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Quantum–Classical Correspondence and Nonclassical State Generation in Dissipative Quantum Optical Systems[¶]

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Abstract—We develop a semiclassical method to determine the nonlinear dynamics of dissipative quantum optical systems in the limit of large number of photons N; it is based on the $1/N$-expansion and the quantum–classical correspondence. The method is used to tackle two problems: the study of the dynamics of nonclassical state generation in higher order anharmonic dissipative oscillators and the establishment of the difference between the quantum and classical dynamics of the second-harmonic generation in a self-pulsing regime. In addressing the first problem, we obtain an explicit time dependence of the squeezing and the Fano factor for an arbitrary degree of anharmonism in the short-time approximation. For the second problem, we analytically find a characteristic time scale at which the quantum dynamics differs insignificantly from the classical one. *© 2000 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The situation when nonlinear interactions involve a large number of photons, N, is quite typical of many problems in quantum and nonlinear optics [1–3]. Heidmann *et al.* suggested [4] the use of the $1/N$-expansion method [5] to describe the nonlinear dynamics of the mean values and second-order cumulants of a quantum system in the $N \gg 1$ limit. Following the general scheme of that method [5], an exact or approximate solution can be found in terms of the coherent state representation in the classical limit as $N \longrightarrow \infty$ and can then be adjusted by adding the quantum corrections. The method proves to be particularly convenient when the dynamics of nonclassical state generation must be determined [4]. We have recently developed the method further to study the enhanced squeezing at the transition to quantum chaos [6–8].

Papers [4, 6, 7] are concerned with the problems of nondissipative quantum systems only. In this paper, we extend the method to dissipative quantum systems. For quantum systems without dissipation, the lowest order of the $1/N$-expansion is equivalent to the linearization in terms of the classical solution [6, 7], whereas in dissipative systems, as is demonstrated in what follows, the solution of the equations of motion for variations near the classical trajectory cannot provide complete information on the dynamics of quantum fluctuations even in the lowest order of $1/N$. We show that the influence of the reservoir on the dynamics of expectation values and dispersions, which is different from the energy dissipation, always exists: It has the quantum nature and cannot be neglected even in the semiclassical limit. However, specific manifestations of the effect depend on the type of the attractor in the underlying classical dynamic system. For systems with a simple attractor in the classical limit, the "quantum diffusion" associated with the quantum fluctuations of the reservoir do not lead to any new physical effects in the dynamics of the main system, at least in the short-time limit. For a stable limit cycle, on the other hand, such a diffusion appears to be the main mechanism responsible for the difference between the classical and quantum dynamics for $N \gg 1$.

Along with the presentation of a general formalism, we consider two typical examples of quantum optical systems with a simple attractor and a stable limit cycle in the classical limit as $N \longrightarrow \infty$: the dissipative higher order anharmonic oscillator and the self-pulsing regime of intracavity second-harmonic generation (SHG). We show how the $1/N$-expansion method can be used to investigate the dynamics of the nonclassical state generation and to determine the time scale for a correct classical description of the dissipative quantum dynamics.

The quantum anharmonic oscillator with a Kerr-type nonlinearity is one of the simplest and most popular models used in the description of quantum statistical properties of light interacting with a nonlinear medium [1, 9]. The Kerr oscillator model with a third-order nonlinearity yields an exact solution in both the nondissipative [10] and dissipative limits [9]. However, because of the complexity of the solution in the dissipative case, numerical methods or special approximate analytic

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methods must be used to determine statistical properties of the radiation in the most relevant experimental case involving a large number of photons. Moreover, there are no exact solutions available for the model of the anharmonic oscillator with a higher order nonlinearity.

In this paper, we analytically obtain a simple and explicit time dependence of the degree of squeezing and the Fano factor in the anharmonic oscillator model of an arbitrary order for the most interesting experimental situation featuring higher intensities ($N \gg 1$) and short-time interactions. As another example of application of the $1/N$-expansion, we consider the self-pulsing in SHG [11]. Such an oscillatory regime corresponding to the limit cycle was observed experimentally in [12]. There are several papers dealing with the development of approximate analytic and numerical methods with the purpose of describing different dynamical regimes in SHG in terms of quantum mechanics [13–17]. In particular, Savage [14] calculated the Gaussian approximation of the Q distribution function about the classical limit cycle. He demonstrated numerically that in the classical limit, the initial rapid collapse of the Q distribution in the neighborhood of the limit cycle is followed by the diffusion around the limit cycle. However, the author did not offer any analytical solution of the problem or an explanation of the physics of the effect observed.

In this paper, we show that the diffusion around the classical limit cycle can be obtained as a solution of the equations of motion for low-order cumulants by using the $1/N$-expansion technique. This enables us to find the time scale $t \ll t^*$ with $t^* \simeq 2N\gamma^{-1}$ (where γ is a damping constant) for a correct classical description of self-oscillations in SHG. The resultant estimate is consistent with that obtained for t^* numerically in [14]. Finally, we interpret the quantum diffusion around the limit cycle as a diffusion caused by the effect of the reservoir vacuum on the SHG dynamics.

This paper is organized as follows. In Section 2, we describe a general formalism of the $1/N$-expansion applicable to an arbitrary single-mode quantum dissipative system and present the solution of the equations of motion for mean values and second-order cumulants obtained in the first order of $1/N$. In Sections 3 and 4, we deal with the nonclassical state generation dynamics in higher order anharmonic oscillators and the quantum–classical correspondence for the self-pulsing regime in SHG, respectively. The final section contains a summary and concluding remarks.

2. 1/N-EXPANSION AND QUANTUM–CLASSICAL CORRESPONDENCE

We begin with generalizing the approach of [7] systems with dissipation. As an illustrative example, we consider a quantum anharmonic oscillator with the Hamiltonian in the interaction picture

$$H = \Delta b^\dagger b + \frac{\lambda_l}{l+1}(b^\dagger b)^{l+1}, \quad [b, b^\dagger] = 1, \qquad (1)$$

where the operators b and $b^\dagger$ describe a single quantum field mode and the constant λ_l is proportional to a $(2l+1)$-order nonlinear susceptibility of a nonlinear medium (l is an integer), Δ is the light frequency detuning from the characteristic quantum transition frequency, and $\hbar \equiv 1$. Everywhere in this paper, we use the normal ordering of operators. The oscillator interacts with an infinite linear reservoir at a finite temperature. The Hamiltonians of the reservoir and of the oscillator-reservoir interaction are defined as

$$\begin{aligned} H_r &= \sum_j \psi_j (d_j^\dagger d_j + 1/2), \\ H_{int} &= \sum_j (\kappa_j d_j b^\dagger + \text{H.c.}), \end{aligned} \qquad (2)$$

where the Bose operator d_j ($[d_j, d_k^\dagger] = \delta_{jk}$) describes an infinite reservoir with the characteristic frequencies ψ_j and κ_j are the coupling constants between reservoir modes and the oscillator. We introduce new scaled operators $a = b/N^{1/2}$ and $c_j = d_j/N^{1/2}$ and their Hermitian conjugates satisfying the commutation relations

$$[a, a^\dagger] = 1/N, \quad [c_j, c_k^\dagger] = \delta_{jk}/N. \qquad (3)$$

In the classical limit as $N \longrightarrow \infty$, we obtain commuting classical c-numbers instead of operators. The full Hamiltonian

$$H = H_0 + H_r + H_{\text{int}}$$

can be rewritten as

$$H = N\mathcal{H},$$

where $\mathcal{H}$ is as in (1) and (2) but with the replacements

$$\begin{aligned} &b \longrightarrow a, \quad b^\dagger \longrightarrow a^\dagger, \quad d_j \longrightarrow c_j, \\ &d_j^\dagger \longrightarrow c_j^\dagger, \quad \lambda_l \longrightarrow g_l(N) \equiv \lambda N^l. \end{aligned} \qquad (4)$$

It can be shown that the photon-number dependent constant $g_l(N)$ provides a correct time scale of oscillations for nonlinear oscillator (1) in the classical limit (for the Kerr nonlinearity with $l = 1$, see, e.g., [18]). We note that $\mathcal{H}$ can have an explicit time dependence in the general case [7]. Within a standard Heisenberg–Langevin approach, the equation of motion has the form ([1], Chap. 7)

$$\dot{a} = -i\left(\Delta - i\frac{\gamma}{2}\right)a + V + L(t), \qquad (5)$$

where $V = \partial\mathcal{H}_0/\partial a^\dagger$, $\gamma = 2\pi|\kappa(\omega)|^2\rho(\omega)$ is the damping constant, with $\rho(\omega)$ being the density function of reservoir oscillators whose spectrum is considered to be flat. The Langevin force operator $L(t)$ is in a standard relation to the operators $\{c_j\}$ of the reservoir [1]. In our notation (4), the properties of $L(t)$ [1] can be rewritten as

$$\begin{aligned}\langle L(t)\rangle_R &= \langle L^\dagger(t)\rangle_R = 0,\\ \langle L^\dagger a\rangle_R + \langle a^\dagger L\rangle_R &= \gamma\frac{\langle n_d\rangle}{N},\\ \langle La\rangle_R + \langle aL\rangle_R &= 0.\end{aligned} \tag{6}$$

Here, the averaging is performed over the reservoir variables and $\langle n_d\rangle$ is a single-mode mean number of the reservoir quanta (phonons) related to temperature T as

$$\langle n_d\rangle = \left[\exp\left(\frac{\omega}{kT}\right) - 1\right]^{-1},$$

where k is the Boltzmann constant and ω is the characteristic phonon frequency. From the Heisenberg–Langevin equations for a, a^2 and the Hermitian conjugated equations, using (5) and (6), we obtain

$$\begin{aligned}i\frac{d}{dt}\langle\alpha\rangle &= \langle V\rangle - i\frac{\gamma}{2}\langle\alpha\rangle,\\ i\frac{d}{dt}\langle(\delta\alpha)^2\rangle &= 2\langle V\delta\alpha\rangle + \langle W\rangle - i\gamma\langle(\delta\alpha)^2\rangle,\\ i\frac{d}{dt}\langle\delta\alpha^*\delta\alpha\rangle &= -\langle V^*\delta\alpha\rangle + \langle\delta\alpha^* V\rangle\\ &\quad - i\gamma\langle\delta\alpha^*\delta\alpha\rangle + i\gamma\frac{\langle n_d\rangle}{N},\end{aligned} \tag{7}$$

where

$$W = (1/N)\partial V/\partial a^\dagger, \quad z \equiv \langle a\rangle,$$
$$\langle(\delta\alpha)^2\rangle = \langle a^2\rangle - z^2, \quad \langle\delta\alpha^*\delta\alpha\rangle = \langle a^\dagger a\rangle - |z|^2,$$

and the averaging is performed over both the reservoir variables and the coherent state

$$|\alpha\rangle = \exp(N\alpha a^\dagger - N\alpha^* a)|0\rangle$$

corresponding to the mean photon number $\simeq N$. In deriving (7), we neglect the insignificant additional detuning introduced to Δ by the interaction with the reservoir [1]. In the absence of damping, $\gamma = 0$, our equations for the mean values and the second-order cumulants (7) are reduced to the corresponding equations in [4, 7].

The set of equations (7) is not closed and is basically equivalent to the infinite dynamical hierarchy system for the cumulants of a different order. To truncate the system to the second-order cumulants, we make the substitution $a \longrightarrow z + \delta\alpha$, where, at least initially, the mean value is $z \simeq 1$ and the quantum correction are

$$|\delta\alpha(t=0)| \simeq N^{-1/2} \ll 1.$$

Using the Taylor expansion of the functions V and W and after some algebra analogous to that used in [7], we obtain from (7) in the first order of $1/N$ the following self-consistent system of equations for the mean value and the second-order cumulants (for details see [19]):

$$i\dot{z} = -i\frac{\gamma}{2}z + \langle V\rangle_z + \frac{1}{N}Q(z, z^*, C, C^*, B), \tag{8}$$

$$i\dot{C} = 2\left(\frac{\partial V}{\partial\alpha}\right)_z C + 2\left(\frac{\partial V}{\partial\alpha^*}\right)_z B - i\gamma C, \tag{9}$$

$$i\dot{B} = -\left(\frac{\partial V^*}{\partial\alpha}\right)_z C + \left(\frac{\partial V}{\partial\alpha^*}\right)_z C^* - i\gamma(B - B^{(0)}). \tag{10}$$

The corresponding equation for $C^*(t)$ can be obtained from (9) by complex conjugation. The quantum correction to the classical motion Q in (8) has the form

$$\begin{aligned}Q &= \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^2}\right)_z C + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^{*2}}\right)_z C^*\\ &\quad + \left(\frac{\partial^2 V}{\partial\alpha^*\partial\alpha}\right)_z\left(B - \frac{1}{2}\right).\end{aligned} \tag{11}$$

In (8)–(11), the subscript z means that the values of V and its derivatives are calculated for the mean value z; we have introduced

$$B = N\langle\delta\alpha^*\delta\alpha\rangle + 1/2, \quad C = N\langle(\delta\alpha)^2\rangle. \tag{12}$$

The initial conditions for system (8)–(10) are of the form

$$B(0) = 1/2, \quad C(0) = 0, \tag{13}$$

and an arbitrary $z(0) \equiv z_0$ which is of the order of unity. The equilibrium value of the cumulant B in (10) is determined by the mean number of the reservoir quanta and its zero-point energy as

$$B^{(0)} = \langle n_d\rangle + 1/2. \tag{14}$$

We note that the zero-point energy of the reservoir appears in the equations of motion for the cumulants, though it is not present in the Heisenberg equations of motion and can even be dropped from the Hamiltonian by redefining a zero of energy. Such a “reappearance” of a zero-point field energy is quite common in other quantum theory problems where the vacuum is responsible for physical effects [20].

The equations of motion for the second-order cumulants B and C [(9), (10)] are linear inhomogeneous equations. Their solution consists of two parts: a general solution of the homogeneous set of equations, (i.e., without the term $+i\gamma B^{(0)}$ in (10)) that we denote as $(\bar{B}(t), \bar{C}(t))$ and the particular solution of the inhomogeneous equations

$$(B(t), C(t)) = (\bar{B}(t), \bar{C}(t)) + (\gamma B^{(0)}t, 0). \quad (15)$$

To find $(\bar{B}(t), \bar{C}(t))$, we use the perturbation theory for $N \gg 1$ and as a first step, neglect the quantum correction Q/N in (8). It is easy to see that the homogeneous equations of motion for cumulants (9) and (10) can be obtained from the classical equation (i.e., from (8) with $Q/N \longrightarrow 0$) by linearization around z (which goes by substituting $z \longrightarrow z + \delta z$, $|\delta z| \ll |z|$), if one writes the dynamical equations for the variables $(\delta z)^2$ and $|\delta z|^2$. The only difference between the linearization of the classical equations of motion and equations for quantum cumulants (9) and (10) lies in the impossibility of obtaining the initial conditions (13) for C and B from only the initial conditions for the linearized classical equations of motion (see also the discussion of this problem in [7]). Hence, we first need to know the classical solution $z_{\mathrm{cl}}(t)$, find the differentials dz_{cl} and dz^*_{cl}, and then use the substitution

$$(\bar{B}(t), \bar{C}(t)) \longrightarrow (|dz|^2, (dz)^2).$$

Thus, it has become apparent that assuming the actual field deviations from the coherent state to be small and treating the small deviation as a first-order correction is not equivalent to the direct linearization around the classical trajectory. Even in the limit as $N \longrightarrow \infty$, we always deal with the effect of reservoir on the dynamics of the quantum system via the second-order cumulant B, which has the form of the quantum diffusion

$$B(t) = \bar{B}(t) + (\langle n_d \rangle + 1/2)\gamma t, \quad (16)$$

where $\bar{B}$ is obtained by linearizing around a large mean field. In particular, as follows from (16), the quantum diffusion also exists for a quiet reservoir $\langle n_d \rangle = 0$.

We now discuss the validity range of the $1/N$-expansion and the role of the quantum diffusion in different classical dynamical regimes. The validity criterion of the $1/N$-expansion can be represented in two forms. First, the $1/N$-expansion works well, provided the difference between the classical and quantum solutions is small,

$$\left|\frac{z(t) - z_{cl}(t)}{z_{cl}(t)}\right| \simeq \frac{1}{N}\frac{\left|\int^{t} Q(t')dt'\right|}{|z(t)|} \ll 1, \quad (17)$$

where $z_{\mathrm{cl}}(t)$ is the solution of (8) for $N \longrightarrow \infty$. To write the second form of the validity criterion of the $1/N$-expansion, we follow [6, 7] in introducing the "convergence radius"

$$R = \{[\mathrm{Re}(\delta\alpha)^2] + [\mathrm{Im}(\delta\alpha)]^2\}^{1/2}.$$

The expansion is then correct over a time interval when

$$\frac{R(t)}{|z(t)|} \simeq \frac{B^{1/2}(t)}{N^{1/2}|z(t)|} \ll 1. \quad (18)$$

As a rule, both conditions (17) and (18) determine the same time interval for the validity of the $1/N$-expansion [6, 7]. (For a physically interesting exception, the problem of SHG, see Section 4.)

For dissipative systems with a simple attractor, the classical field intensity $|z_{\mathrm{cl}}(t)|^2$ as well as the cumulants $\bar{B}(t)$ and $C(t)$ and the quantum correction $Q(t)$ are proportional to the factor $\exp(-\gamma t)$; therefore, as follows from (17) and (18), with (16) taken into account, the $1/N$-expansion is well defined only in the time interval of the order of several relaxation times:

$$t^* \simeq \gamma^{-1}$$

(see [19]). Moreover, during this time interval, the effect of quantum diffusion on the system dynamics is small.

A quite different behavior is characteristic of the stable limit cycle. Here, a variation near the classical trajectory collapses to zero ($\delta\alpha \longrightarrow 0$), hence,

$$\bar{B}(t) \simeq |\delta\alpha|^2 \longrightarrow 0, \quad C(t) \simeq (\delta\alpha)^2 \longrightarrow 0.$$

However, $|z_{\mathrm{cl}}(t)| \simeq 1$ for the limit cycle and, as a result, the time interval of the validity of the $1/N$-expansion is rather large,

$$t^* \simeq N\gamma^{-1}.$$

It is important that the diffusion is a major physical mechanism responsible for the difference between the classical and quantum dynamics for a stable limit cycle. In the following two sections, we consider two typical examples of dissipative optical systems with a simple attractor and a limit cycle.

3. NONCLASSICAL STATE GENERATION IN HIGHER ORDER ANHARMONIC OSCILLATORS

We start by defining the squeezing and the Fano factor. We define the general field quadrature as

$$X_\theta = a\exp(-i\theta) + a^\dagger\exp(i\theta),$$

where θ is the local oscillator phase. A state is called squeezed if there exists a value of θ for which the variance of X_θ is smaller than the variance for the coherent state or the vacuum [1, 9]. Minimizing the variance of

X_θ over θ, we obtain the condition of the so-called principal squeezing [1, 9, 10] in the form

$$S \equiv 1 + 2N(\langle|\delta\alpha|^2\rangle - |\langle(\delta\alpha)^2\rangle|) = 2(B - |C|) < 1. \quad (19)$$

The determination of the principal squeezing S is very useful because it gives the maximum squeezing measurable by the homodyne detection [1, 9].

Another important characteristic of nonclassical properties of light is the Fano factor

$$F = (\langle n^2\rangle - \langle n\rangle^2)/\langle n\rangle$$

that determines the deviation of the probability distribution from the Poisson distribution [1, 9]. After the substitution $a \longrightarrow z + \delta\alpha$ in the expressions

$$\langle n\rangle = N\langle a^\dagger a\rangle$$

and

$$\langle n^2\rangle = N^2\langle a^\dagger a a^\dagger a\rangle = N^2\langle a^{\dagger 2}a^2\rangle + \langle n\rangle,$$

and after the Taylor expansions to the first order of $1/N$, we obtain

$$F = 2B + \left(\frac{z^*}{z}C + \text{c.c.}\right). \quad (20)$$

We see that in order to determine the time dependence of the principal squeezing S in (19) and the Fano factor (20) for nonlinear oscillators, we must find the time dependence of z, C, and B in (8)–(10) for Hamiltonian (1). Following the general procedure described in previous section, we first neglect the quantum correction Q/N in (8). In this case, equation (8) has the exact solution

$$z(t) = z_0\exp[(-i\Delta - \gamma/2)t]\exp[-ig_l|z_0|^{2l}\mu_l(t)], \quad \mu_l(t) \equiv [1 - \exp(-\gamma l t)]/\gamma l. \quad (21)$$

We find the differentials dz and dz^* of classical solution (21) and using the substitutions $|dz|^2 + \tilde{B} \longrightarrow B$ and $(dz)^2 \longrightarrow C$, we obtain

$$\begin{aligned} C(t) &= -lz_0^2|z_0|^{2(l-1)}g_l\mu_l(t)(l|z_0|^{2l}g_l\mu_l(t) + i) \\ &\quad\times\exp[(-\gamma - i2\Delta)t - i2|z_0|^{2l}g_l\mu_l(t)], \\ B(t) &= \exp(-\gamma t)[1/2 + l^2|z_0|^{4l}g_l^2\mu_l^2(t)] \\ &\quad + (\langle n_d\rangle + 1/2)\gamma t, \end{aligned} \quad (22)$$

where we took the initial conditions for B and C, (13), into account. Inserting (22) in (19), we obtain in the limits $\tau \equiv g_l(N)t \ll 1$ and $\gamma t \ll 1$ a very simple time dependence of S,

$$S(t) = 1 - [lx_0^{2l} - (\gamma/g_l)\langle n_d\rangle]2\tau < 1, \quad (23)$$

where, for the sake of simplicity, we assume that the initial value z_0 is real, $x_0 = \text{Re}z_0$, and only the terms that are linear in τ and γt are taken into account. The short-time approximation $\tau \ll 1$ and the limit of a large photon number $N \gg 1$ are quite realistic for a nonlinear medium modeled by the anharmonic oscillators (for numerical estimates, see [1, Chap. 10] and [10]). It should be noted that our formula (23) coincides with the corresponding formula for $S(t)$ in [10] for the Kerr nonlinearity ($l = 1$) with zero loss ($\gamma = 0$). In the case where $\gamma = 0$, our formula (23) shows that the rate of squeezing is determined by the factor

$$2lx_0^{2l}\lambda_l N^t \equiv 2l\mathcal{P}^{(2l+1)}.$$

Since λ_l is proportional to the $(2l + 1)$-order nonlinear susceptibility, the factor $\mathcal{P}^{(2l+1)}$ has a physical meaning of nonlinear polarization. Therefore, the stronger the nonlinear polarization induced by light in the medium, the greater the possibility of effective squeezing of light. For a finite dissipation $\gamma \neq 0$, the squeezing is determined by an interplay between the polarization of nonlinear medium modeled by the anharmonic oscillator and the thermal fluctuations of the reservoir. As follows from (23), there exists a critical number of phonons

$$\langle n_d\rangle^{(\text{cr})} = (l/\gamma)\mathcal{P}^{(2l+1)}$$

such that the squeezing is no longer possible for $\langle n_d\rangle \geq \langle n_d\rangle^{(\text{cr})}$.

In the same approximation, we obtain from (20) the following time dependence of the Fano factor

$$F(t) = 1 + 2\langle n_d\rangle\gamma t. \quad (24)$$

Thus, the statistics is super-Poissonian for any $\gamma \neq 0$ and is independent of the degree of nonlinearity l. This is in good agreement with the earliest result of [9] for a dissipative Kerr oscillator ($l = 1$), where the impossibility of sub-Poissonian statistics and antibunching were found from the exact solution.

We now discuss the validity ranges of our approach. It is easy to see that in terms of this approach, the time dependence of the number of quanta for $l = 1$ is

$$\langle n\rangle(t) + 1/2 = N|z|^2 + B \approx N|z_0|^2(1 - \gamma t) + \langle n_d\rangle\gamma t, \quad \gamma t \ll 1, \quad g_l t \ll 1, \quad (25)$$

where we have used (22) for cumulants B and C. It is instructive to compare (25) with the exact solution for $\langle n\rangle(t)$ for the Kerr nonlinearity [9],

$$\langle n\rangle(t) = \langle n_0\rangle\exp(-\gamma t) + [1 - \exp(-\gamma t)]\langle n_d\rangle. \quad (26)$$

Equations (25) and (26) both describe the evolution of an initially coherent state to a final chaotic state that is characteristic of the reservoir. It is evident that (26) and (25) coincide when $\gamma\tau \ll 1$ and $\langle n_0\rangle \simeq N \gg 1$. A more accurate analysis of the validity condition of the $1/N$-expansion should include a comparison of the solution of quantum motion equation (8), which takes into account the quantum correction Q/N given by (11), with the solution of classical motion equation (21). It may be shown after some algebra, that if $\gamma t \ll 1$ and $\tau \ll 1$, the effect of the quantum correction Q/N on the dynamics of the mean value z is of the order of $1/N$ and, therefore, our cumulant expansion is well defined for $N \gg 1$. The same conclusion could be obtained from another criterion of validity (18).

4. QUANTUM–CLASSICAL CORRESPONDENCE IN SELF-PULSING REGIME OF SECOND-HARMONIC GENERATION

We now consider another example of a quantum optical system, namely intracavity SHG. The Hamiltonian describing two interacting quantum modes in the interaction picture has the form [11, 14]

$$H = \sum_{j=1}^{2}\Delta_j b_j^\dagger b_j + iEN^{1/2}(b_1^\dagger - b_1) + \frac{i\chi}{2}(b_1^{\dagger 2}b_2 - b_1^2 b_2^\dagger), \quad (27)$$

where the boson operators b_j ($j = 1, 2$) describe the fundamental and second-harmonic modes, respectively, Δ_j is the cavity detuning of mode j, $EN^{1/2}$ is the classical field driving first mode (E is of the order of unity), and χ is a second-order nonlinear susceptibility. The linear reservoir and its interaction with a second-order nonlinear medium are described by Hamiltonians (2). Now, we can rewrite the full Hamiltonian of the problem as $H = N\mathcal{H}$, where $\mathcal{H}$ has the same form as (27) and (2) with replacements analogous to (4) taking into account and with the new coupling constant defined by

$$g = \chi\sqrt{N}, \quad (28)$$

which is of the order of unity. Formally, the $1/N$-expansion procedure developed in Section 2 cannot be applied to the problem of SHG; however, its straightforward generalization to two interacting modes gives in the first order of $1/N$ the following self-consistent set of equations

$$\dot{z}_1 = -\frac{\gamma_1}{2}z_1 + E + gz_1^* z_2 + \frac{1}{N}gB_{12}, \quad (29)$$

$$\dot{z}_2 = -\frac{\gamma_2}{2}z_2 - \frac{g}{2}z_1^2 - \frac{1}{N}\frac{g}{2}C_1, \quad (30)$$

$$\dot{B}_1 = -\gamma_1(B_1 - B^{(0)}) + gB_{12}^* z_1 + gB_{12}z_1^* + C_1^* z_2 + C_1 z_2^*, \quad (31)$$

$$\dot{B}_2 = -\gamma_2(B_2 - B^{(0)}) - gB_{12}^* z_1 - gB_{12}z_1^*, \quad (32)$$

$$\dot{C}_1 = -\gamma_1 C_1 + 2g(C_{12}z_1^* + B_1 z_2), \quad (33)$$

$$\dot{C}_2 = -\gamma_2 C_2 - 2gC_{12}z_1, \quad (34)$$

$$\dot{C}_{12} = -0.5(\gamma_1 + \gamma_2)C_{12} + gB_{12}z_2 - C_1 z_1 + C_2 z_1^*, \quad (35)$$

$$\dot{B}_{12} = -0.5(\gamma_1 + \gamma_2)B_{12} + gC_{12}z_2^* + gz_1(B_2 - B_1), \quad (36)$$

where

$$z_j \equiv \langle a_j\rangle = N^{1/2}\langle b_j\rangle, \quad B_j = N\langle\delta\alpha_j^*\delta\alpha_j\rangle + 0.5,$$

$$C_j = N\langle(\delta\alpha_j)^2\rangle \ (j = 1, 2),$$

$$B_{12} = N\langle\delta\alpha_1^*\delta\alpha_2\rangle, \quad C_{12} = N\langle\delta\alpha_1\delta\alpha_2\rangle,$$

and $B^{(0)}$ is defined in (14). The initial conditions for system (29)–(36) are

$$B_j(0) = 1/2, \quad C_j(0) = C_{12}(0) = B_{12}(0) = 0,$$
$$z_2(0) = 0, \quad z_1(0) = z_0,$$

where z_0 is of the order of unity. In this work, we limit ourselves by the values of the field strength z_0 corresponding to self-oscillations [11] and $\Delta_1 = \Delta_2 = 0$.

It is easy to see that in the limit as $N \longrightarrow \infty$ and for $g = \text{const} \simeq 1$, we obtain from (29) and (30) the correct classical equations of motion for the scaled field amplitudes. The solution of equations of motion (31)–(36) for the second-order cumulants has the form

$$\mathbf{X}(t) = \bar{\mathbf{X}}(t) + (\gamma B^{(0)}t, \gamma B^{(0)}t, 0, 0, 0, 0),$$
$$\mathbf{X}(t) \equiv [B_1(t), B_2(t), C_1(t), C_2(t), B_{12}(t), C_{12}(t)], \quad (37)$$

where the vector $\bar{\mathbf{X}}$ describes the part of $\mathbf{X}$ that can be obtained by linearization around the classical trajectory. Variations near a stable limit cycle rapidly approach zero and, therefore, $\bar{\mathbf{X}}(t) \longrightarrow 0$. As a result, we have only a diffusive growth of cumulants B_j ($j = 1, 2$) as

$$B_j(t) = 0.5\gamma_j t, \quad (38)$$

where we considered the case of a quiet reservoir $\langle n_d\rangle$. This result indicates that the effect of reservoir zero-point energy on the dynamics of the nonlinear system is the principal physical mechanism responsible for the difference between the classical and quantum dynamics in the semiclassical limit. A time scale t^* for a correct description of the quantized SHG dynamics in terms of classical electrodynamics can be found using criterion (18). Taking into account that $|z(t)| \simeq 1$, we obtain $t^* \simeq 2N\gamma^{-1}$.

We note that the quantum corrections to the classical equations of motion (29) and (30) do not include the cumulants $B_{1,2}$. Therefore, in the first order of $1/N$, there is no difference between the evolution of quantum mean values and the classical limit cycle dynamics. In other words, the quantum correction $Q \longrightarrow 0$, and therefore, criterion (17) of the $1/N$-expansion validity does not work. In this respect, the quantized SHG is a somewhat singular problem. In other quantum optical systems, for instance, for a nonlinear oscillator with $l \geq 1$, both validity criteria (18) and (17) typically give the same result.

Over a decade ago, Savage addressed the same quantum–classical correspondence problem for self-oscillations in SHG numerically [14]. He calculated the Q distribution function in the Gaussian approximation centered at a deterministic trajectory corresponding to a limit cycle. He worked in a large field and small nonlinearity limits, $\chi/\gamma_{1,2} \longrightarrow 0$, which correspond to the classical limit [14]. It is easy to see that the condition $\chi/\gamma_{1,2} \longrightarrow 0$ is consistent with our condition $N \gg 1$, if one additionally considers the natural condition of a not very strong dissipation in (29)–(36), $\gamma_{1,2}/g < 1$ together with $g \simeq 1$ (28). In other words, Savage's small parameter χ/γ corresponds to our large parameter N as $\chi/\gamma \longrightarrow N^{-1/2}$. To establish the difference between the classical and quantum dynamics, the equations of motion for low-order cumulants were obtained in [14] and solved numerically for particular values of the parameters [21]. Based on the results of numerical simulations, Savage concluded that it is a quantum diffusion that is mostly responsible for the difference between the classical and quantum dynamics in the semiclassical limit. Moreover, his numerical estimate for a characteristic time for the classical description scales as $(\gamma/\chi)^2$, which is in a good agreement with our analytic result $t^* = 2\gamma^{-1}N$. In summary, our analytic results for the quantum–classical correspondence at self-pulsing in SHG are consistent with the previous numerical investigation of same problem in [14].

5. CONCLUSION

We developed the $1/N$-expansion method to consider the nonlinear dynamics and nonclassical properties of light in dissipative optical systems in the limit of a large number of photons. The method was applied to the investigation of squeezing in higher order dissipative nonlinear oscillators. We would like to note that our method can also be directly applied to an important case of the generation of nonclassical states in a medium involving competing nonlinearities [22].

We found a time scale of validity of the $1/N$-expansion for a classical description of the dynamics of nonlinear optical systems with a simple attractor and with a limit cycle. For systems with a simple attractor, this time scale is of the order of unity, and for the limit cycle, is proportional to large N. Qualitatively, this result can be understood as follows. For time of the order of unity, the trajectory spirals around a stable stationary point with a small amplitude, and therefore, by virtue of the uncertainty principle, the contribution of quantum corrections to the classical equations of motion becomes very important. Unlike the previous case, the oscillations corresponding to the limit cycle are often close to harmonic and, thus, their quantum and classical descriptions can coincide for a sufficiently long period of time. The basic difference between the classical and quantum dynamics in the latter case originates from the influence of reservoir zero-point fluctuations, which, in our notation, are of the order of $1/N$. This result is in a good agreement with the result of earlier numerical simulations of self-oscillations in the quantized second-harmonic generation [14]. Finally, it should be noted that our findings are of a rather general nature and can be applied to the investigations of self-oscillations in other optical systems, for example, in those involving optical bistability [23–25].

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