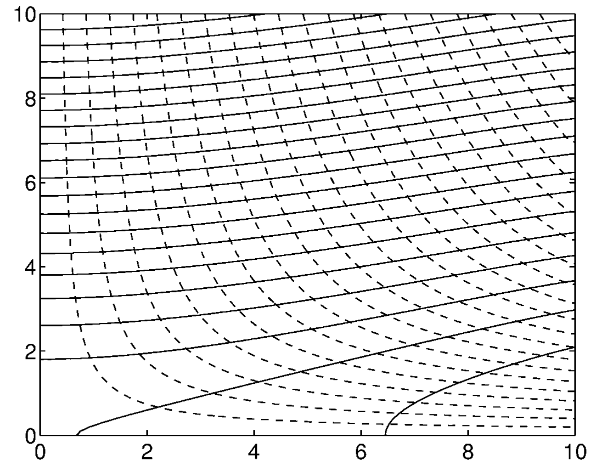

FIG. 3. Computer plots of reconnecting magnetic field lines (solid line) and stream lines (dashed line) in the Sweet-Parker regime.

state (see Fig. 3) with much less reconnection rate $\tilde{\varepsilon} \sim 0.25$ even if the Petschek solution has been used as the initial configuration (see also [7,9]). This seems to imply that Petschek-type reconnection is possible only if the resistivity of the plasma is localized in a small region, and for constant resistivity the Sweet-Parker regime is realized.

The size of the diffusion region l_d can be defined as the size of the region where the convective electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ (which is zero at the origin) reaches the asymptotic value $\tilde{\varepsilon}$ (or, some level, say $0.95\tilde{\varepsilon}$). For the case of a localized resistivity l_d practically coincides with the scale of the inhomogeneity of the conductivity. In principle, there might be a possibility to produce Petschek-type reconnection with a constant resistivity using a highly inhomogeneous behavior of the MHD parameters at the upper boundary (narrow stream, for example, see [14]), and then l_d has the meaning of the scale of this shearing flow or other boundary factor which causes the reconnection.

Matching procedure.—We have only a numerical solution for the diffusion region, and this makes it difficult for the matching procedure because the latter needs an analytical presentation of the solutions to be matched. The only remaining way out is to continue the diffusion region solution to the inflow region using data known from the simulation distribution of the B_y component along the upper boundary of the diffusion region. Then try to match the solutions in the current free inflow region at the distance $r \sim l_d$ (see Fig. 1).

As can be seen from Eq. (2) the B_x component of the Petschek solution diverges at the origin $B_x \to -\infty$ when $r = \sqrt{x^2 + y^2} \to 0$. This singularity is a consequence of the fact that dissipation actually has not been taken into account for the solution [(1)–(3)] which is nevertheless still valid until the distance is of the order of the size of the diffusion region.

In order to be adjusted to the Petschek solution, the B_y' component must have the following limit for $x/l_d \to \infty$ at the upper boundary of the diffusion region:

$$B_y'(x/l_d) \to 2\varepsilon \,. \tag{11}$$

To obtain the asymptotic behavior of B_x' for $r > l_d$, we take into account that B_x' obeys Laplace's equation with the Neumann boundary condition $\partial B_x'/\partial y' = \partial B_y'/\partial x'$. Hence, we can use the Poisson integral [15]:

$$\begin{aligned} B_x'(x,y) &= B_0' + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial B_y'^{(1)}(\tilde{x},0)}{\partial \tilde{x}} \ln \frac{\sqrt{(x-\tilde{x})^2 + y^2}}{l_d}\, d\tilde{x} \\ &= B_0' + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial B_y'^{(1)}(\xi,0)}{\partial \xi} \left\{ \ln \frac{\sqrt{x^2+y^2}}{l_d} + \frac{\xi^2 - 2x\xi}{x^2+y^2} \right\} d\xi = B_0' + \frac{4\varepsilon}{\pi} \ln \frac{r}{l_d} + O(1/r)\,, \end{aligned} \tag{12}$$

where $\xi = x/l_d$. This gives an outer expansion for the inner solution. On the other hand, a convective solution (2) can be rewritten in the following form in order to determine the inner expansion of the outer solution:

$$B_x = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{r} = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d} - \frac{4\varepsilon}{\pi} \ln \frac{l_d}{r}\,. \tag{13}$$

Setting these two asymptotic expansions equal, we obtain the matching relation:

$$B_0' = 1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d}\,. \tag{14}$$

Now everything is prepared in order to determine the reconnection rate. The electric field must be constant in the whole inflow region, hence,

$$v' B_0' = v B_0\,, \tag{15}$$

$$\varepsilon' B_0'^2 = \varepsilon B_0^2\,, \tag{16}$$

where the definition of the reconnection rates $\varepsilon' = v'/B_0'$, $\varepsilon = v/B_0$ has been used. Bearing in mind that

$\varepsilon' = \tilde{\varepsilon}/\sqrt{R}\, e'_m$ [see scaling (6)], we obtain

$$\tilde{\varepsilon} B_0'^{3/2} = \varepsilon B_0^{3/2} \sqrt{\frac{l_d B_0}{\eta}} \,. \qquad (17)$$

Substituting B_0' from Eq. (14), we determine finally the following equation for the reconnection rate ε:

$$\tilde{\varepsilon}\left(1 - \frac{4\varepsilon}{\pi} \ln \frac{L}{l_d}\right)^{3/2} = \varepsilon \sqrt{Re_m \frac{l_d}{L}} \,, \qquad (18)$$

where $Re_m = V_A L/\eta$, and the internal reconnection rate $\tilde{\varepsilon}$ has to be found from the simulation of the diffusion region problem.

For small ε, there is an analytical expression:

$$\varepsilon = \frac{\tilde{\varepsilon}}{\sqrt{Re_m \frac{l_d}{L}} + \frac{6}{\pi} \tilde{\varepsilon} \ln \frac{L}{l_d}} \,. \qquad (19)$$

Here, $\tilde{\varepsilon}$ is an internal reconnection rate determined from the numerical solution: $\tilde{\varepsilon} \sim 0.7$ for the Petschek-type solution and $\tilde{\varepsilon} \sim 0.25$ for the Parker-Sweet regime.

Discussion and conclusion.—Equations (18) and (19) give the unique reconnection rate for the known parameters of the current sheet L, B_0, V_A, η, l_d. For sufficiently long diffusion region ($l_d \sim L$), Eq. (19) corresponds to the Sweet-Parker regime $\varepsilon \sim \tilde{\varepsilon}/\sqrt{Re_m(l_d/L)}$. In the opposite case of a resistivity constrained in a small region $\varepsilon \sim \frac{\pi}{6} \ln(L/l_d)$, one obtains Petschek type reconnection. Hence, the reconnection rate [(18) and (19)] naturally incorporates both regimes obtained in simulations [6,7,9].

According to our simulations a strongly localized resistivity is needed for the Petschek state to exist, while for the spatially homogeneous resistivity $l_d \sim L$ the Sweet-Parker regime seems to be always the case. This result resolves the old question about the conditions that are necessary for Petschek-type reconnection to appear.

It is interesting that for the deriving of Eqs. (18) and (19) the only value that has been actually used is the internal reconnection rate $\tilde{\varepsilon}$ obtained form the numerical solution, but the distribution of the B_y component along the upper boundary of the diffusion region does not contribute at all [besides asymptotic behavior (12)] in the zero-order approximation considered above. Of course, from the mathematical point of view it is important that the diffusion region solution exists and has Petschek-like asymptotic behavior [(1)–(3)].

The strongly localized resistivity is often the relevant case in space plasma applications, but for the laboratory experiments where the size of a device is relatively small the Sweet-Parker regime is expected. Both Petschek and Sweet-Parker reconnection rates can be enhanced considerably by including anomalous resistivity [16].

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