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The 1/*N*-expansion, Quantum-Classical Correspondence and Nonclassical States Generation in Dissipative Higher-Order Anharmonic Oscillators

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Abstract

We develop a method for the determination of the dynamics of dissipative quantum systems in the limit of large number of quanta N, based on the $1/N$-expansion of Heidmann *et al.* [Opt. Commun. **54**, 189 (1985)] and the quantum-classical correspondence. Using this method, we find analytically the dynamics of nonclassical states generation in the higher-order anharmonic dissipative oscillators for an arbitrary temperature of a reservoir. We show that the quantum correction to the classical motion increases with time quadratically up to some maximal value, which is dependent on the degree of nonlinearity and a damping constant, and then it decreases. Similarities and differences with the corresponding behavior of the quantum corrections to the classical motion in the Hamiltonian chaotic systems are discussed. We also compare our results obtained for some limiting cases with the results obtained by using other semiclassical tools and discuss the conditions for validity of our approach.

1. Introduction

The quantum anharmonic oscillator with the Hamiltonian in the interaction picture ($\hbar \equiv 1$)

$$H = \Delta b^\dagger b + \frac{\lambda_l}{l+1}\left(b^\dagger b\right)^{l+1}, \quad [b, b^\dagger] = 1 \qquad (1)$$

is one of the simplest and most popular models describing the quantum statistical properties of light interacting with a nonlinear medium [1,2]. In Eq. (1), the operators b and $b^\dagger$ describe a single mode of quantum field and the constant λ_l is proportional to the $(2l+1)$-order nonlinear susceptibility of a nonlinear medium (l is an integer), Δ is the detuning of the light frequency from the characteristic frequency of quantum transition. We adopt the normal ordering of operators. For the case of a cubic nonlinearity ($l = 1$), this model was first introduced by Tanaś [3] for the investigation of self-squeezing of light propagating through a nonlinear Kerr medium without loss. Because such a model is exactly integrable, the explicit time dependence of the quadrature variances necessary for the determination of the squeezing condition has been found for any moments of time and for any number of photons [3, 4]. The problem of a dissipative anharmonic oscillator is much more difficult. Nevertheless, Milburn and Holms obtained the exact solution for the damped Kerr oscillator ($l = 1$) interacting with a reservoir of zero temperature [5]. This result has been further generalized to the case of a reservoir of non zero temperature in [6].

In the conditions of an experiment, as a rule, a large number of photons are involved in a nonlinear interaction between light and a nonlinear medium modelled by the anharmonic oscillator [1,3,4]. The determination of squeezing conditions from the exact solution in this limiting case is straightforward for the model of Kerr oscillator without loss [4]. In contrast, due to the complex form of the exact solution for the damped Kerr oscillator, the determination of photon statistics for the large number of photons in this model demands an application of numerical methods or special approximate analytical methods (for a review see [2]). Moreover, there are no exact solutions for the model of the quantum dissipative oscillator with higher-order nonlinearity and a very small amount of information on its dynamics is documented in the literature [2].

In general, the situation when a large number of photons N are involved in nonlinear interactions is quite typical for many problems of quantum optics [7,8]. Heidmann *et al.* suggested [9] to use the method of $1/N$-expansion for the determination of nonclassical states generation dynamics. They originally applied the $1/N$-expansion technique to the problem of squeezing and antibunching of an electromagnetic field interacting with a collection of Rydberg atoms inside a high-Q cavity [10], where a large number of atoms is of the same order as the number of photons N. The general scheme of the $1/N$-expansion method states that an exact or an approximate solution of the problem can be found in the classical limit $N \to \infty$ and then the quantum corrections could be added [11]. Because this method allows to find the motion equations for the mean values and the lower-order cumulants, it could also be considered as a variant of the cumulant expansion [9,12]. Recently we further developed and applied the $1/N$-expansion technique [9] to the investigation of an enhanced squeezing at the transition to quantum chaos[1] [13–15]. It should be noticed that only nondissipative quantum systems have been considered in papers [9, 13–15].

In this paper, using the method of $1/N$-expansion, we consider a dynamics of squeezing and a deviation from the Poissonian statistics for the damped anharmonic oscillators

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[1] Recently a good agreement between the predictions for dynamics of squeezing obtained using the method of $1/N$-expansion and the results of numerical simulation for the model of kicked quantum rotator with $N \simeq 10^5$ levels were found in the works [16,17].

with arbitrary degree of nonlinearity l in the limit of a large number of photons $N \gg 1$. We find the explicit time dependencies for the squeezing and the Fano factor for an arbitrary degree of nonlinearity and for an arbitrary temperature of a reservoir. Our consideration is based on the quantum-classical correspondence and the fact that the solution of classical equations of motion, obtained within the zero-order approximation of $1/N$, could be found analytically for the case of any linear damping. We show that for weak damping the degree of squeezing is mainly determined by the nonlinear polarization of a nonlinear medium, modelled by the nonlinear oscillator. For the case of no damping, our time dependencies for squeezing are transformed to the corresponding formulas of work [4], which have been found from an exact solution of the Hamiltonian problem. A finite damping decreases the degree of squeezing. The consideration of the Fano factor demonstrates that the quantum statistics is always a super-Poissonian for dissipative oscillators. Another restrictive factor influencing the time dependencies of squeezing and the Fano factor are the thermal fluctuations of the reservoir.

Note that in spite of the fact that we find our main results for the the model of the higher-order oscillator, we present our self-consistent system of motion equations for the first- and the second-order cumulants in a form which is valid for the description of any single-mode quantum system in the semiclassical limit. One of the main findings of this general consideration consists in the influence of a specific quantum diffusive term on the dynamics of the expectation values and dispersions. We interpret this diffusion, which is proportional to the damping constant, as an influence of the zero-point energy of the reservoir on the quantum system. Although the influence of quantum diffusion around the classical solution is insufficient for the description of time dependencies of squeezing and mean values for the particular system under study, and especially in the most interesting case of a short time of interaction, we think that the account of this quantum diffusion is important for the correct description of other dissipative quantum systems in the limit of large quantum number.

We compare our basic equations of motion for dissipative systems with the equations arising within the so-called generalized Gaussian approximation [1,12,18] and find a one-to-one correspondence up to terms of $1/N^2$ for several popular models of quantum optics [18–21].

We also discuss the conditions for validity of the cumulant expansion in the form of the $1/N$-expansion for the description of the dissipative dynamics of nonlinear oscillators. This problem is related to the problem of finding the time interval for the quantum-classical correspondence, which attracts large attention nowadays, and especially in connection with the studies of quantum chaotic systems (see [22] and the references cited therein). For the Hamiltonian systems with regular dynamics, the quantum corrections to the corresponding classical equations grow in the course of time as power laws [13–15,22–24]. As a result, the time interval for the classical description has a power-wise dependence on the semiclassical parameter N [22–24]. In contrast, for the case of nondissipative quantum systems which are chaotic in the classical limit, the quantum corrections grow exponentially in time due to underlying local instability

in the classical system [13–15,22–24]. Therefore, the time interval for the validity of the $1/N$-expansion method and the classical description is logarithmic in the semiclassical parameter N [22–24].

Our finding for the dissipative nonlinear oscillators is that the quantum correction first increases, then reaches some maximum and finally decreases in the course of time. As a consequence, the time interval for validity of the $1/N$-expansion could be divided into two subintervals with completely different dependencies on the semiclassical parameter N. The first subinterval scales up power-wise with N and the second one has a logarithmic dependence on N. While a power dependence on N originates from the same time behavior as in the Hamiltonian systems with regular dynamics, the $\log N$ scale has a different nature. It appears because of an exponential damping of the underlying classical dynamics.

Our work is organized as follows. We consider a single mode dissipative quantum system and present the derivation of motion equations for the first and second-order cumulants in section 2. In the same section we also compare our approach based on the $1/N$-expansion method with other semiclassical methods and find analytically the time dependencies for the mean value and the second-order cumulants for the model of the higher-order dissipative oscillator. Using these results, we focus on the time dependencies of the squeezing and the Fano factor in section 3. Section 4 is devoted to the discussion of conditions for the validity of the $1/N$-expansion in the description of nonlinear oscillators dynamics. In the concluding section, we briefly summarize our results and outlook the main directions for the future developments. Some details related to the solution of motion equations for the cumulants and the determination of time intervals for the validity of our approach are presented in two Appendices.

2. Basic equations

2.1. *The $1/N$-expansion method and comparison with other approaches*

First of all, we need to generalize the approach of [9, 14, 15] to the case of a single-mode quantum system with dissipation. Consider an oscillator with the Hamiltonian (1) which interacts with an infinite linear reservoir of finite temperature. The Hamiltonians of a reservoir and its interaction with an oscillator are the following

$$H_r = \sum_j \psi_j (d_j^\dagger d_j + 1/2), \quad H_{\text{int}} = \sum_j \left(\kappa_j d_j b^\dagger + \text{H.c.} \right), \quad (2)$$

where the Bose operator d_j ($[d_j, d_k^\dagger] = \delta_{jk}$) describes an infinite reservoir with characteristic frequencies ψ_j, and κ_j are coupling constants between the reservoir modes and the oscillator. Introduce new scaled operators $a = b/N^{1/2}$, $c_j = d_j/N^{1/2}$, and Hermitian conjugats to them with commutation relations

$$[a, a^\dagger] = 1/N, \quad [c_j, c_k^\dagger] = \delta_{jk}/N. \quad (3)$$

In the classical limit $N \to \infty$, we have commuting classical c-numbers instead of operators. Now the full Hamiltonian $H = H_0 + H_r + H_{\text{int}}$ may be rewritten as $H = N\mathcal{H}$, where

$\mathcal{H}$ has the same form as in the formulas (1) and (2) with an account of the following replacements

$$b \to a, \quad b^\dagger \to a^\dagger, \quad d_j \to c_j, \quad d_j^\dagger \to c_j^\dagger, \quad \text{and}$$
$$\lambda_l \to g_l(N) \equiv \lambda N^l. \tag{4}$$

It could be shown that dependent on photon number the constant $g_l(N)$ correctly gives the time scale of energy oscillations for the nonlinear oscillator (1) in the classical limit (for the case of the Kerr nonlinearity, see, *e.g.* [25]).

Within the standard Heisenberg-Langevin approach the equation of motion has the form ([1], chap. 7; [26])

$$i\dot{a} = \left(\Delta - i\frac{\gamma}{2}\right)a + V + L(t), \tag{5}$$

where $V = \partial\mathcal{H}_0/\partial a^\dagger$, $\gamma = 2\pi|\kappa(\omega)|^2\rho(\omega)$ is a damping constant, $\rho(\omega)$ being the density function of reservoir oscillators, which spectrum is considered to be flat. The Langevin force operator $L(t) = \sum_j \kappa_j d_j(0) \exp(-i\psi_j t)$ has properties [1, 26], which in our notations (4) may be rewritten as

$$\langle L(t)\rangle_R = \langle L^\dagger(t)\rangle_R = 0, \quad \langle L^\dagger a\rangle_R + \langle a^\dagger L\rangle_R = \gamma\frac{\langle n_d\rangle}{N},$$
$$\langle La\rangle_R + \langle aL\rangle_R = 0, \tag{6}$$

where the average is performed over the reservoir variables and $\langle n_d\rangle$ is a mean number of reservoir quanta (phonons), related to the temperature T as $\langle n_d\rangle = \langle c^\dagger(0)c(0)\rangle = \left[1 - \exp\left(\frac{\omega}{kT}\right)\right]^{-1}$, where k is the Boltzmann's constant. From the Heisenberg-Langevin equations for a, a^2 and Hermitian conjugated equations, we get using Eqs. (5) and (6)

$$i\frac{d}{d\tau}\langle\alpha\rangle = \langle V\rangle - i\frac{\Gamma}{2}\langle\alpha\rangle,$$
$$i\frac{d}{d\tau}\langle(\delta\alpha)^2\rangle = 2\langle V\delta\alpha\rangle + \langle W\rangle - i\Gamma\langle(\delta\alpha)^2\rangle,$$
$$i\frac{d}{d\tau}\langle\delta\alpha^*\delta\alpha\rangle = -\langle V^*\delta\alpha\rangle + \langle\delta\alpha^* V\rangle - i\Gamma\langle\delta\alpha^*\delta\alpha\rangle + i\Gamma\frac{\langle n_d\rangle}{N}, \tag{7}$$

where

$$W(\alpha, \alpha^*) = \frac{1}{N}\frac{\partial V}{\partial a^\dagger}, \quad z \equiv \langle a\rangle, \quad \langle(\delta\alpha)^2\rangle = \langle a^2\rangle - z^2,$$
$$\langle\delta\alpha^*\delta\alpha\rangle = \langle a^\dagger a\rangle - |z|^2, \tag{8}$$

and we have introduced the scaled variables $\tau = g_l t$, $\Gamma = \gamma/g_l$, $\bar{\Delta} = \Delta/g_l$. Averaging in Eqs. (7) is performed over both the reservoir variables and the coherent state $|\alpha\rangle = \exp(N\alpha a^\dagger - N\alpha^* a)|0\rangle$ corresponding to the mean photon number $\simeq N$. Such kind of minimum-uncertainty states are most suitable for a consideration of the semiclassical limit $N \gg 1$ [11]. In derivation of Eqs. (7) we neglect an insufficient additional detuning introduced to Δ by the interaction with the reservoir [1, 26].

The set of equations (7) is not closed and actually is equivalent to the infinite hierarchic dynamical system for the cumulants of different order. To truncate it up to the cumulants of the second order, we make the substitution $a \to z + \delta\alpha$, where at least initially the mean $z \simeq 1$ and the quantum correction $|\delta\alpha(0)| \simeq N^{-1/2} \ll 1$. Using the Taylor expansion of the functions $V(\alpha, \alpha^*)$ and $W(\alpha, \alpha^*)$

around the mean values z and z^*,

$$V = V_z + \left(\frac{\partial V}{\partial\alpha}\right)_z \delta\alpha + \left(\frac{\partial V}{\partial\alpha^*}\right)_z \delta\alpha^* + \cdots,$$
$$W = W_z + \left(\frac{\partial W}{\partial\alpha}\right)_z \delta\alpha + \left(\frac{\partial W}{\partial\alpha^*}\right)_z \delta\alpha^* + \cdots, \tag{9}$$

and after some algebra analogous to that used in [14,15], we get from (7) in the first order of $1/N$ the following self-consistent system of equations for the mean value and the second-order cumulants

$$i\frac{d}{d\tau}z = -i\frac{\Gamma}{2}z + \langle V\rangle_z + \frac{1}{N}Q\left(z, z^*, \langle(\Delta\alpha)^2\rangle, \langle(\Delta\alpha^*)^2\rangle, \langle|\Delta\alpha|^2\rangle\right), \tag{10}$$

$$i\frac{d}{d\tau}\langle(\Delta\alpha)^2\rangle = 2\left(\frac{\partial V}{\partial\alpha}\right)_z \langle(\Delta\alpha)^2\rangle + 2\left(\frac{\partial V}{\partial\alpha^*}\right)_z \langle|\Delta\alpha|^2\rangle + \langle w\rangle_z - i\Gamma\langle(\Delta\alpha)^2\rangle, \tag{11}$$

$$i\frac{d}{d\tau}\langle|\Delta\alpha|^2\rangle = -\left(\frac{\partial V^*}{\partial\alpha}\right)_z \langle(\Delta\alpha)^2\rangle + \left(\frac{\partial V}{\partial\alpha^*}\right)_z \langle(\Delta\alpha^*)^2\rangle - i\Gamma\langle|\Delta\alpha|^2\rangle + i\Gamma\langle n_d\rangle, \tag{12}$$

where we have introduced the scaled second-order cumulants as $\langle(\Delta\alpha)^2\rangle = N\langle(\delta\alpha)^2\rangle$ and $\langle|\Delta\alpha|^2\rangle = N\langle\delta\alpha^*\delta\alpha\rangle$, as well as the function $w(z, z^*) = NW(z, z^*)$ (for definition of W see Eq. (8)). Now both the first-order cumulant z and the second-order cumulants $\langle(\Delta\alpha)^2\rangle$, $\langle|\Delta\alpha|^2\rangle$ are of the order of unity and small parameter $1/N$ arises only as a prefactor for the quantum correction Q to the classical motion equation in (10). The expression for the quantum correction has the form of second order differential $Q = \frac{1}{2}d^2 V|_z$. In this formula and in Eqs. (10)–(12), the subscript z means that the values of V and its derivatives are calculated at the mean value z.

The equations (11), (12) are nonlinear due to the presence of the nonlinear term $\langle w\rangle_z$ in Eq. (11). However, introducing new variables for the second-order cumulants

$$B = \langle|\Delta\alpha|^2\rangle + \frac{1}{2}, \quad C = \langle(\Delta\alpha)^2\rangle, \tag{13}$$

the term $\langle w\rangle_z$ could be removed and the self-consistent equations (10)–(12) may be rewritten as

$$i\dot{z} = -i\frac{\Gamma}{2}z + \langle V\rangle_z + \frac{1}{N}Q(z, z^*, C, C^*, B), \tag{14}$$

$$i\dot{C} = 2\left(\frac{\partial V}{\partial\alpha}\right)_z C + 2\left(\frac{\partial V}{\partial\alpha^*}\right)_z B - i\Gamma C, \tag{15}$$

$$i\dot{B} = -\left(\frac{\partial V^*}{\partial\alpha}\right)_z C + \left(\frac{\partial V}{\partial\alpha^*}\right)_z C^* - i\Gamma\left(B - B^{(0)}\right), \tag{16}$$

$$Q(z, z^*, C, C^*, B) = \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^2}\right)_z C + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^{*2}}\right)_z C^* + \left(\frac{\partial^2 V}{\partial\alpha^*\partial\alpha}\right)_z \left(B - \frac{1}{2}\right). \tag{17}$$

If initially the oscillator is in the coherent state, then the

initial conditions for system (14)–(16) are

$$B(0) = 1/2, \quad C(0) = 0, \tag{18}$$

and some arbitrary $z(0) \equiv z_0$ which is of the order of unity. Involved into equation (16) an equilibrium value of cumulant B is determined by the mean number of reservoir's quanta and its zero-point energy as

$$B^{(0)} = \langle n_d \rangle + 1/2. \tag{19}$$

Note that the zero-point energy of a reservoir appears in the equations of motion for the cumulants though it was not presented in the Heisenberg equations of motion and even may be dropped from the Hamiltonian redefining a zero of the energy. Such "reappearence" of a zero-point field energy is rather often in other problems of quantum theory where a vacuum is responsible for the physical effects [27].

It is sufficient that now our basic equations for the second-order cumulants (15), (16) are linear and this fact allows us to find analytically the solution of whole set (14)–(16) using the quantum-classical correspondence in the limit $N \to \infty$. The substitution (13) describes, of course, nothing but the transformation from a normal to a symmetric ordering. The finding that equations of motion for the lower-order cumulants in the semiclassical limit are solvable just within the symmetric ordering is an illustration of the fact that symmetric ordering is most suitable for the consideration of the quantum-classical correspondence [8].

Before we will find a solution of equations (14)–(16), let us make several comments and make a comparison with other approaches.

(*i*) It should be noticed that we derive equations (7), (14)–(16) within the Heisenberg-Langevin approach, but alternatively, the same equations could be obtained [28] starting from the corresponding generalized Fokker-Planck equation [1,26].

(*ii*) In the case of no damping $\Gamma = 0$, our equations for the mean values and the second order cumulants (14)–(16) are reduced to the corresponding equations of ref. [14,15]. On other hand, for the Hamiltonians which can be presented as sum of a kinetic and a potential energy, our approach gives the same motion equations as the cumulant expansion method introduced in [23]. Moreover, Sundaram and Milonni showed [23] that the first-order cumulant approximation to the Heisenberg equations of motion gives the expectation values identical to those obtained by the methods of Gaussian wavepacket dynamics [29,30], semiquantum [31,32], and based on the time-dependent variational principle [33]. In this respect, our present work could be considered as some variant for the generalization of the semiquantum methods to the case of a dissipative dynamics.

(*iii*) Another approach to the description of the dynamics of quantum fluctuations based on almost Gaussian wavepackets is the so-called generalized Gaussian approximation (GGA) [1,12,18]. This approximation is valid for both the Hamiltonian and the dissipative systems and consists in an assumption that the Fourier transform of quantum distribution function, *i.e.* quantum characteristic function, is Gaussian for any moment of time [12]. Within GGA only the first- and the second-order cumulants are non zero, and therefore it presents the higher-order

cumulants in terms of only the first and the second-order cumulants and to truncate an infinite dynamic system for the cumulants. We compare our system (14)–(16) with the corresponding dynamic equations for the mean values and the cumulants obtained within GGA for a several popular dissipative models of quantum optics: a second harmonic generation [19], a nondegenerate optical three-wave mixing[2] and a forced nonlinear oscillator with cubic nonlinearity [21]. For the forced nonlinear oscillator, we found that our self-consistent set of equations (14)–(16) coincides with the corresponding basic equations of [21] up to the terms of the order of $1/N^2$, and for the problems of a nondegenerate and a degenerate optical 3-wave mixing, our approach gives equations which are identical to the corresponding motion equations obtained within GGA.

There are yet the principal differences between our approach and GGA: First, our approach works well in the semiclassical limit $N \gg 1$, in contrast, GGA is suitable also for the quantum limit $N \simeq 1$. Second, in the general case, the equations of motion obtained within GGA are nonlinear and cannot be solved analytically. As we shall see in the next subsection, the solution of our system (14)–(16) could be obtained analytically and has simple physical meaning.

2.2. The quantum-classical correspondence and the dynamics of cumulants

We need to find the time dependencies of z, C, and B from the equations (14)–(16). First, suppose that we know the solution $z(\tau)$ of the motion equation (14) for the mean value. Then, the equations of motion for the second-order cumulants (15) and (16) form the linear inhomogeneous equation for the vector variable $X(\tau) = [C(\tau), C^*(\tau), B(\tau)]$ as

$$\mathrm{i}\dot{X} = -\mathrm{i}(\Gamma/2)X + \hat{A}X + \varepsilon X_0, \tag{20}$$

where $X_0 = (0, 0, \mathrm{i}\Gamma B^{(0)})$ and $B^{(0)}$ is given by Eq. (19). The matrix $\hat{A}$ is formed by the partial derivatives of V calculated at the mean value z, its form can be easily obtained from Eqs. (15) and (16) (to save space we do not present an explicit form of $\hat{A}$ here). We also introduced a parameter ϵ for the discussion of solutions of Eq. (20) (actual value $\epsilon \equiv 1$ in (20)). The solution $X(\tau)$ of the linear Eq. (20) consists of two parts

$$X(\tau) = \bar{X}(\tau) + \tilde{X}(\tau), \tag{21}$$

where $\bar{X}(\tau)$ is a general solution of the homogeneous equation (i.e., Eq. (20) with $\epsilon = 0$) and $\tilde{X}(\tau)$ is a particular solution of the inhomogeneous equation. It easy to see that

$$\tilde{X}(\tau) = \left(0, \Gamma B^{(0)}\tau\right). \tag{22}$$

To find $\bar{X}(\tau)$ we need to solve the self-consistent set of equations (20) for $\epsilon = 0$ together with (14) and (17). We seek a solution by the perturbation theory using the smallness of the parameter $1/N$. Substituting the expression

$$z(\tau) = z_{\mathrm{cl}}(\tau) + \frac{1}{N}z^{(1)}(\tau), \quad \frac{1}{N}|z^{(1)}(\tau)| \ll |z_{\mathrm{cl}}(\tau)| \tag{23}$$

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[2] The equations of motion within GGA for the model of nondegenerate three-wave mixing could be easily obtained in analogy with the case of second harmonic generation [19]. The study of lossless three-wave mixing within GGA has been presented in [20].

into the motion equation (14) for the mean value, we get in zero-order of $1/N$ the classical motion equation

$$\mathrm{i}\dot{z}_{\mathrm{cl}} = -\mathrm{i}\frac{\Gamma}{2}z_{\mathrm{cl}} + V(z_{\mathrm{cl}}, z_{\mathrm{cl}}^*). \tag{24}$$

Now, it could be shown that the solution $\bar{X}(\tau)$ of the equation of motion for the cumulants (20) with $\epsilon = 0$ can be obtained directly from the solution of the classical equation (24) by the linearization near z_{cl} (the substitution $z_{\mathrm{cl}} \rightarrow z_{\mathrm{cl}} + \delta z$, $|\delta z| \ll |z|$), if one writes the dynamical equations for the variables $(\delta z)^2$ and $|\delta z|^2$ (for the details of derivation, see Appendix I). Thus, the solution for $\bar{X}(\tau)$ is

$$\bar{X}(\tau) = \left[(\mathrm{d}z_{\mathrm{cl}})^2, (\mathrm{d}z_{\mathrm{cl}}^*)^2, |\mathrm{d}z_{\mathrm{cl}}|^2\right], \tag{25}$$

where $\mathrm{d}z_{\mathrm{cl}}(\tau)$ is the differential of the classical variable $z_{\mathrm{cl}}(\tau)$ governed by Eq. (24), and initial conditions for (25) are (18), *i.e.* one should take

$$(\mathrm{d}z_{\mathrm{cl}})^2(0) = 0, \quad |\mathrm{d}z_{\mathrm{cl}}|^2(0) = 1/2. \tag{26}$$

Now we can demonstrate how to find the dynamics of cumulants C and B, if the classical dynamics $z_{\mathrm{cl}}(\tau)$ is known. Combining formulas (21), (22), (25) and (19), we get

$$C = (\mathrm{d}z_{\mathrm{cl}})^2, \quad B = |\mathrm{d}z_{\mathrm{cl}}|^2 + (\langle n_d \rangle + 1/2)\Gamma\tau. \tag{27}$$

Turn to the determination of the influence of the quantum correction (17) on the dynamics of the mean value z. Substituting the expansion (23) into equation (14), we have in the first order of $1/N$ the motion equation for $z^{(1)}$ as

$$\mathrm{i}\dot{z}^{(1)} = -\mathrm{i}\frac{\Gamma}{2}z^{(1)} + Q(\tau), \quad Q(\tau) \equiv Q[z_{\mathrm{cl}}(\tau), z_{\mathrm{cl}}^*(\tau),$$
$$C(\tau), C^*(\tau), B(\tau)], \tag{28}$$

where the cumulants $C(\tau)$, $C^*(\tau)$, and $B(\tau)$ are determined by the formulas (27) and $z_{\mathrm{cl}}(\tau)$ is a solution of the classical equation (24). A formal solution of Eq (28) is

$$z^{(1)}(\tau) = z^{(1)}(0)\exp\left(-\frac{\Gamma}{2}\tau\right) - \mathrm{i}\int_0^\tau \mathrm{d}\tau' Q(\tau'). \tag{29}$$

For the initial conditions (18), the initial value of quantum correction (17) is zero. Therefore, $z(0) = z_{\mathrm{cl}}(0) = z_0$ and $z^{(1)}(0) = 0$. The condition of smallness of $z^{(1)}(\tau)$ in comparison with $z_{\mathrm{cl}}(\tau)$ [Eq. (23)] takes the form

$$R(\tau) \equiv \left|\frac{z(\tau) - z_{\mathrm{cl}}(\tau)}{z_{\mathrm{cl}}(\tau)}\right| = \frac{1}{N}\left|\frac{z^{(1)}(\tau)}{z_{\mathrm{cl}}(\tau)}\right| = \frac{1}{N}\left|\frac{\int_0^\tau \mathrm{d}\tau' Q(\tau')}{z_{\mathrm{cl}}(\tau)}\right| \ll 1. \tag{30}$$

We shall see when the condition (30) is valid for the model of anharmonic oscillator (1) in the next section.

We turn to the determination of the dynamics of the cumulants and the mean value in the case of anharmonic oscillator with the Hamiltonian (1). Here, the expression for $V(z, z^*)$ takes the form

$$V(z, z^*) = \bar{\Delta}z + |z|^{2l}z. \tag{31}$$

The exact solution of the classical motion equation (24) with

V of the form (31) is

$$z(\tau) = z_0 \exp\left[(-\mathrm{i}\bar{\Delta} - \Gamma/2)\tau\right]\exp\left[-\mathrm{i}|z_0|^{2l}\mu_l(\tau)\right],$$
$$\mu_l(\tau) = [1 - \exp(-\Gamma l\tau)]/\Gamma l. \tag{32}$$

Using the expression (32), we have from Eqs. (27) the following time dependencies for the cumulants

$$C(\tau) = -\mu_l(\tau)lz_0^2|z_0|^{2(l-1)}\left(\mu_l(\tau)l|z_0|^{2l} + \mathrm{i}\right)$$
$$\qquad \exp\left[(-\Gamma - \mathrm{i}2\bar{\Delta})\tau - \mathrm{i}2|z_0|^{2l}\mu_l(\tau)\right],$$
$$B(\tau) = \exp(-\Gamma\tau)\left[1/2 + l^2|z_0|^{4l}\mu_l^2(\tau)\right] + (\langle n_d \rangle + 1/2)\Gamma\tau, \tag{33}$$

where we took into an account the initial conditions (26). We shall use these time dependencies for the cumulants in the consideration of the squeezing and the Fano factor in the next section.

3. The squeezing and the Fano factor

3.1. *The dynamics of squeezing*

Define the general field quadrature as $X_\theta = a\exp(-\mathrm{i}\theta) + a^\dagger \exp(\mathrm{i}\theta)$, where θ is a local oscillator phase. A state is said to be squeezed if there exists some θ such that the variance of X_θ is smaller than the variance for a coherent state or the vacuum [1,2]. Minimizing the variance of X_θ over θ, we get the condition for so-called principal squeezing [1,2,4]

$$S \equiv 1 + 2N(\langle|\delta\alpha|^2\rangle - |\langle(\delta\alpha)^2\rangle|) = 2(B - |C|) < 1. \tag{34}$$

The determination of the principal squeezing S is very useful because it gives the maximal squeezing measurable by the homodyne detection [1,2]. Substituting expressions (33) for the cumulants B and C into the definition (34), we find

$$S(\tau) = \exp(-\Gamma\tau)\left[1 + \phi_l(x_0, \tau)\right] + (\langle n_d \rangle + 1/2)2\Gamma\tau, \tag{35}$$

$$\phi_l(x_0, \tau) = 2a\left[a - (1 + a^2)^{1/2}\right], \quad a(\tau) = lx_0^{2l}\mu_l(\tau), \tag{36}$$

where $\mu_l(\tau)$ is defined in (32) and we assumed for the sake of simplicity that the initial condition z_0 is real, $x_0 = \mathrm{Re}z_0$. First, consider some interesting particular cases.

3.1.1. The weak dissipation In the case of weak dissipation $\Gamma\tau \ll 1$, we have $\exp(-\Gamma\tau) \approx 1 - \Gamma\tau$ and $\mu_l(\tau) \approx \tau$. Thus, in this limit, from Eqs. (35) and (36), we get

$$S(\tau) = 1 + (1 - \Gamma\tau)\phi_l(x_0, \tau) + 2\langle n_d \rangle\Gamma\tau,$$
$$a(\tau) \approx lx_0^{2l}\tau \equiv l\mathcal{P}^{(2l+1)}t. \tag{37}$$

In the case of no loss ($\Gamma = 0$), formula (37) shows that the rate of squeezing is determined by the factor $2lx_0^{2l}\lambda_l N^l \equiv 2l\mathcal{P}^{(2l+1)}$. Because λ_l is proportional to the $(2l + 1)$-order nonlinear susceptibility, the factor $\mathcal{P}^{(2l+1)}$ has a physical meaning of the nonlinear polarization. Therefore, the stronger the nonlinear polarization induced by a light in the medium, the more effective squeezing of light is possible. For a finite dissipation $\Gamma \neq 0$, the squeezing is determined by an interplay between dissipation, the polarization of the nonlinear medium modelled by the anharmonic oscillator and the thermal fluctuations of a reservoir. Both dissipation and a finite temperature of the reservoir are destructive factors for the squeezing. It is
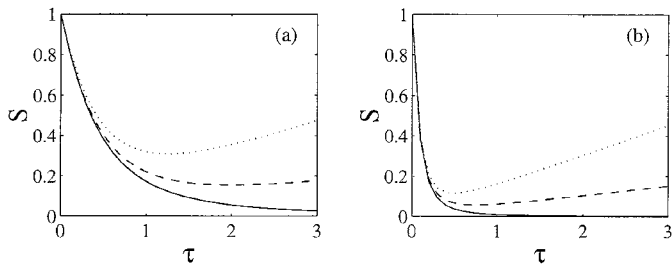
Fig. 1. The dependence of squeezing S on scaled time τ for the Kerr nonlinearity with $l = 1$ (a) and with $l = 5$ (b). Other parameters are following: $\Gamma = 0$, $\langle n_d \rangle = 0$ (solid curve); $\Gamma = 5 \times 10^{-2}$, $\langle n_d \rangle = 0$ (dashed curve); and $\Gamma = 5 \times 10^{-2}$, $\langle n_d \rangle = 1$ (dotted curve). The initial condition is $z_0 = 1$.

interesting to note that the vacuum term does not appear in Eq. (37), which is characteristic of the weak dissipation limit.

3.1.2. The short-time approximation $\tau \ll 1$.

The short-time approximation $\tau \ll 1$, as well as the limit of large photon number $N \gg 1$, are quite realistic for a nonlinear medium modelled by anharmonic oscillators (for numerical estimates, see [1], chap. 10, and [4]). In the limits of $\tau \ll 1$ and $\Gamma \tau \ll 1$, we have from Eq. (36): $\phi_l(x_0, \tau) \approx -2lx_0^{2l}\tau = -2l\mathcal{P}^{(2l+1)}t$. Thus, substituting this expression into (37), we get a very simple dependence of S on time as

$$S(t) = 1 - \left[lx_0^{2l} - \Gamma\langle n_d\rangle\right]2\tau + O(\tau^2). \tag{38}$$

As follows from Eq. (38), there exists a critical number of phonons $n_d^{(cr)} = (l/\gamma)\mathcal{P}^{(2l+1)}$ such that at $n_d \geq n_d^{(cr)}$ any degree of squeezing is impossible. More detailed consideration of the short-time approximation is presented in [34].

3.1.3. The lossless case $\Gamma = 0$.

In the case of no loss, the time dependence of S is

$$S(\tau) = 1 + \phi_l(x_0, \tau), \tag{39}$$

where ϕ_l is defined in (36) and we should take into account that now $\mu_l(\tau) = \tau$ and thus $a(\tau) = lx_0^{2l}\tau = l\mathcal{P}^{(2l+1)}t$. For the case of Kerr nonlinearity ($l = 1$), the formula (39) coincides with the corresponding formula from [4] obtained in the limit $N \gg 1$ from the exact solution for the Hamiltonian case.

Now we consider the time dependence of S [Eq. (39)] in the limit of large time $\tau \gg 1$, which corresponds to $a \gg 1$. Rewriting ϕ_l in the form $\phi_l(x_0, \tau) = 2a^2\left[1 - (1 + a^{-2})^{1/2}\right]$ and expanding $(1 + a^{-2})^{1/2}$ up to the term of order of a^{-3}, we get $\phi_l \approx -1 + (6a)^{-1}$ for $a \gg 1$. Therefore

$$S(\tau) \approx \left(lx_0^{2l}\tau\right)^{-1} = \left(l\mathcal{P}^{(2l+1)}t\right)^{-1} \tag{40}$$

for $\tau \gg 1$. This is in good agreement with the statement that the principal squeezing S is a power-wise decreasing function of time for the general class of integrable systems in the semiclassical limit [15].

Return to the case of arbitrary dissipation. The dependence of S on τ [Eqs. (35), (36)] is shown in Fig. 1, where the solid curve corresponds to the lossless case ($\Gamma = 0$), the dashed curve corresponds to the finite damping constant $\Gamma = 5 \times 10^{-2}$ with a reservoir of zero temperature $\langle n_d \rangle = 0$, and, finally, the dotted curve corresponds to a reservoir with

finite temperature ($\langle n_d \rangle = 1$). As is evident from this figure, the squeezing is stronger for higher nonlinearity (compare Fig. 1a and Fig. 1b), a dissipation slows down the rate of squeezing (compare the solid and the dashed curves in Fig. 1). Moreover, a finite temperature of a reservoir fastly destroys the squeezing.

3.2. The Fano factor

Another important characteristic of the nonclassical properties of light is the Fano factor

$$F = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle}, \tag{41}$$

which determines the deviation of a probability distribution from the Poissonian distribution with $F = 1$ [1,2]. In (41) the mean foton number $\langle n \rangle$ and the mean square foton number $\langle n^2 \rangle$ are

$$\langle n \rangle \equiv \langle b^\dagger b \rangle = N\langle a^\dagger a \rangle,$$
$$\langle n^2 \rangle = N^2\langle a^\dagger a a^\dagger a \rangle = N^2\langle a^{\dagger 2} a^2 \rangle + \langle n \rangle. \tag{42}$$

Substituting expression $a \to z + \delta\alpha$ ($|\delta\alpha| \ll |z| \simeq 1$) into formulas (42), we have after Taylor expansions in the first order of $1/N$

$$\langle a^\dagger a \rangle = |z|^2 + N^{-1}(B - 1/2),$$
$$\langle a^{\dagger 2} a^2 \rangle = |z|^4 + N^{-1}(z^*C + \text{c.c.}) + N^{-1}4|z|^2(B - 1/2), \tag{43}$$

and, therefore, the dependence of the Fano factor on the cumulants and the mean values in the first order of $1/N$ is[3]

$$F = 2B + \left(\frac{z^*}{z}C + \text{c.c.}\right). \tag{44}$$

Substituting expressions (33) for B and C into Eq. (44), we find the following time dependence for the Fano factor

$$F(\tau) = \exp(-\Gamma\tau) + (\langle n_d \rangle + 1/2)2\Gamma\tau. \tag{45}$$

As it is evident from Eq. (45), the statistics is super-Poissonian $F(\tau) > 1$ at any time. The most simple form the time dependence for the Fano factor takes in the case of weak dissipation

$$F(\tau) = 1 + 2\langle n_d \rangle\Gamma\tau, \quad \Gamma\tau \ll 1. \tag{46}$$

Note that the vacuum term does not appear in Eq. (46), which is a characteristic of the weak dissipation limit.

Thus, the statistics is super-Poissonian for any $\Gamma \neq 0$ and is independent on the degree of nonlinearity l. This is in good agreement with the previous result of [2,6] for the case of the dissipative Kerr oscillator ($l = 1$), where from the exact solution the impossibility of sub-Poissonian statistics and antibunching has been predicted.

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[3] Note that this expression coincides with the corresponding result obtained within GGA (see formulas (3.153) and (10.42) in [1] up to the terms of order of $1/N^2$).
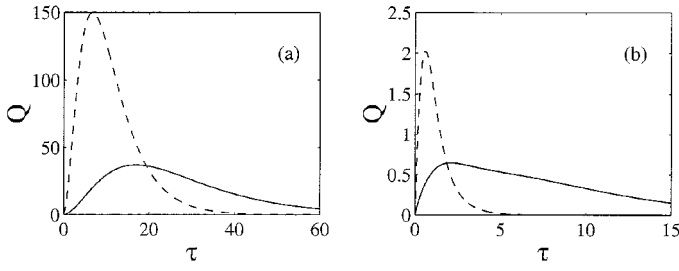
Fig. 2. The dependence of the quantum correction Q on the scaled time τ for weak damping $\Gamma = 0.05$ (a), and for strong damping $\Gamma = 0.5$ (b). Solid curve corresponds to the Kerr nonlinearity with $l = 1$ and dashed curve – to nonlinearity with $l = 3$. Everywhere $\langle n_d \rangle = 0$, $\Delta = 0$, $z_0 = 1$.
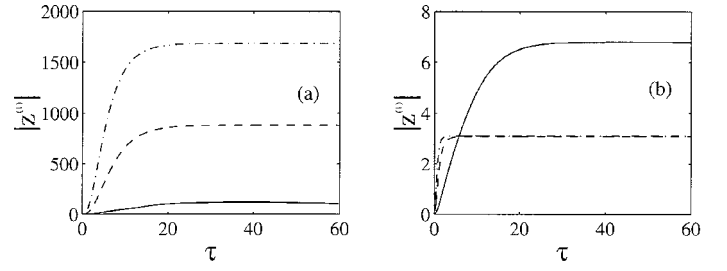
Fig. 3. The scaled difference between the quantum and the classical expectation values $|z^{(1)}| = N|z - z_{cl}|$ plotted as a function of time τ for weak damping $\Gamma = 0.05$ (a), and strong damping $\Gamma = 0.5$ (b). The Kerr nonlinearity with $l = 1$ is described by the solid curve, dashed curve corresponds to the nonlinearity with $l = 3$, and dashed and dotted curve corresponds to the nonlinearity with $l = 5$. Other parameters and initial condition are the same as in Fig. 2.

4. The conditions of validity of the $1/N$-expansion and the quantum-classical correspondence

The procedure of the $1/N$-expansion may be considered self-consistently, if the influence of the quantum correction to the classical motion on the dynamics of the mean values is small, *i.e.* condition (30) is satisfied. From Eqs. (17) and (31), we have the following expression for the quantum correction Q in the case of an arbitrary nonlinearity l

$$Q(z, z^*) = \frac{1}{2}l(l+1)z^{*l}z^{l-1}C + \frac{1}{2}l(l-1)z^{*(l-2)}z^{l+1}C^* \\ + l(l+1)z^{*(l-1)}z^l(B - 1/2). \tag{47}$$

We start the consideration of the condition of validity (32) with some particular cases.

4.1. The short-time approximation

First, we consider the short-time approximation. In this case, the integral over $Q(\tau)$ in (30) could be replaced by the product and we have

$$\left| \frac{z(\tau) - z_{cl}(\tau)}{z_{cl}(\tau)} \right| = \frac{\tau}{N} \left| \frac{Q(\tau)}{z_{cl}(\tau)} \right| \ll 1. \tag{48}$$

It is easy to see that for $\tau \ll 1$, $Q(\tau) \simeq Q(0)$ and thus $|(z - z_{cl})/z_{cl}|$ is of the order of $1/N$. Therefore, the condition of validity within the short-time approximation (48) is fulfilled for any number of photons $N \gg 1$ large enough.

4.2. The lossless case $\Gamma = 0$

In Appendix II we show that the time interval of validity of our approach for the lossless case $\Gamma = 0$ is

$$\tau \ll \tau^*_{ham} = N^{1/2} \frac{|\bar{\Delta} + |z_0|^{2l}|^{1/2}}{l^{3/2}|z_0|^{3l-1}}. \tag{49}$$

This result is in good agreement with our previous finding [15] that for the Hamiltonian integrable models the time scale for the validity of the $1/N$-expansion has a power-wise dependence in N in the semiclassical limit $N \gg 1$. Now we turn to the general case $\Gamma \neq 0$.

4.3. The case of an arbitrary dissipation

The condition of validity (30) cannot be obtained analytically for an arbitrary time and for an arbitrary damping Γ, but combining the computations and simple analytic esti-

mates for some limiting case, we can qualitatively understand the behavior of $R(\tau)$ [Eq. (30)]. We start with the time dependence of the quantum correction $Q(\tau)$ [Eqs. (28), (47), (32), (33)]. This function is shown in Fig. 2 for the different degrees of nonlinearity (the solid line for $l = 1$ and the dashed line for $l = 3$), as well as for a rather week (Fig. 2a) and for a relatively strong (Fig. 2b) damping constants. The quantum correction $Q(\tau)$ first increases, then reaches some maximum and finally decreases. The maximum of Q increases with the increase of nonlinearity degree (compare the solid and the dashed curves in Fig. 2) and shifts to a shorter time with the increase of dissipation (compare Fig. 2a and Fig. 2b).

Such behavior is in sharp contrast to the lossless (Hamiltonian) nonlinear systems, where the quantum correction always increases with time either power-wise for regular dynamics [22] or exponentially for chaotic dynamics [22]. Before $Q(\tau)$ gets to a maximum, during some time interval $[0, \tau_1]$ it could be well approximated by the time dependence for the lossless case $Q(\tau < \tau_1) \simeq l^3\tau^2$ (see formula (67)). If we estimate the time scale τ_1 from the behavior of the fastest decreasing function $\mu_l(\tau)$ [Eq. (32)], we have

$$\tau_1 = (\Gamma l)^{-1}, \quad Q(\tau_1) \simeq l^3\tau_1^2 = l/\Gamma^2 \tag{50}$$

for the position and the value of maximum of $Q(\tau)$. As is easy to see from Fig. 2, this naive estimate, however, well represents all main features of the time behavior of $Q(\tau)$.

Consider now the time dependence of the difference between the classical and the quantum mean values $z^{(1)}(\tau) = N|z(\tau) - z_{cl}(\tau)|$ caused by the existence of a quantum correction to the classical motion. Computing the integral over $Q(\tau)$ [Eq. (29)], we plot the time dependence $z^{(1)}(\tau)$ in Fig. 3 for weak (Fig. 3a) and for strong (Fig. 3b) damping, as well as for the different degrees of nonlinearity. This plot shows that the difference $z^{(1)}(\tau)$ first increases and then saturates at some level determined by the degree of nonlinearity and dissipation.

We start our analysis with the case of weak dissipation (Fig. 3a). For weak dissipation, the level of saturation of $z^{(1)}(\tau)$ increases with consequent consideration of higher nonlinearities. This result may be qualitatively understood as follows. For weak dissipation, the time interval of almost lossless behavior, $\tau \lesssim \tau_1$ (50), is rather long and therefore the asymptotic behavior of $z^{(1)}(\tau)$ sufficiently depends on

its behavior during the time $0 \leq \tau \lesssim \tau_1$. For $\tau \lesssim \tau_1$, we can use the approximation (68), i.e. $|z^{(1)}(\tau_1)| \simeq \tau_1^2 l^3 = l/\Gamma^2$. This estimate shows that the level of saturation for $z^{(1)}(\tau)$ should increase with the growth of l.

Turn to the case of strong dissipation (Fig. 3b). For strong dissipation, the level of saturation of $z^{(1)}(\tau)$ is sufficiently lower in comparison with the case of weak dissipation (compare Fig. 3a and Fig. 3b). Moreover, the level of saturation for $l = 1$ is slightly higher than for nonlinearities with $l > 1$ and the difference of the saturation levels for the different nonlinearities with $l > 1$ is practically invisible. To understand such behavior we note that for a strong enough dissipation the time interval for almost lossless dynamics, $\tau \lesssim \tau_1$, is short and the behavior of $Q(\tau)$ for another time interval $[\tau_1, \infty]$ is more sufficient (Fig. 2b). For the nonlinearity with $l = 1$, $Q(\tau)$ has a slowly decreasing tail, while it decays rapidly for $l > 1$ (Fig. 2b). As a result, the area under the curve $Q(\tau)$ for $l = 1$ is greater than the areas under the curves corresponding to different l with $l > 1$. Yet, because all $Q(\tau)$ are fastly decreasing functions at $\tau > \tau_1$ for all $l > 1$, the levels of saturation of $z^{(1)}(\tau)$ corresponding to different l are indistinguishable in the scale of Fig. 3.

Till now we studied the time dependence of the difference between quantum mean value and classical solution $|z(\tau) - z_{cl}(\tau)|$. However, the criterion of validity of the $1/N$-expansion in the form (32) includes also the $z_{cl}(\tau)$ as $R(\tau) = N^{-1}|z^{(1)}/z_{cl}|$. During the time interval of order of τ_1, when $z^{(1)}(\tau)$ is growing, z_{cl} is a slowly decreasing and oscillating function. But when $z^{(1)}(\tau)$ saturates, simultaneously the classical solution should be considered as a rapidly decreasing function $z_{cl} \simeq \exp(-\Gamma\tau/2)$. Therefore, for the plateaus in Fig. 3, $R(\tau) \simeq N^{-1}\exp(\Gamma\tau/2)$ resulting in the corresponding time scale for validity of our approach on a plateau as

$$\tau < \tau^* = \Gamma^{-1} \ln N. \tag{54}$$

Formally, this result looks like the corresponding estimate for the time scale for validity of the semiclassical description ("breaking time") in the quantum chaotic systems $\tau^* = \lambda^{-1} \ln N$, where λ is the maximal Lyapunov exponent [13–15,22,23]. However, the physical reason for the appearance of a very short, logarithmic breaking time is quite different in our case. When a classical oscillator spiraled around an equilibrium state, the amplitude of the oscillations becomes very small and the quantum description starts to differ significantly from the classical one due to the uncertainty principle. This is the reason for the logarithmically short time scale of the semiclassical description of strongly damped oscillations.

However, the behavior of an oscillator near an equilibrium point is a physically not very interesting process. On the opposite, the physically interesting case when the oscillations are not overdamped, we can again apply the estimate (63) during the time interval $\tau \simeq \tau_1 = (\Gamma l)^{-1}$ and get the criterion of validity in the form

$$R(\tau) = \frac{1}{N}\left|\frac{z^{(1)}(\tau_1)}{z_{cl}(\tau_1)}\right| \simeq \frac{l}{N\Gamma^2} \ll 1. \tag{52}$$

For large enough N, the criterion (52) could always be satisfied and our approach works well.

We can summarize our findings concerning the quantum-classical correspondence and the conditions for validity of the $1/N$-expansion as follows. In a lossless case, the quantum correction to the classical motion equation grows quadratically with time. For finite dissipation, the correction $Q(\tau)$ first increases and then decreases with time. The difference between the solution for the quantum expectation value and the classical solution $|z(\tau) - z_{cl}(\tau)|$ first increases power-wise with time and then saturates at some level, which is dependent on the degree of nonlinearity and on the dissipation strength. The condition for validity has different scaling dependences on the semiclassical parameter N for two different time intervals. During the time interval of growth of $|z(\tau) - z_{cl}(\tau)|$, the criterion of validity has the form (55), i.e. $R \simeq 1/N \ll 1$, and it is certainly satisfied for large N. During the time interval when $|z(\tau) - z_{cl}(\tau)|$ saturates, our approach is valid only for a short time $\tau^* \simeq \ln N$. We believe that the described above conditions for the semiclassical description of the quantum system are common for all dissipative systems with simple attractors.

5. Conclusion

In summary, using the quantum-classical correspondence, we find analytically the time dependencies of squeezing and the Fano factor for the dissipative anharmonic oscillators of an arbitrary degree of nonlinearity in the limit of large number of fotons. It should be noticed that our basic equations are rather general and could be considered as some generalization of the Gaussian wavepacket dynamics methods to the dissipative systems. Moreover, some effects mentioned in the present work are also rather general. For instance, quantum diffusion around a classical trajectory in the dissipative system, which we interpret as the influence of the reservoir's vacuum on the quantum system, should be observable not only in the dissipative nonlinear oscillators. In this respect, we would like to mention the work of Savage published ten years ago [35]. The author of [35] studied numerically, within the Gaussian approximation, the quantized version of the second harmonic generation problem for the self-oscillating regime [36]. He had found numerically a diffusive growth of the quantum variances near a classical limit cycle. However, neither an analytic treatment of the problem, nor a physical explanation of this "diffusion" were presented in [35]. We plan to devote our future publication to the investigation of this problem together with more accurate study of the validity conditions for the application of the $1/N$-expansion to dissipative quantum systems.

Appendix I

In this Appendix we show that the equations of motion for the cumulants (20) at $\epsilon = 0$ could be obtained from the classical motion equation (24). This proof is valid for any dissipative system with one degree of freedom.

Linearization of classical equations (24) near $z_{\rm cl}$ by means of the substitution $z_{\rm cl} \to z_{\rm cl} + \delta z$ gives

$$
\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\delta z = -\mathrm{i}\frac{\Gamma}{2}\delta z + \frac{\partial V}{\partial z}\delta z + \frac{\partial V}{\partial z^*}\delta z^*,\\
&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\delta z^* = -\mathrm{i}\frac{\Gamma}{2}\delta z^* - \frac{\partial V^*}{\partial z}\delta z - \frac{\partial V^*}{\partial z^*}\delta z^*,
\end{aligned}
\tag{53}
$$

where all derivatives are taken on the classical trajectory $z_{\rm cl}(\tau)$. Using the equality $(\partial V/\partial z) = (\partial V^*/\partial z^*)$, we have from (56) for the quadratic variables $(\delta z)^2$ and $|\delta z|^2$ the following equations of motion

$$
\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta z)^2 = -\mathrm{i}\Gamma(\delta z)^2 + 2\frac{\partial V}{\partial z}(\delta z)^2 + 2\frac{\partial V}{\partial z^*}|\delta z|^2,\\
&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}|\delta z|^2 = -\mathrm{i}\Gamma|\delta z|^2 - \frac{\partial V^*}{\partial z}(\delta z)^2 + \frac{\partial V}{\partial z^*}(\delta z^*)^2.
\end{aligned}
\tag{54}
$$

It is easy to see that Eqs. (54) are equivalent to Eqs. (20) with $\epsilon = 0$, if one makes substitutions $|\delta z|^2 \to B$ and $(\delta z)^2 \to C$.

It should be noticed that yet exists one difference between the linearization of the classical motion equations and the equations for quantum cumulants (15), (16): It is impossible to get the initial conditions (18) for C and B from only initial conditions for the linearized classical equations of motion (see also discussion of this problem in [14, 15]).

Appendix II

In this Appendix, we present the derivation of the time scale for the validity of the $1/N$-method for lossless $(\Gamma = 0)$ oscillators. Start with the case of Kerr nonlinearity $l = 1$ and then generalize obtained results for arbitrary l. From Eqs. (32) and (33) for $\Gamma = 0$ and $l = 1$, we get

$$
z_{\rm cl}(\tau) = z_0 \exp(-\mathrm{i}\Omega\tau), \quad \Omega \equiv \bar{\Delta} + |z_0|^2,
\tag{55}
$$

$$
\begin{aligned}
C(\tau) &= -\tau z_0^2\big[|z_0|^2\tau + \mathrm{i}\big]\exp(-\mathrm{i}2\Omega\tau),\\
B(\tau) &= 1/2 + |z_0|^4\tau^2.
\end{aligned}
\tag{56}
$$

The expression (47) for quantum correction in this case is

$$
Q(z, z^*, C, C^*, B) = z^*C + 2zB - z.
\tag{57}
$$

Substituting (55) and (56) into (57), we have for defined in Eq. (28) function $Q(\tau)$:

$$
Q(\tau) = z_0|z_0|^4\tau^2\exp(-\mathrm{i}\Omega\tau) - \mathrm{i}z_0|z_0|^2\tau\exp(-\mathrm{i}\Omega\tau),
\tag{58}
$$

and involved in (30) integral is

$$
\begin{aligned}
\int_0^\tau \mathrm{d}\tau' Q(\tau') = {}&z_0|z_0|^4\Omega^{-3}\big[2\Omega\tau\exp(-\mathrm{i}\Omega\tau) + \mathrm{i}(\Omega^2\tau^2 - 2)\\
&\exp(-\mathrm{i}\Omega\tau) + 2\mathrm{i}\big] - \mathrm{i}z_0|z_0|^2\Omega^{-2}[\exp(-\mathrm{i}\Omega\tau)\\
&+ \mathrm{i}\Omega\tau\exp(-\mathrm{i}\Omega\tau) - 1].
\end{aligned}
\tag{59}
$$

The most rapidly increasing term in (59) for $\tau \gg 1$ is $\Omega^{-1}z_0|z_0|^4\tau^2\exp(-\mathrm{i}\Omega\tau)$. Substituting this expression into Eq. (30), we have condition of validity in the form

$$
\frac{|z_0|^4}{|\bar{\Delta} + |z_0|^2|}\frac{\tau^2}{N} \ll 1,
\tag{60}
$$

and therefore the time scale τ^* of validity is

$$
\tau \ll \tau^* = N^{1/2}\frac{|\bar{\Delta} + |z_0|^2|^{1/2}}{|z_0|^2}.
\tag{61}
$$

Turn to the lossless case for an arbitrary l. Analogously to the case $l = 1$, it could be shown that the expression for quantum correction (28) in the limit of large time $\tau \gg 1$ is

$$
Q(\tau) \approx z_0|z_0|^{2(3l-1)}l^3\tau^2\exp[-\mathrm{i}\Omega_l\tau], \quad \Omega_l \equiv \bar{\Delta} + |z_0|^{2l},
\tag{62}
$$

and the influence of quantum correction on mean value (29) is

$$
z^{(1)}(\tau) = \int_0^\tau \mathrm{d}\tau' Q(\tau') \approx \frac{l^3 z_0|z_0|^{2(3l-1)}\exp(-\mathrm{i}\Omega_l\tau)}{\Omega_l}\tau^2,
\tag{63}
$$

where we again presented only the term which is leading in time for $\tau \gg 1$. Finally, substituting (63) to (30) with account of (32), we obtain (49).

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