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Magnetic monopoles with generalized quantization condition

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Abstract

Theory of pointlike magnetic monopole with an arbitrary magnetic charge is considered. It is shown that a proper description requires making use of nonunitary representations of the rotation group and the nonassociative generalization of the gauge group and fibre bundle theory.

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1. Introduction

In his remarkable paper Dirac [1] showed that a proper description of the quantum mechanics of a charged particle of the charge e in the field of the magnetic monopole of the charge q requires the quantization condition $2\mu \in \mathbb{Z}$ (we set $\mu = eq$ and $\hbar = c = 1$). There are strong mathematical and physical arguments why this condition must be fulfilled [1–10]. For instance, it restores associativity of the translation group for the charge-monopole system, ensures the absence of an Aharanov–Bohm effect produced by a Dirac string, arises as natural condition of the description pointlike Abelian magnetic monopole in the framework of fibre bundle theory. Finally, Dirac's quantization condition can be derived employing the unitary representation of the rotation group.

In our Letter we show that there exists the consistent theory of the magnetic monopole with an arbitrary magnetic charge. It requires nonunitary representations of the rotation group and nonassociative generalization of gauge transformations and fibre bundles theory, where a gauge group is replaced by gauge loop.

2. Preliminaries

A magnetic field of the monopole is

$$\mathbf{B} = q\frac{\mathbf{r}}{r^3}, \tag{1}$$

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and as well-known any choice of the vector potential $\mathbf{A}$ being compatible with Eq. (1) must have singularities. For instance, Dirac introduced the vector potential as

$$\mathbf{A}_{\mathbf{n}} = q\frac{\mathbf{r}\times\mathbf{n}}{r(r-\mathbf{n}\cdot\mathbf{r})}, \tag{2}$$

where the unit vector $\mathbf{n}$ determines the direction of a string $S_{\mathbf{n}}$ passing from the origin of coordinates to ∞ [1]. Schwinger's choice is

$$\mathbf{A}^{\mathrm{SW}} = \frac{1}{2}(\mathbf{A}_{\mathbf{n}} + \mathbf{A}_{-\mathbf{n}}) = q\frac{(\mathbf{n}\cdot\mathbf{r})\mathbf{r}\times\mathbf{n}}{r(r^2-(\mathbf{n}\cdot\mathbf{r})^2)}, \tag{3}$$

and the string is propagated from $-\infty$ to ∞ [2].

It is easy verify that

$$\operatorname{rot}\mathbf{A}_{\mathbf{n}} = \mathbf{B} - \mathbf{h}_{\mathbf{n}}, \qquad \operatorname{rot}\mathbf{A}^{\mathrm{SW}} = \mathbf{B} - \mathbf{h}^{\mathrm{SW}},$$

where

$$\mathbf{h}_{\mathbf{n}} = 4\pi q\mathbf{n}\int_0^\infty \delta^3(\mathbf{r}-\mathbf{n}\tau)\,d\tau, \tag{4}$$

$$\mathbf{h}^{\mathrm{SW}} = 2\pi q\mathbf{n}\int_{-\infty}^\infty \delta^3(\mathbf{r}-\mathbf{n}\tau)\,d\tau, \tag{5}$$

determine the magnetic field of the respective strings. Both vector potentials yield the same magnetic monopole field, however the quantization is different, while the Dirac condition is $2\mu = p$, the Schwinger one is $\mu = p$, $p\in\mathbb{Z}$.

These two strings are members of a family $\{\mathbf{S}_{\mathbf{n}}^{\kappa}\}$ with the magnetic field given by

$$\mathbf{h}_{\mathbf{n}}^{\kappa} = \kappa\mathbf{h}_{\mathbf{n}} + (1-\kappa)\mathbf{h}_{-\mathbf{n}}, \tag{6}$$

where κ is a weight of a semi-infinite Dirac's string. Further we call $\mathbf{S}_{\mathbf{n}}^{\kappa}$ a *weighted string*.

For a non relativistic charged particle in the field of a magnetic monopole the equations of motion

$$\ddot{\mathbf{r}} = \frac{\mu}{r^3}\mathbf{r}\times\dot{\mathbf{r}} \tag{7}$$

imply that the total angular momentum

$$\mathbf{J} = \mathbf{r}\times(\mathbf{p}-e\mathbf{A}) - \mu\frac{\mathbf{r}}{r} \tag{8}$$

is conserved. The last term in Eq. (8) usually is interpreted as the contribution of the electromagnetic field, which carries an angular momentum [11–13]

$$\mathbf{L}_{\mathrm{em}} = \frac{1}{4\pi}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})\,d^3r = -\mu\frac{\mathbf{r}}{r}.$$

The operator

$$\mathbf{J} = \mathbf{r}\times(-i\nabla - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r}, \tag{9}$$

representing the angular momentum $\mathbf{J}$, has the same properties as a standard angular momentum and obeys the following commutation relations:

$$\left[H,\mathbf{J}^2\right] = 0, \qquad [H, J_i] = 0, \qquad \left[\mathbf{J}^2, J_i\right] = 0, \tag{10}$$

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \tag{11}$$

where H is the Hamiltonian. Notice that the commutation relations fail on the string, however, H and $\mathbf{J}$ may be extended to self-adjoint operators satisfying the commutation relations of Eqs. (10), (11) for any value of μ [14–16].

Now following [3,4], let us cover the two-dimensional sphere S^2 of fixed radius $r>0$ by two neighborhoods $0\leqslant\theta<\pi/2+\varepsilon$ and $\pi/2-\varepsilon\leqslant\theta<\pi$. The vector potential is taken to be

$$\mathbf{A}_N = q\frac{1-\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi, \qquad \mathbf{A}_S = -q\frac{1+\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi, \tag{12}$$

where (r,θ,φ) are the spherical coordinates. Notice that $\mathbf{A}_{N,S}$ have singularities on (S,N) pole of the sphere and in the overlap of the neighborhoods $\mathbf{A}_N$ and $\mathbf{A}_S$ are related by a gauge transformation.

Choosing the vector potential as $\mathbf{A}_N$ we have

$$J_\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\mu\sin\theta}{1+\cos\theta}\right), \tag{13}$$

$$J_0 = -i\frac{\partial}{\partial\varphi} - \mu, \tag{14}$$

$$\mathbf{J}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + i\frac{2\mu}{1+\cos\theta}\frac{\partial}{\partial\varphi} + \mu^2\frac{1-\cos\theta}{1+\cos\theta} + \mu^2 \tag{15}$$

where $J_\pm = J_x \pm iJ_y$ are the raising and the lowering operators for $J_0 = J_z$.

Schrödinger's equation written in the spherical coordinates as

$$\left(-\frac{1}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{(\mathbf{J}^2-\mu^2)}{2mr^2}\right)\Psi = E\Psi, \tag{16}$$

admits the separation of variables and, putting $\Psi = R(r)Y(\theta,\varphi)$ into Eq. (16), we get

$$\left(-\frac{1}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{l(l+1)-\mu^2}{2mr^2}\right)R(r)$$

$$= ER(r),$$
$$\mathbf{J}^2 Y(\theta,\varphi) = l(l+1)Y(\theta,\varphi). \tag{17}$$

Starting from $J_0 Y_\mu = mY_\mu$ and writing

$$Y_\mu = e^{i\alpha\varphi} z^{\alpha/2}(1-z)^{\beta/2} F,$$
$$\alpha = m+\mu,\ \beta = m-\mu,$$

where $z = (1-\cos\theta)/2$, we obtain the resultant equation in the standard form of the hypergeometric equation,

$$z(1-z)\frac{d^2F}{dz^2} + \big(c-(a+b+1)z\big)\frac{dF}{dz} - abF = 0, \tag{18}$$

where

$$c = m+\mu+1, \qquad a+b = 2m+1,$$
$$ab = (m-l)(l+m+1). \tag{19}$$

The hypergeometric function $F(a,b;c;z)$ diverges when $\Re(c-b-a) \leqslant -1$, and it reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0,1,2,\ldots)$. For a being negative integer we find that the corresponding solution of Eq. (18) is of the form [17,18]

$$F = z^\delta (1-z)^\gamma p_n(z), \tag{20}$$

where $p_n(z)$ is a polynomial in z of degree n.

Here we are looking for the regular solutions, like (20), of the Schrödinger equation (17). The requirement of the wave function being single valued force us to take $m+\mu$ as an integer. The respective regular solution is given by

$$Y_\mu = C_{lm\mu} e^{i(m+\mu)\varphi} z^{\alpha/2}(1-z)^{\beta/2} F(a,b;c;z),$$
$$\alpha = m+\mu, \quad \beta = m-\mu, \quad c = m+\mu+1, \tag{21}$$

where $C_{lm\mu}$ is the normalization and for the parameters a and b we have:

$$a = -n, \qquad b = n+\alpha+\beta+1, \quad \text{if } \alpha = 0,1,2,\ldots,$$
$$a = n+1, \qquad b = -n-\alpha-\beta, \quad \text{if } \alpha = -1,-2,\ldots.$$

It follows that F reduces to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ so that Y_μ takes the form (compare with [4,11])

$$Y_l^{(\mu,n)} = C_{ln\mu} e^{i\alpha\varphi}(1-u)^{|\alpha|/2}(1+u)^{|\beta|/2} P_n^{(|\alpha|,|\beta|)}(u),$$

$\alpha = l+\mu-n$, $\beta = l-\mu-n$ and $l = m+n$. Since $m+\mu$ is an integer we conclude that $l+\mu$ must be an integer too.

The function $Y_l^{(\mu,n)}$ is a member of a family $\{Y_{\kappa,l}^{(\mu,n)}\}$ such that

$$Y_{\kappa,l}^{(\mu,n)} = e^{-i2\kappa\mu\varphi} Y_l^{(\mu,n)} \tag{22}$$

is a solution of the Schrödinger equation corresponding to the vector potential

$$\mathbf{A}^\kappa = \kappa \mathbf{A}_S + (1-\kappa)\mathbf{A}_N.$$

The requirement $Y_{\kappa,l}^{(\mu,n)}$ being single valued yields $2\kappa\mu$ being integer. Thus, for a given μ a weight κ is quantizied parameter in units of μ.

The wave functions $Y_{\kappa,l}^{(\mu,n)}$ form a complete set of orthonormal solutions that implies any solution $\Psi(\theta,\varphi;\mu,\kappa)$ can be expanded as

$$\Psi = \sum_{ln} C_{ln} Y_{\kappa,l}^{(\mu,n)}, \qquad C_{ln} = \langle Y_{\kappa,l}^{(\mu,n)}|\Psi\rangle. \tag{23}$$

Similar consideration can be done for the vector potential $\mathbf{A}_S$. In this case $(l-\mu) \in \mathbb{Z}$ and the corresponding wave functions being $Y_{\kappa,l}^{(-\mu,n)} = Y_{1-\kappa,l}^{(\mu,n)}$ form a complete set of orthonormal solutions as well.

For $(l \pm \mu)$ and $2\kappa\mu$ all being integers we call the functions $Y_{\kappa,l}^{(\pm\mu,n)}$ *weighted monopole harmonics*. They are regular for the all allowed values of l, n and μ. When $n+\alpha$, $n+\beta$ and $n+\alpha+\beta$ all are integers $\geqslant 0$ and $\kappa = 0$ the weighted monopole harmonics are reduced to the *monopole harmonics* introduced by Wu and Yang [4], and the imposed here restrictions on the values of n, α and β yield the Dirac quantization condition.

3. Nonunitary representations of the rotation group and solution of Dirac's monopole problem

It is known that the unitary representations of the rotation group leads to Dirac's quantization condition, $2\mu \in \mathbb{Z}$ [14–16,21]. Thus, the unique way to avoid the Dirac's rule is to consider nonunitary representations. In what follows, assuming μ being arbitrary parameter, we are looking for nonunitary representations of the rotation group relating to an arbitrary magnetic charge [19][1].

For $l(l+1)$ being value of the Casimir operator

$$C = J_0^2 + \frac{1}{2}(J_- J_+ + J_+ J_-), \tag{24}$$

we denote the states by $|l,n\rangle$, $n = 0, 1, \ldots, \infty$. For the representations bounded below we obtain

$$J_+|l,n\rangle = \sqrt{(2l+n)(n+1)}\,|l,n+1\rangle, \tag{25}$$

$$J_-|l,n\rangle = -\sqrt{n(2l+n-1)}\,|l,n-1\rangle, \tag{26}$$

$$J_0|l,n\rangle = (l+n)|l,n\rangle. \tag{27}$$

The representation is characterized by the eigenvalue l of the highest-weight state: $|l,0\rangle$ such that $J_-|l,0\rangle = 0$ and $J_0|l,0\rangle = l|l,0\rangle$. Comparing Eq. (27) with $J_0 Y_l^{\mu,m} = m Y_l^{\mu,m}$ and remembering that $m + \mu \in \mathbb{Z}$ (see Section 2) we conclude that $l + \mu$ is an integer. Thus, the representation bounded below also can be characterized by $l + \mu$ being integer. Taking into account the restriction following from the Schrödinger equation: $l(l+1) - \mu^2 \geqslant 0$, we find that the allowed values of l are

$$l = |\mu| + \{-(\mu + |\mu|)\} + k, \quad k = 0, 1, 2, \ldots. \tag{28}$$

For the representation bounded above we have

$$J_+|l,n\rangle = -\sqrt{n(2l+n-1))}\,|l,n-1\rangle, \tag{29}$$

$$J_-|l,n\rangle = \sqrt{(n+1)(2l+n)}\,|l,n+1\rangle, \tag{30}$$

$$J_0|l,n\rangle = -(l+n)|l,n\rangle. \tag{31}$$

This representation is characterized by the eigenvalue $-l$ of the highest-weight state: $|l,0\rangle$ such that $J_+|l,0\rangle = 0$ and $J_0|l,0\rangle = -l|l,0\rangle$. We found that in this case $l - \mu$ is an integer and the allowed values of l are

$$l = |\mu| + \{\mu - |\mu|\} + k, \quad k = 0, 1, 2, \ldots. \tag{32}$$

The obtained representations can be realized in the space of holomorphic functions of a complex variable z. Following [20] we assign a "wave function" $\langle z|l,n\rangle$ by

$$(l+\mu) \Rightarrow \langle z|l,n\rangle = A z^n, \tag{33}$$

$$(l-\mu) \Rightarrow \langle z|l,n\rangle = A z^{-2l-n}, \tag{34}$$

where $A = \sqrt{\Gamma(2l+n)/\Gamma(n+1)\Gamma(2l-1)}$ is a normalization, Γ being the Gamma function. The monomials (33) and (34) form the basis for the analytic functions in the unit disc D: $|z| \leqslant 1$ and in $\widetilde{D}$: $|z| \geqslant 1$, respectively.

The Lie algebra is realized by the differential operators:

$$J_+ = z^2\partial_z + 2lz, \qquad J_- = -\partial_z, \qquad J_0 = z\partial_z + l, \tag{35}$$

$$[J_+, J_-] = 2J_0, \qquad [J_0, J_\pm] = \pm J_\pm, \tag{36}$$

and an arbitrary state of the representation is of the form

$$f(z) = \sum_{n=0}^{\infty} f_n \langle z|l,n\rangle. \tag{37}$$

The inner product of two holomorphic functions is defined as follows:

$$(l+\mu) \Rightarrow \langle f|g\rangle = \frac{1}{2\pi i}\int_D d\bar{z}\,dz\,\frac{\bar{f}g}{(1-|z|^2)^{2-2l}}, \tag{38}$$

$$(l-\mu) \Rightarrow \langle f|g\rangle = \frac{1}{2\pi i}\int_{\widetilde{D}} d\bar{z}\,dz\,\frac{\bar{f}g}{(|z|^2-1)^{2-2l}}. \tag{39}$$

With the introduced inner product the group representation is infinite dimensional, irreducible and nonunitary.

Finite-dimensional representation arises when l takes the exceptional values $2l = p$ with p being positive integer. In this case the representation is unitary and bounded from above and below. One has the standard selectional rules: $l = |\mu| + k$, $k = 0, 1, 2, \ldots$, $m = -l, \ldots, l$, and the Dirac quantization condition holds [4].

Returning to the eigenvalues equations

$$\mathbf{J}^2 Y(z) = l(l+1)Y(z), \tag{40}$$

$$J_0 Y(z) = \pm(l+n)Y(z) \tag{41}$$

we see that their solutions given by eigenfunctions $Y_l^{(\mu,n)}(z)$ of Eqs. (33), (34) satisfy the Schrödinger equation (17). Introducing the wave function as follows: $\Psi(r,z) = R(r)Y(z)$, where $Y(z)$ is a holomorphic function:

$$Y(z) = \sum_{n=0}^{\infty} f_n \langle z|l,n\rangle \tag{42}$$

[1] We follow the ideas of [20] where the description of infinite dimensional unitary representations has been done for the group SU(1,1)

we obtain the solution of the monopole problem inside of the unit disc and for an arbitrary monopole charge.

For $D_\pm$ being unit disc we relate $z \in D_+$ to the points of the upper semi sphere Σ_+ via the stereographic projection from the south pole and $z \in D_-$ to the points of the lower semi-sphere Σ_- via the stereographic projection from the north pole. Covering the two-sphere S^2 as follows: $S^2 = D_+ \cup D_-$, we have the solution of the Schrödinger equation of the form $\Psi(r,\theta,\varphi) = \Psi_+ \cup \Psi_-$ for the whole sphere. In the intersection $D_+ \cap D_-$ the functions $\Psi_\pm$ must satisfy the relation: $\Psi_+ = \Psi_-$.

4. Gauge transformations and monopole charge quantization

Before proceeding let us note that with the representations $(l \pm \mu)$ are related two string families: $\{S^\kappa_{\mathbf{n}}\}$ and $\{S^{\tilde\kappa}_{-\mathbf{n}}\}$. Their respective vector potentials are

$$A^\kappa_{\mathbf{n}} = \kappa A_{\mathbf{n}} + (1-\kappa)A_{-\mathbf{n}}, \quad 2\kappa\mu \in \mathbb{Z}, \tag{43}$$

$$A^{\tilde\kappa}_{-\mathbf{n}} = \tilde\kappa A_{-\mathbf{n}} + (1-\tilde\kappa)A_{\mathbf{n}}, \quad 2\tilde\kappa\mu \in \mathbb{Z}, \tag{44}$$

and the change $S^\kappa_{\mathbf{n}} \to S^{\tilde\kappa}_{-\mathbf{n}}$ is given by the following gauge transformation:

$$A^{\tilde\kappa}_{-\mathbf{n}} = A^\kappa_{\mathbf{n}} - d\chi^\gamma_{\mathbf{n}}, \quad \tilde\kappa = 1-\kappa-\gamma, \tag{45}$$

$$d\chi^\gamma_{\mathbf{n}} = 2\gamma q \frac{(\mathbf{r}\times\mathbf{n})\cdot d\mathbf{r}}{r^2 - (\mathbf{n}\cdot\mathbf{r})^2}, \tag{46}$$

$\chi_{\mathbf{n}}$ being polar angle in the plane orthogonal to $\mathbf{n}$.

We start with an observation that due to the string quantization one has the equivalence relation: $2\kappa'\mu = 2\kappa\mu \bmod \mathbb{Z}$. Therefore, further we restrict ourselves by the gauge transformations, that do not change the weight of the string, $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$. It produces the transformation $(\mathbf{A}^\kappa_{\mathbf{n}}, \Psi^\kappa_{\mathbf{n}}) \to (\mathbf{A}^\kappa_{\mathbf{n}'}, \Psi^\kappa_{\mathbf{n}'})$ given by [2,24]

$$\Psi^\kappa_{\mathbf{n}'}(\mathbf{r}) = \exp\bigl(-ie\Phi^\kappa_{\mathbf{n},\mathbf{n}'}(\mathbf{r})\bigr)\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \tag{47}$$

where the function $\Phi^\kappa_{\mathbf{n},\mathbf{n}'}(\mathbf{r})$ satisfies

$$\mathbf{A}^\kappa_{\mathbf{n}}(\mathbf{r}) - \mathbf{A}^\kappa_{\mathbf{n}'}(\mathbf{r}) = \nabla\Phi^\kappa_{\mathbf{n},\mathbf{n}'}(\mathbf{r}). \tag{48}$$

Let denote by $\mathbf{n}' = g\mathbf{n}$, $g \in \mathrm{SO}(3)$, the left action of the rotation group induced by $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$. From rotational symmetry of the theory it follows immediately that an arbitrary gauge transformation $\Psi^\kappa_{\mathbf{n}} \to \Psi^\kappa_{\mathbf{n}'}$ can be undone by rotation $\mathbf{r} \to \mathbf{r}g$. Using this fact and adopting results of [4,5,22] we find that an arbitrary gauge transformation U_g, producing the rotation of the string $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$, is given by nonintegrable phase factor,

$$U_g\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) = \exp\bigl(i\alpha^\kappa_1(\mathbf{r},\mathbf{n};g)\bigr)\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \tag{49}$$

$$\alpha_1(\mathbf{r};g) = e\int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}^\kappa_{\mathbf{n}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \quad \mathbf{r}' = \mathbf{r}g, \tag{50}$$

where the integration is performed along the geodesic $\widehat{\mathbf{r}\mathbf{r}'} \subset S^2$ and α_1 is the so-called first cochain [5,6,23]. Actually, U_g is an operator of the parallel transport along the geodesics on the two-dimensional sphere of the fixed radius r.

For a given cochain α_1 a 2-cocycle α_2 is defined by

$$\begin{aligned}\alpha_2(\mathbf{r};g_1,g_2) &= \delta\alpha_1 = \alpha_1(\mathbf{r}g_1;g_2)\\ &\quad - \alpha_1(\mathbf{r};g_1g_2) + \alpha_1(\mathbf{r};g_1)\end{aligned} \tag{51}$$

which satisfies $\delta\alpha_2 = 0$, and, for α_2 being 2-cochain, a 3-cocycle $\alpha_3 = \delta\alpha_2$ is given by

$$\begin{aligned}\alpha_3(\mathbf{r};g_1,g_2,g_3) &= \alpha_2(\mathbf{r}g_1;g_2,g_3) - \alpha_2(\mathbf{r};g_1g_2,g_3)\\ &\quad + \alpha_2(\mathbf{r};g_1,g_2g_3) - \alpha_2(\mathbf{r};g_1,g_2).\end{aligned}$$

Similarly one can introduce n-cocycle $\alpha_n(\mathbf{r};g_1,g_2,\dots,g_n)$ [6,23].

Following [5,6] let us define a 2-cochain, α_2, by

$$\alpha_2(\mathbf{r};g_1,g_2) = e\int_\Sigma \mathbf{B}\, d\mathbf{s} = e\Phi|_\Sigma, \tag{52}$$

where $\Phi|_\Sigma$ is a magnetic flux through the geodesic triangle $\Sigma \subset S^2$ spanned by $(\mathbf{r}, \mathbf{r}g_1, \mathbf{r}g_1g_2)$. Since $\mathbf{B} = \nabla\times\mathbf{A}$ locally, but not globally then α_2 is a 2-cochain and not a 2-cocycle. Indeed, applying Stokes' theorem we get

$$\alpha_2(\mathbf{r};g_1,g_2) = \delta\alpha_1(\mathbf{r};g_1,g_2) + \sigma\bigl(S^\kappa_{\mathbf{n}},\Sigma\bigr), \tag{53}$$

where $\sigma = \int_\Sigma \mathbf{h}^\kappa_{\mathbf{n}}\cdot d\mathbf{s}$ being contribution of the string is not zero if and only if the string crosses Σ.

For computing σ let us divide R^3 into R^3_+ and R^3_- by the plane passing through the origin of coordinates and orthogonal to $\mathbf{n}$. Assuming that the string $S^\kappa_{\mathbf{n}}$ crosses Σ at a point p_0, we find

$$\sigma = \begin{cases} 4\pi(1-\kappa)\mu, & p_0 \in \Sigma\cap R^3_-,\\ 4\pi\kappa\mu, & p_0 \in \Sigma\cap R^3_+.\end{cases} \tag{54}$$

Since $2\kappa\mu$ is an integer, one has

$$\alpha_2 = \delta\alpha_1 + 4\pi\mu|_{p_0} \bmod 2\pi\mathbb{Z}. \tag{55}$$

Similar consideration of the gauge transformations $S^\kappa_{\mathbf{n}} \to S^\kappa_{-\mathbf{n}}$, related with the reflections, yields

$$\alpha_2 = \delta\alpha_1 + 4\pi(1-2\kappa)\mu = \delta\alpha_1 + 4\pi\mu \mod 2\pi\mathbb{Z}.$$

Examining the composition of two operators U_{g_1} and U_{g_2}, we find that 2-cochain α_2 occurs in its composition law as follows:

$$U_{g_1}U_{g_2}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) = \exp\bigl(i\alpha_2(\mathbf{r}; g_1, g_2)\bigr)U_{g_1g_2}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \quad (56)$$

where $g_1, g_2 \in \mathrm{SO}(3)$.

Consider now three elements $g_1, g_2, g_3 \in \mathrm{O}(3)$ producing the transformations $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}_1}$, $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}_2}$, $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}_3}$, respectively. Then the product of the three operators is given by

$$\begin{aligned} &U_{g_1}(U_{g_2}U_{g_3})\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) \\ &\quad = \exp\bigl(i\alpha_3(\mathbf{r}; g_1, g_2, g_3)\bigr)(U_{g_1}U_{g_2})U_{g_3}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \end{aligned}$$

where α_3 is a *three cocycle*.

From Eqs. (52) and (55) it follows $\alpha_3 = 4\pi\mu \mod 2\pi\mathbb{Z}$ if the monopole is enclosed by the geodesic simplex with vertices $(\mathbf{r}, \mathbf{r}g_1, \mathbf{r}g_1g_2, \mathbf{r}g_1g_2g_3)$ or zero otherwise.[2]

We turn now to Eq. (56) and rewrite the product of the two transformations as

$$U_{g_1}U_{g_2}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) = U_{\varphi(g_1,g_2;\mathbf{r})}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \quad (57)$$

where φ is defined by

$$\begin{aligned} \varphi(g_1, g_2; \mathbf{r}) &= \alpha^\kappa_1(\mathbf{r}; g_1g_2) + \alpha_2(\mathbf{r}; g_1, g_2) \\ &= \alpha_1(\mathbf{r}; g_1) + \alpha_1(\mathbf{r}; g_2) + \sigma(S^\kappa_{\mathbf{n}}, \Sigma). \end{aligned} \quad (58)$$

It is easy verify that the following identity of quasi-associativity holds:

$$\varphi\bigl(g_1, \varphi(g_2, g_3; \mathbf{r}); \mathbf{r}\bigr) = \varphi\bigl(\varphi(g_1, g_2; \mathbf{r}g_3), g_3; \mathbf{r}\bigr). \quad (59)$$

We say that Eqs. (57)–(59) define a *gauge loop*. This is a special case of transformation quasigroup introduced by Batalin [25] and a 3-cocycle, being a "measure" of nonassociativity, can be related with an associator in theory of quasigroups and loops [26–30].

[2] Similar analysis related with the modification of the translation group in the presence of the magnetic monopole has been done in [5] (see also [6–9]).

The gauge loop is associated also with the loop QU(1) defined as a loop of multiplication by unimodular complex numbers [26–28]:

$$\begin{aligned} &e^{i\alpha} * e^{i\beta} = e^{i\alpha*\beta}, \\ &\alpha * \beta = \alpha + \beta + F(\alpha, \beta), \\ &F(\alpha, 0) = F(0, \beta) = 0. \end{aligned} \quad (60)$$

Before proceeding notice that QU(1) is isomorphic to the group U(1) if

$$\begin{aligned} &F(\alpha, \beta) + F(\alpha * \beta, \gamma) - F(\beta, \gamma) \\ &\quad - F(\alpha, \beta * \gamma) = 0 \mod 2\pi\mathbb{Z}, \end{aligned} \quad (61)$$

that is a 2-cocycle condition $\delta\alpha_2 = 0 \mod 2\pi\mathbb{Z}$.

Assuming QU(1) to be a local loop we define a respective gauge loop over S^2 by

$$U_{\alpha(\mathbf{r})}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) = \exp\bigl(i\alpha(\mathbf{r})\bigr)\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \quad (62)$$

$$U_{\alpha(\mathbf{r})}U_{\beta(\mathbf{r})}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}) = U_{\alpha(\mathbf{r})*\beta(\mathbf{r})}\Psi^\kappa_{\mathbf{n}}(\mathbf{r}). \quad (63)$$

Here the operation $\alpha(\mathbf{r}) * \beta(\mathbf{r})$ is given by Eq. (60) with $F(\alpha, \beta; \mathbf{r})$ determined as follows: $F = \sigma(S^\kappa_{\mathbf{n}}, \Sigma)$ where the geodesic triangle $\Sigma \subset S^2$ is spanned by $(\mathbf{r}, \mathbf{r}g_\alpha, \mathbf{r}g_\beta)$, $g_\alpha, g_\beta \in \mathrm{SO}(3)$. For computing g_α we employ the rotational symmetry of the theory. This implies that for a given string $\mathbf{S}^\kappa_{\mathbf{n}}$ and gauge function $\alpha(\mathbf{r})$ the following equation holds:

$$\begin{aligned} &\Psi^\kappa_{\mathbf{n}}(\mathbf{r}') = \Psi^\kappa_{\mathbf{n}'}(\mathbf{r}) = \exp\bigl(i\alpha(\mathbf{r})\bigr)\Psi^\kappa_{\mathbf{n}}(\mathbf{r}), \\ &\mathbf{r}' = \mathbf{r}g_\alpha, \ \mathbf{n}' = g_\alpha\mathbf{n}, \quad g_\alpha \in \mathrm{SO}(3). \end{aligned} \quad (64)$$

It should be considered as the equation for finding g_α. Returning now to Eq. (63) we see that the local loop QU(1) becomes the gauge loop defined by Eqs. (57), (58).

5. Discussion and concluding remarks

We deduced a consistent pointlike monopole theory, with an arbitrary magnetic charge, involving nonunitary representations of the rotation group and making use of nonassociative QU(1) bundle over S^2, where QU(1) is the structure loop [26–28]. From our approach it follows a generalized quantization condition, $2\kappa\mu \in \mathbb{Z}$, that can be considered as quantization of the weight string instead of the monopole charge. In particular cases $\kappa = 1$ and $\kappa = 1/2$ it yields the Dirac and Schwinger selectional rules, respectively.

At first sight our results are in contradiction with well-known topological and geometrical arguments in behalf of Dirac quantization rule [2,3,10]. For the better understanding of the problem let us notice that known proofs are based on employing unitary finite-dimensional representations of the rotation group or classical fibre bundle theory. One can remove the effect of 3-cocycle imposing the Dirac quantization condition, however, this arises only from a realization of the monopole as U(1) bundle over S^2 [3,4,6]. This implies that there exists the division of space into overlapping regions $\{U_i\}$ such that nonsingular vector potential can be defined and yields the correct monopole magnetic field in each region. On each intersection $U_i \cap U_j$ can be defined the transition functions $q_{ij} = e\Phi_{\mathbf{n}_i\mathbf{n}_j}$ such that $U_i \cap U_j \to \mathrm{U}(1)$. On the triple overlap $U_i \cap U_j \cap U_k$ it holds

$$\exp\bigl(i(q_{ij} + q_{jk} + q_{ki})\bigr) = \exp(i4\pi\mu), \tag{65}$$

and the consistency condition requires $q_{ij} + q_{jk} + q_{ki} = 0 \mod 2\pi\mathbb{Z}$. This gives $2\mu \in \mathbb{Z}$ and the Dirac quantization condition appears again, now as a *necessary condition to have a consistent* U(1)*-bundle over* S^2. Notice that it is consequence of the dynamics and not of the representation theory [6].

While the Jacobi identity holds for the generators of the rotation group [14–16] the situation with the translations in the background of the monopole is quite different. The difference has a topological nature and arises from the non-trivial topology of the orbit space. In the case of the rotations, the orbit space is just a two-dimensional sphere S^2. For the translations the orbit space is three-dimensional space R^3 with one point removed and its non-trivial topology provides the non-vanishing three-cocycle [8]. Thus, the Jacobi identity fails for the gauge invariant algebra of translations and for the finite translations $\{U_{\mathbf{a}}\}$ one has [5,6]

$$(U_{\mathbf{a}}U_{\mathbf{b}})U_{\mathbf{c}}\Psi(\mathbf{r}) = \exp\bigl(i\alpha_3(\mathbf{r};\mathbf{a},\mathbf{b},\mathbf{c})\bigr)U_{\mathbf{a}}(U_{\mathbf{b}}U_{\mathbf{c}})\Psi(\mathbf{r}). \tag{66}$$

For the Dirac quantization condition being satisfied one has $\alpha_3 = 0 \mod 2\pi\mathbb{Z}$, and (66) provides an associative representation of the translations, in spite of the fact that the Jacobi identity continues to fail.

Since a conventional quantum mechanics deals with linear Hilbert space operators, the Dirac quantization rule is a necessary condition for the consistency of quantum mechanics in the presence of a monopole. Avoiding this condition forces us to go beyond the standard quantum mechanical approach and introduce a *nonassociative algebra of observables* [5–9]. Notice that in ordinary quantum mechanics the Schrödinger and Heisenberg pictures are equivalent, but the same is not true in a nonassociative quantum mechanics. Indeed, whilst the concept of the Hilbert space failed for nonassociative algebras, the Heisenberg approach could be still realized [31–33]. In a possible nonassociative quantum mechanics one must give up a conventional description of the quantum mechanics provided by Hilbert space concept and look for the generalization based on the Heisenberg approach and maybe only in terms of density matrix [9,32,33].

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