Effects of one- and three-dimensional inhomogeneities on the wave spectrum of multilayers with finite interface thicknesses

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To describe a partially randomized multilayer structure with arbitrary thicknesses of the interfaces between layers, we introduce a model in which the dependence of a material parameter along the axis of such a superlattice is described by a Jacobian elliptic sine function with a random spatial modulation of its period. Both one- and three-dimensional inhomogeneities of the period are considered. We develop the correlation function for this model, and investigate the dispersion law and damping of averaged waves in this superlattice. The dependencies of the widths of the gaps in the spectrum and the damping at the boundaries of all odd Brillouin zones, on the thicknesses of the interfaces, and on the dimensionality, intensity, and correlation wave number of the inhomogeneities are found. It is shown that experimental investigations of the widths of the gaps and damping for several Brillouin zones could permit, in principle, determining all parameters of the superlattice as well as the parameters of the inhomogeneities from these spectral characteristics.

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I. INTRODUCTION

Multilayered media—one-dimensional superlattices—are promising materials for use in various devices of high technology such as resonators, filters, mirrors, etc., for electromagnetic, spin, and elastic waves. However, real multilayers are not ideal periodic systems: they contain random violations of the periodicity caused by technological and other reasons. That is why investigations of the spectrum of waves in partially randomized superlattices have been carried out very intensively in recent years. Several methods exist now for developing a theory of such superlattices.

The modeling of the randomization by altering the order of successive layers of two different materials A and B (of different or the same thickness) is in wide use now. It is assumed that neither the parameters of the materials nor the layer thicknesses change when the system is randomized: only the periodicity $ABAB\ldots$ in the arrangement of the layers corresponding to the ideal superlattice is destroyed. The different versions of this method differ in the types of disruptions of the periodicity in the arrangement of the layers: in some versions the layers A and B are arranged according to the Fibonacci or Thue-Morse sequence rule; in others they form either partially correlated or totally uncorrelated random sequences. A number of important and interesting results have been derived with the help of this method in studies of the propagation of elastic,[1–3] spin[4–6] and electromagnetic[7] waves. In several papers the study of wave propagation in a superlattice was conducted in the framework of a method that consists in the numerical modeling of the random deviations of the interfaces from their initial periodic arrangement.[8–10] Another method was suggested independently in Refs. 11 and 12, where a form of the correlation function of a superlattice with inhomogeneities was postulated and then the wave spectrum and damping were calculated analytically. In Ref. 13 the propagation of the electromagnetic waves in disordered media was considered in the frameworks of geometrical optics. In Ref. 14 the dynamic composite elastic medium theory was suggested for calculating the wave spectrum in randomly layered one-dimensional media and media with three-dimensional inclusions.

One more method for investigating the influence of inhomogeneities on the wave spectrum of a superlattice was suggested in Ref. 15, the method of the random spatial modulation (RSM) of the period of the superlattice. This method is an extension of the well-known theory of the random frequency (phase) modulation of a radio signal[16,17] to the case of spatial inhomogeneities in the superlattice. Laws of the dispersion and damping of the averaged spin, elastic, and electromagnetic waves were determined by this method for two models of superlattices: superlattices with an initial sinusoidal dependence of material parameters on a coordinate,[15,18] and superlattices with a dependence in the form of rectangular spatial pulses.[19,20] These models correspond to the two limiting cases of the relation between the thickness of the interfaces d and the period l of the multilayer structure. For the second model (the model of the sharp interface) $d/l=0$. For the first model, which is the limiting case of smooth interfaces, the thickness of the ''layers'' and ''interfaces'' is the same; the ratio $d/l=1/4$ corresponds to this model. It must be emphasized that not only in our papers[15,18–20] but in practically all works carried out to date only these two models have been used in studies of the wave spectrum in ideal as well as in randomized superlattices.

However, in real multilayers the ratio d/l can have an arbitrary value between these limiting cases. To describe such multilayer structures we have introduced in Ref. 21 a model in which the dependence of a material parameter along the superlattice axis is described by a Jacobian elliptic function. Depending on the value of the modulus κ of the elliptic function, the model describes the limiting cases of multilayers with sharp interfaces ($\kappa=1, d/l=0$) and of sinu-

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soidal superlattices ($\kappa=0, d/l=1/4$), as well as all intermediate situations. We have also investigated the wave spectrum for this model in the absence of any inhomogeneities.

The aim of the present paper is to calculate the spectrum and damping of the averaged waves in a partially randomized superlattice by the method of RSM[15] using the model of a superlattice[21] with an arbitrary ratio d/l. In Sec. II we develop a correlation function for such a superlattice for one- and three-dimensional inhomogeneities. In Sec. III we develop the analytical equations for the spectrum and damping of the superlattice. In Sec. IV we calculate the width of the gaps in the spectrum and the damping at the boundaries of odd Brillouin zones due to one- and three-dimensional inhomogeneities, and discuss the results obtained.

II. CORRELATION FUNCTION

Any superlattice is characterized by the dependence of some material parameter A on coordinates $\mathbf{x}=\{x,y,z\}$. The physical nature of the parameter $A(\mathbf{x})$ can be different. This parameter can be a density of matter or a force constant for the elastic system of a medium, the value of the magnetization, anisotropy, or exchange for a magnetic system, and so on. We represent $A(\mathbf{x})$ in the form

$$A(\mathbf{x})=A[1+\gamma\rho(\mathbf{x})], \tag{1}$$

where A is the average value of the parameter, γ is its relative rms variation, $\rho(\mathbf{x})$ is a centered ($\langle\rho(\mathbf{x})\rangle=0$) and normalized ($\langle\rho(\mathbf{x})^2\rangle=1$) function. The function $\rho(\mathbf{x})$ describes the periodic dependence of the parameter along the superlattice axis z, as well as the random spatial modulation of this parameter which, in the general case, can be a function of all three coordinates $\mathbf{x}=\{x,y,z\}$. We represent this function in the form

$$\rho(\mathbf{x})=\kappa\left(\frac{\mathbf{K}}{\mathbf{K}-\mathbf{E}}\right)^{1/2}\mathrm{sn}\left[\frac{\pi}{2d}(z-u(\mathbf{x}))+\psi\right], \tag{2}$$

which has the form of the Jacobian elliptic sine function in the absence of disorder ($u\equiv 0$). Here $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and second kind, respectively, κ is the modulus of these integrals, and l is the period of the superlattice. The coefficient multiplying the elliptic function is the normalization constant, which follows from the condition $\langle\rho(\mathbf{x})^2\rangle=1$. The parameter $d=\pi l/8\mathbf{K}$ has been introduced in Ref. 21 as an effective thickness of the interfaces in the initial ideal superlattice; the numerical coefficient has been chosen so that $d/l=1/4$ for the limiting case of the sinusoidal superlattice; in so doing the main variation of the value of the parameter $A(\mathbf{x})$ occurs over the length d for all values of d/l. As in Refs. 19,20 for the superlattice with sharp interfaces we assume here that the function $\rho(\mathbf{x})$ can be represented in the form of a Fourier series even for $u(\mathbf{x})\neq 0$

$$\rho(\mathbf{x})\simeq\sqrt{2}\sum_{m=0}^{\infty}B_p\,\sin p[q(z-u(\mathbf{x}))+\psi]. \tag{3}$$

This can be done if the function $u(\mathbf{x})$ is smoother in all directions than the first harmonic of the Fourier series. Here $q=2\pi/l$, $p=2m+1$, and $\sqrt{2}B_p$ are the exact Fourier coefficients for the function $\rho(z)$ of the ideal superlattice,

$$B_p=\frac{\sqrt{2}\pi}{\sqrt{\mathbf{K}(\mathbf{K}-\mathbf{E})}}\frac{R^{p/2}}{1-R^p}, \tag{4}$$

where

$$R=\exp\left(-\frac{\pi\mathbf{K}'}{\mathbf{K}}\right),\quad \mathbf{K}'(\kappa)=\mathbf{K}(\kappa'),\quad \kappa'=\sqrt{1-\kappa^2}. \tag{5}$$

As in Refs. 15,19,20 we have introduced a coordinate independent random phase ψ, which is characterized by a uniform distribution in the interval $(-\pi,\pi)$. This permits us to satisfy the condition of ergodicity even in the case where $u\equiv 0$.

The product of the function $\rho(\mathbf{x})$ and $\rho(\mathbf{x}+\mathbf{r})$ can be represented in the form

$$\begin{aligned}\rho(\mathbf{x}+\mathbf{r})\rho(\mathbf{x})&\\ =\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}&B_pB_{p'}\{\cos q[pr_z-p'u(\mathbf{x})+pu(\mathbf{x}+\mathbf{r})\\ &+(p-p')(z+\psi/q)]+\cos q[pr_z-p'u(\mathbf{x})\\ &-pu(\mathbf{x}+\mathbf{r})+(p+p')(z+\psi/q)]\},\end{aligned} \tag{6}$$

where $p'=2m'+1$. The second summand vanishes after averaging over the phase ψ. The terms with $p'\neq p$ in the first summand vanish as well, and after this averaging we have

$$\langle\rho(\mathbf{x}+\mathbf{r})\rho(\mathbf{x})\rangle_\psi=\sum_{m=0}^{\infty}B_p^2\cos p(qr_z+\chi), \tag{7}$$

where

$$\chi(\mathbf{x},\mathbf{r})=q[u(\mathbf{x}+\mathbf{r})-u(\mathbf{x})]. \tag{8}$$

Averaging Eq. (7) over χ with a Gaussian distribution function for χ, we obtain a general expression for the correlation function in the form

$$K(\mathbf{r})=\sum_{m=0}^{\infty}B_p^2\cos pqr_z\exp\left[-\frac{p^2}{2}Q(\mathbf{r})\right], \tag{9}$$

where

$$Q(\mathbf{r})=q^2\langle[u(\mathbf{x}+\mathbf{r})-u(\mathbf{x})]^2\rangle \tag{10}$$

is the structure function of the random displacements $u(\mathbf{x})$. This function does not depend on the model of a superlattice. It has been found in Ref. 15 for the cases of one-, two-, and three-dimensional inhomogeneities.

In Ref. 22 some refinements of these results have been carried out, according to which the parameter σ figuring in Ref. 15 has been represented in the form

$$\sigma=\gamma_u(k_\parallel^2+2k_\perp^2)^{1/2}/q, \tag{11}$$

where $k_{\parallel}$ and $k_{\perp}$ are the correlation wave number of inhomogeneities along the z axis and in the xy plane, respectively, and $\gamma_u = \pi\gamma_u'$, where γ_u' is the rms fluctuation of $u(z)$. Taking into account Eq. (11) the structure function for the case of one-dimensional inhomogeneities obtained in Ref. 15 can be represented in the form

$$Q(r_z) = 2\gamma_u^2[\exp(-k_{\parallel} r_z) + k_{\parallel} r_z - 1] \tag{12}$$

or in the limiting cases of large and small r_z

$$Q(r_z) \approx \begin{cases} 2\gamma_u^2(k_{\parallel} r_z - 1), & k_{\parallel} r_z \gg 1, \\ (\gamma_u k_{\parallel} r_z)^2, & k_{\parallel} r_z \ll 1. \end{cases} \tag{13}$$

According to the results of Ref. 15 the approximate equations (13) can be used for the superlattice in the entire region of variation of the variable r_z for the limiting cases of small ($p\gamma_u \ll 1$) and large ($p\gamma_u \gg 1$) rms fluctuation of $u(z)$. They lead to the following approximate expression for the correlation function of the superlattice in the one-dimensional case

$$K(r_z) = \sum_{m=0}^{\infty} B_p^2 \cos pqr_z \Phi_p, \tag{14}$$

where

$$\Phi_p = \begin{cases} \exp[-p^2\gamma_u^2(k_{\parallel} r_z - 1)], & p\gamma_u \ll 1, \\ \exp[-(p\gamma_u k_{\parallel} r_z)^2/2], & p\gamma_u \gg 1. \end{cases} \tag{15}$$

For isotropic three-dimensional inhomogeneities the structure function has the form

$$Q(r) = 6\gamma_u^2\left[1 - \frac{2}{k_0 r} + \left(1 + \frac{2}{k_0 r}\right)\exp(-k_0 r)\right], \tag{16}$$

or in the limiting cases of large and small r

$$Q(r) \approx \begin{cases} 2\gamma_u^2\left(1 - \dfrac{2}{k_0 r}\right), & k_0 r \gg 1, \\ (\gamma_u k_0 r)^2, & k_0 r \ll 1, \end{cases} \tag{17}$$

where $k_0 = k_{\parallel} = k_{\perp}$ is the correlation wave number of the random function $u(\mathbf{x})$.

An approximate equation for the correlation function of the superlattice in the three-dimensional case for $p\gamma_u \gg 1$ can be written in the entire region of variation of the variable r in the same way as this has been done in the one-dimensional case. But for $p\gamma_u \ll 1$ this way is impossible in the three-dimensional case, because the equation for $Q(r)$ diverges when $r \to 0$. To overcome this difficulty we used in Ref. 20 the exact Eq. (16) for $Q(r)$, and represented the exponent in Eq. (9) for $K(\mathbf{r})$ as a power series in γ_u. Here we use another approach that leads to a simpler form of the equation for the wave spectrum of the superlattice. Namely, we approximate the correlation function for $p\gamma_u \ll 1$ by the sum of an exponential function and a constant. In so doing we obtain an approximate equation for $K(r)$ in the form

$$K(r) = \sum_{m=0}^{\infty} B_p^2 \cos pqr_z F_p, \tag{18}$$

where

$$F_p = \begin{cases} (1 - D_p)\exp(-p^2\gamma_u^2 k_0 r) + D_p, & p\gamma_u \ll 1, \\ \exp[-(p\gamma_u k_0 r)^2/2], & p\gamma_u \gg 1, \end{cases} \tag{19}$$

where $D_p = \exp(-3\gamma_u^2 p^2)$. Numerical analysis shows that these expressions approximate the exact expression of $K(r)$ well enough for the corresponding values of $p\gamma_u$.

III. SPECTRUM AND DAMPING OF WAVES

We consider the equation for waves in the superlattice in the form

$$\nabla^2\mu + (\nu - \varepsilon\rho(z))\mu = 0, \tag{20}$$

where the expressions for the variable μ and the parameters ε and ν are different for waves of different nature. For spin waves when $\mu = M_x + iM_y$ describes the circular projection of the transverse components of the magnetization $\mathbf{M}$, and the parameter of the superlattice $A(\mathbf{x})$ is the magnetic anisotropy $\beta(\mathbf{x})$, we have[15]

$$\nu = \frac{\omega - \omega_0}{\alpha g M_0}, \quad \varepsilon = \frac{\gamma\beta}{\alpha}, \tag{21}$$

where ω is the frequency, $\omega_0 = g(H + \beta M_0)$, g is the gyromagnetic ratio, α is the exchange parameter, M_0 is the value of the magnetization, β is the average value of the anisotropy, and γ is its relative rms variation. We assume here that only the value of the anisotropy depends on coordinates, while the direction of the anisotropy axis coincides with the direction of the external magnetic field $\mathbf{H}$ and does not depend on coordinates. In the scalar approximation both the spectrum of elastic waves in a medium with an inhomogeneous density and the spectrum of electromagnetic waves in a medium with an inhomogeneous dielectric permeability are also described by this equation with redefinitions of the parameters. For elastic waves we have

$$\nu = (\omega/v)^2, \quad \varepsilon = \nu\gamma, \tag{22}$$

where γ is the rms fluctuation of the density of the material and v is the wave velocity. For an electromagnetic wave we have

$$\nu = \varepsilon_e(\omega/c)^2, \quad \varepsilon = \nu\gamma, \tag{23}$$

where ε_e is the average value of the dielectric permeability, γ is its rms deviation, and c is the speed of light. Equation (20) becomes more complicated when inhomogeneities of the elastic modulus, of the exchange parameter, or of the magnetization are considered: terms of the form $(\nabla\mu)(\nabla\rho)$ appear in the equation in these cases. Inhomogeneities of the direction of the anisotropy axis also complicate the equation because they lead to the appearance of a stochastic magnetic structure in a ferromagnet, which interacts with spin waves. In this paper we do not concern ourselves with such cases.

Replacing to the right hand side of Eq. (20) by the density of a point source $\delta(\mathbf{x}-\mathbf{x}_0)$ we obtain the equation for the Green function $G(\mathbf{x},\mathbf{x}_0)$

$$\nabla^2 G(\mathbf{x},\mathbf{x}_0)+[\nu-\varepsilon\rho(\mathbf{x})]G(\mathbf{x},\mathbf{x}_0)=\delta(\mathbf{x}-\mathbf{x}_0). \tag{24}$$

Representing the Green function in the form of a Fourier integral,

$$G(\mathbf{x},\mathbf{x}_0)=\int G_{\mathbf{k}}e^{i\mathbf{kx}}\,d\mathbf{k}, \tag{25}$$

we obtain the integral equation for the Fourier components of the Green function

$$(\nu-k^2)G_{\mathbf{k}}=\varepsilon\int G_{\mathbf{k}_1}\rho_{\mathbf{k}-\mathbf{k}_1}\,d\mathbf{k}_1+e^{i\mathbf{kx}_0}. \tag{26}$$

Averaging this equation over the ensemble of the random realizations of the function $\rho(\mathbf{x})$ we obtain

$$(\nu-k^2)\langle G_{\mathbf{k}}\rangle=\varepsilon\int \langle G_{\mathbf{k}_1}\rho_{\mathbf{k}-\mathbf{k}_1}\rangle d\mathbf{k}_1+e^{i\mathbf{kx}_0}. \tag{27}$$

Increasing the subscripts on $\mathbf{k}$ by unity in Eq. (26), expressing $G_{\mathbf{k}_1}$ from this equation, and substituting it into Eq. (27), we obtain

$$(\nu-k^2)\langle G_{\mathbf{k}}\rangle=\varepsilon^2\int\int\frac{\langle \rho_{\mathbf{k}-\mathbf{k}_1}\rho_{\mathbf{k}_1-\mathbf{k}_2}G_{\mathbf{k}_2}\rangle}{\nu-k_1^2}d\mathbf{k}_1\,d\mathbf{k}_2+e^{i\mathbf{kx}_0}. \tag{28}$$

Decoupling the averaged product in the integrand of Eq. (28) in an approximation corresponding to the Bourret approximation,[23]

$$\langle \rho_{\mathbf{k}-\mathbf{k}_1}\rho_{\mathbf{k}_1-\mathbf{k}_2}G_{\mathbf{k}_2}\rangle\simeq\langle \rho_{\mathbf{k}-\mathbf{k}_1}\rho_{\mathbf{k}_1-\mathbf{k}_2}\rangle\langle G_{\mathbf{k}_2}\rangle, \tag{29}$$

we obtain the averaged Green function in the form

$$\langle G_{\mathbf{k}}\rangle=\left[\nu-k^2-\epsilon^2\int\frac{S(\mathbf{k}-\mathbf{k}_1)d\mathbf{k}_1}{\nu-k_1^2}\right]^{-1}e^{i\mathbf{kx}_0}. \tag{30}$$

Here $S(\mathbf{k})$ is the spectral density of the superlattice defined by the formula

$$\langle \rho_{\mathbf{k}}\rho_{\mathbf{k}'}\rangle=S(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}'), \tag{31}$$

or by the inverse Fourier transformation of the correlation function of the superlattice $K(\mathbf{r})$:

$$S(\mathbf{k})=\frac{1}{(2\pi)^3}\int K(\mathbf{r})e^{-i\mathbf{kr}}\,d\mathbf{r}. \tag{32}$$

Laws of the dispersion and damping of the averaged waves are determined by the equation for the complex frequency $\nu=\nu'+i\xi$, which follows from the vanishing of the denominator of the Green function:

$$\nu-k^2=\epsilon^2\int\frac{S(\mathbf{k}-\mathbf{k}_1)d\mathbf{k}_1}{\nu-k_1^2}. \tag{33}$$

We restrict ourselves to the case of small values of the function $u(\mathbf{x})$ corresponding to $p\gamma_u\ll 1$. First we consider one-dimensional inhomogeneities. Substituting Eq. (14) with Φ_p corresponding to the upper line of Eq. (15) into Eq. (32) and performing the integration we obtain the spectral density

$$S(\mathbf{k})=\delta(k_x)\,\delta(k_y)S_0(k_z), \tag{34}$$

where $\delta(k_i)$ are Dirac delta functions, and

$$S_0=\frac{\gamma_u^2k_\parallel}{2\pi}\sum_p B_{|p|}^2\frac{\exp(p\gamma_u)^2}{(p^2\gamma_u^2k_\parallel)^2+(k_z-pq)^2}. \tag{35}$$

Here $p=\pm 1,\ \pm 3,\ \ldots$. Substituting Eqs. (34) and (35) into Eq. (33) and performing the integration with respect to $\mathbf{k}_1$ we obtain the equation for the complex variable $\nu=\nu'+i\xi$ in the form

$$\nu-k^2=\frac{\Lambda^2}{4}\sum_p B_{|p|}^2\frac{\exp(p\gamma_u)^2(1-ip^2\gamma_u^2k_\parallel/\sqrt{\nu_1})}{(\sqrt{\nu_1}-ip^2\gamma_u^2k_\parallel)^2+(k_z-pq)^2}, \tag{36}$$

where $\Lambda=\sqrt{2}\varepsilon,\ \nu_1=\nu-k_x^2-k_y^2$. In the case of the absence of inhomogeneities ($\gamma_u=0$) this equation reduces to the equation obtained in Ref. 21 for the wave spectrum in the ideal superlattice with finite thicknesses of interfaces.

The complete equation (36) is very complicated for analytical analysis. But with the proviso that $\Lambda/\nu\ll 1$, the resonances corresponding to different p in the sum in Eq. (36) influence one another only slightly. That is why we can restrict ourselves to the two-wave approximation in the vicinity of each odd Brillouin zone boundary $k\simeq k_{rn}=nq/2$, keeping in the sum only the term $p=n$ corresponding to the Brillouin zone n considered:

$$(\nu-k^2)[(\sqrt{\nu_1}-in^2\gamma_u^2k_\parallel)^2-(nq-k_z)^2] = \frac{\Lambda^2}{4}B_{|n|}^2\exp(n\gamma_u)^2(1-in^2\gamma_u^2k_\parallel/\sqrt{\nu_1}). \tag{37}$$

Let us make some further simplifications in this equation. We will consider the waves to be propagating along the z axis ($k_z=k,\ \nu_1=\nu$). Under the condition $n\gamma_u^2k_\parallel/q\ll 1$ we can neglect both the imaginary part of the coupling parameter and the shift of the crossing resonance point and obtain the equation in the form

$$(\nu-k^2)[\nu-in^3\gamma_u^2k_\parallel q-(nq-k)^2]=\frac{\Lambda^2}{4}B_{|n|}^2\exp(n\gamma_u)^2. \tag{38}$$

Solutions of this equation have been well investigated for the case of the model of the sinusoidal superlattice[15] ($n=1$, $B_{|n|}=1$) as well as for the model of the sharp interfaces[19,20] ($B_{|n|}=2\sqrt{2}/\pi n$). In the absence of inhomogeneities ($\gamma_u=0$) the gaps $\Delta\nu_n=\nu_+-\nu_-=\Lambda B_{|n|}$ exist in the wave spectrum at the Brillouin zone boundaries $k=k_{rn}=nq/2$; here $\nu_\pm$ are the solutions of Eq. (38). In the vicinity of k_{rn} the spectrum has the form shown schematically in Fig. 1 by two solid curves (we use the extended zone scheme). When inhomogeneities appear the solutions become complex. The gap in

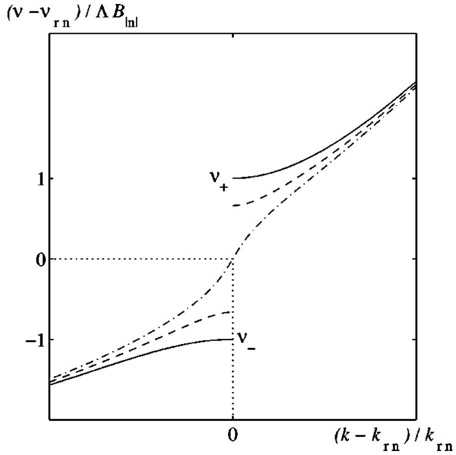

FIG. 1. Dispersion curves near the Brillouin zone boundaries $k = k_{rn}$ for the ideal superlattice (solid curves) and for partially randomized superlattices (dashed curve for the small and dotted-dashed curve for large intensity of inhomogeneities).

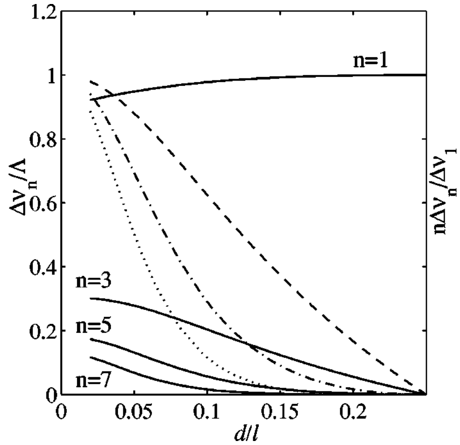

FIG. 2. The dependencies of the gap widths in the spectrum $\Delta\nu_n$ on d/l for the different Brillouin zones $n=1$, 3, 5, and 7 (solid curves) for the ideal superlattice. The relations $n\Delta\nu_n/\Delta\nu_1$ are also shown for $n=3$ (dashed curve), $n=5$ (dotted-dashed curve), and $n=7$ (dotted curve).

the spectrum $\Delta\nu_n = \nu'_+ - \nu'_-$ decreases with the increasing γ_u (dashed curves in Fig. 1), and at last closes: the spectrum of the averaged waves is described now by a continuous curve with a point of inflection at $k=k_{rn}$ (the dotted-dashed curve in Fig. 1). Simultaneously with the increase of γ_u the damping ξ increases, whose dependence on k has a maximum at $k=k_{rn}$. So, the Brillouin zone boundaries are the most sensitive points of the spectrum with respect to the influence of inhomogeneities. At these points the expressions for $\Delta\nu_n$ and ξ_n have the forms

$$\frac{\Delta\nu_n}{\Lambda} = \mathrm{Re}\, L_n, \tag{39}$$

$$\frac{\xi_n}{\Lambda} = \frac{1}{2}(n^3\gamma_u^2\eta \pm \mathrm{Im}\, L_n). \tag{40}$$

Here

$$L_n = [B_{|n|}^2 \exp(n\gamma_u)^2 - (n^3\gamma_u^2\eta)^2]^{1/2}, \tag{41}$$

where $\eta = k_{\|}q/\Lambda$. Both the width of the gap $\Delta\nu_n$ and value of the damping ξ_n depend on the parameters of the initial ideal superlattice (q, Λ, and d/l), characteristics of the inhomogeneities (γ_u and η), and the Brillouin zone number n.

We come now to a consideration of the case of three-dimensional inhomogeneities. Substituting Eq. (18), with F_p corresponding to the upper line of Eq. (19), into Eq. (32), and performing the integration we obtain the spectral density

$$S(\mathbf{k}) = \frac{1}{2\pi^2}\sum_p B_{|p|}^2 \left\{ \frac{\gamma_u^2 p^2 k_0 (1-D_p)}{[(\gamma_u^2 p^2 k_0)^2 - (\mathbf{k}-p\mathbf{q})^2]^2} + \pi^2 D_p \delta(\mathbf{k}-p\mathbf{q}) \right\}, \tag{42}$$

where $p = \pm 1, \pm 3, \ldots$. Substituting Eq. (42) into Eq. (33) and performing the integration with respect to $\mathbf{k}_1$ we obtain the equation for the complex variable ν in the form

$$\nu - k^2 = \frac{\varepsilon^2}{2}\sum_p B_{|p|}^2 \left[\frac{1-D_p}{(\sqrt{\nu} - i\gamma_u^2 p^2 k_0)^2 - (\mathbf{k}-p\mathbf{q})^2} + \frac{D_p}{\nu - (\mathbf{k}-p\mathbf{q})^2} \right]. \tag{43}$$

In the two-wave approximation in the vicinity of each odd Brillouin zone boundary $k \approx k_{rn} = nq/2$, keeping in the sum only the term $p=n$, and using the same simplification that yielded Eq. (38), we obtain

$$\nu - k^2 = \frac{\Lambda^2}{4} B_{|n|}^2 \left[\frac{1-D_n}{\nu - i\gamma_u^2 n^3 k_0 q - (\mathbf{k}-n\mathbf{q})^2} + \frac{D_n}{\nu - (\mathbf{k}-n\mathbf{q})^2} \right]. \tag{44}$$

This equation is a cubic equation in ν and in contrast to Eq. (38) for the one-dimensional case its solution cannot be represented in an explicit form analogous to Eqs. (39)–(41).

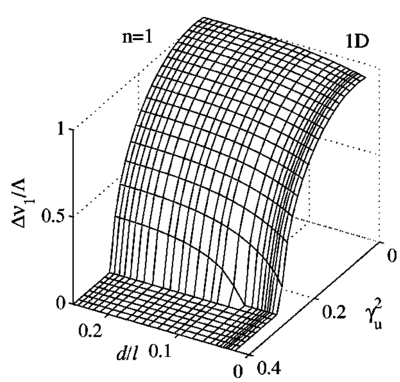

FIG. 3. The dependence of the width of the gap $\Delta\nu_1$ on d/l and γ_u^2 at $\eta=4$ for the 1D case.

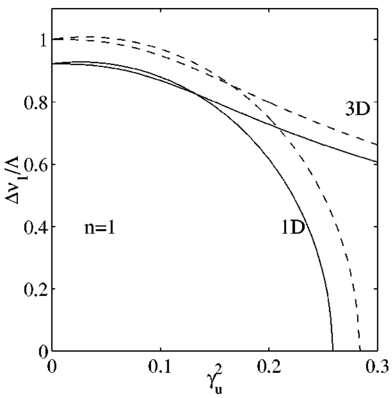

FIG. 4. The dependence of the width of the gap $\Delta\nu_1$ on γ_u^2 at $\eta=4$ for the 1D and 3D cases for the superlattice with sharp interfaces ($d/l=0$, solid curves) and sinusoidal superlattice ($d/l=1/4$, dashed curves).

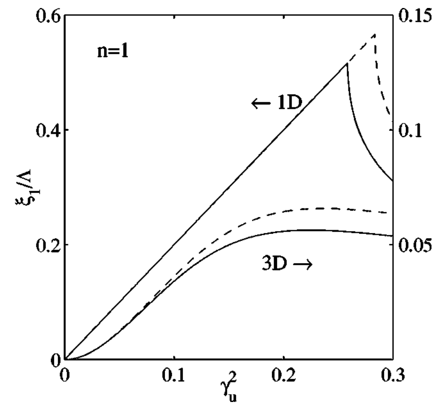

FIG. 6. The dependence of the damping ξ_1 on γ_u^2 at $\eta=4$ for the 1D and 3D cases for the superlattice with sharp interfaces ($d/l=0$, solid curves) and sinusoidal superlattice ($d/l=1/4$, dashed curves). The scales for the 1D and 3D cases are shown by the arrows.

IV. NUMERICAL RESULTS AND THEIR DISCUSSION

The quantity $B_{|n|}$ in Eqs. (38)–(44) is a transcendental function of d/l. That is why we investigate these equations by numerical methods. Results of these investigations at the points $k_{rn}=nq/2$ are shown in Figs. 2–8.

The spectrum of waves in an ideal superlattice with an arbitrary ratio d/l in the absence of inhomogeneities ($\gamma_u=0$) has been studied in Ref. 21. Widths of the gaps in this case are determined by the dependence on n and d/l of the coefficients of the expansion of the function $\rho(z)$ in the Fourier series:

$$\frac{\Delta\nu_n}{\Lambda}=B_{|n|}. \tag{45}$$

These dependencies are depicted in Fig. 2 for $n=1$, 3, 5, and 7 (solid curves). From this figure we notice that the width of the gap for the first Brillouin zone depends slightly on d/l because the coefficient before the first Fourier harmonic depends slightly on the form of the function $\rho(z)$:$B_1=2\sqrt{2}/\pi\approx0.9$ for the limiting case of a superlattice with sharp interfaces ($d/l=0$), and $B_1=1$ for the other limiting case of the sinusoidal superlattice ($d/l=1/4$). The widths of the gaps $\Delta\nu_n$ for $n>1$ strongly depend on n as well as on d/l. The dependence on n for the limiting case $d/l=0$ is determined by the expression $B_{|n|}=2\sqrt{2}/\pi n$, and in the limiting case of the sinusoidal superlattice ($d/l=1/4$) all the Fourier harmonics, except the first, vanish. This means that in this case the first order of perturbation theory does not give a contribution to the gap widths. The latter are determined by terms of higher orders which were not taken into account in our analysis. The ratios $n\Delta\nu_n/\Delta\nu_1$ are shown also in Fig. 2 for $n=3$, 5, and 7. As has been noticed in Ref.

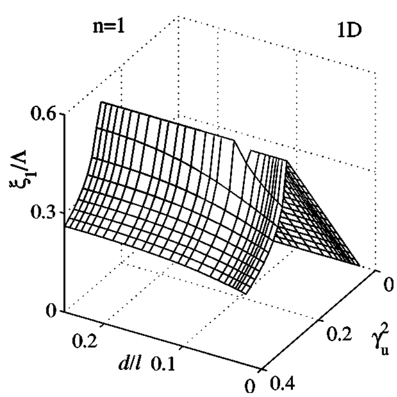

FIG. 5. The dependence of the damping ξ_1 on d/l and γ_u^2 at $\eta=4$ for the 1D case.

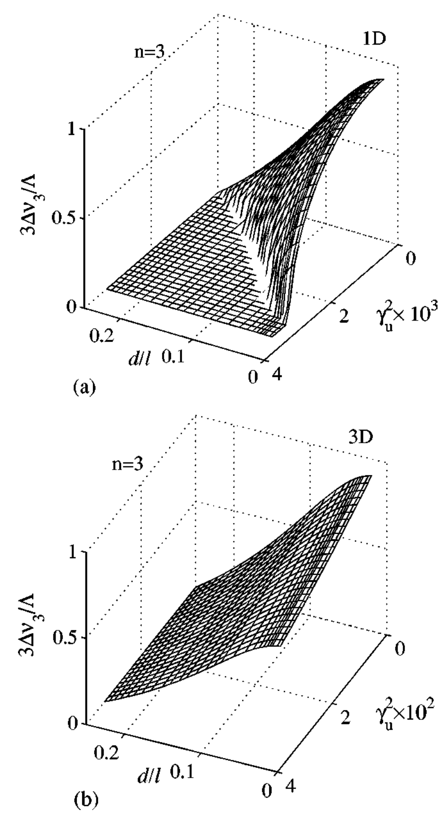

FIG. 7. The dependence of the width of the gap $\Delta\nu_3$ on d/l and γ_u^2 at $\eta=4$ for the 1D (a) and 3D (b) cases.

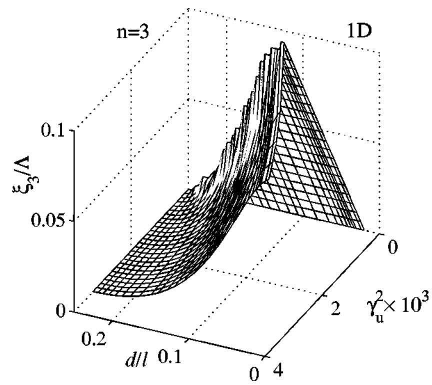

FIG. 8. The dependence of the damping ξ_3 on d/l and γ_u^2 at $\eta=4$ for the 3D case.

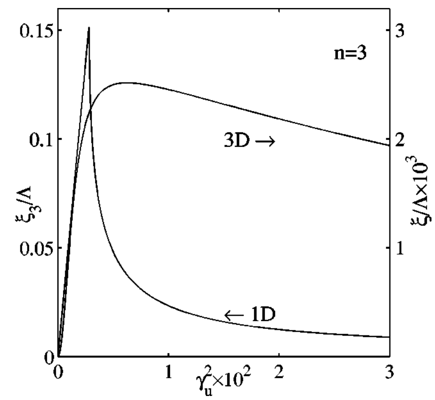

FIG. 9. The dependence of the damping ξ_3 on γ_u^2 at $\eta=4$ and $d/l=0$ for the 1D and 3D cases. The scales for the 1D and 3D cases are shown by the arrows.

21, the experimental measurement of the ratio between the widths of the gaps at the boundaries of the first and any other Brillouin zones offers the possibility of determining the thickness of the interfaces in a multilayered medium.

When inhomogeneities appear the width of the gap in addition to its dependence on d/l and n, now depends also on the intensity of the inhomogeneities γ_u^2 and on their dimensionless correlation wave number η. The latter is determined for the one-dimensional (1D) and three-dimensional (3D) inhomogeneities by the expressions

$$\eta=\begin{cases} k_{\parallel}q/\Lambda, & \text{1D},\\ k_0 q/\Lambda, & \text{3D}.\end{cases} \tag{46}$$

In Figs. 3, 4, 5, and 6 the dependencies of $\Delta\nu_1$ and ξ_1 on d/l and γ_u^2 are shown for $\eta=4$. The width of the gap $\Delta\nu_1$ for the first zone for the one-dimensional inhomogeneities (Fig. 3) decreases with the increase of γ_u^2, and $\Delta\nu_1$ closes at some critical value of γ_u^2 that is approximately the same for the superlattice with any ratio d/l. In Fig. 4 the dependencies of $\Delta\nu_1$ on γ_u^2 are depicted for the one- and three-dimensional inhomogeneities for the limiting cases of the superlattice with the sharp interfaces ($d/l=0$, solid curves) and the sinusoidal superlattice ($d/l=1/4$, dashed curves). All curves corresponding to intermediate values of d/l align themselves between these two limiting curves. One can see that $\Delta\nu_1$ decreases much more slowly with the increase of 3D inhomogeneities as compared with the 1D case: the gap for 3D has a rather large value when the gap for 1D is already closed.

In Fig. 5 the damping ξ_1 is shown for the 1D case as a function of d/l and γ_u^2 for $\eta=4$. The graph in this figure corresponds to the choice of the minus sign before the second term in Eq. (40). This choice of the sign is justified in Ref. 18, where susceptibilities of the sinusoidal superlattice are investigated. With the increase of γ_u^2 a linear increase of the damping ξ_1 occurs at the beginning, which results from the first term of Eq. (40); the second term of this equation is equal to zero this time because L_n has only a real component. The gap closes at some critical γ_u^2 and the second term of Eq. (40) subtracts from the first one leading to a decrease of the damping with the further increase of γ_u^2. In Fig. 6 the dependencies of ξ_1 on γ_u^2 are depicted for both the 1D and 3D cases for the limiting cases $d/l=0$ (solid curves) and $d/l=1/4$ (dashed curves). The dependence of ξ_1 for the 1D case has a sharp peak at the same value of γ_u^2 where the gap $\Delta\nu_1$ closes (Fig. 4). It should be noted that the scales for the 1D and 3D cases on the graph are different, and the difference between the values of the damping for the cases of one- and three-dimensional inhomogeneities reaches an order of magnitude.

We come now to a consideration of the values of $\Delta\nu_n$ and ξ_n for the third Brillouin zone $(n=3)$. In Fig. 7 the dependencies of $\Delta\nu_3$ on d/l and γ_u^2 are shown for $\eta=4$ for the one- and three-dimensional cases [Figs. 7(a) and 7(b), respectively]. The width of the gap $\Delta\nu_3$ for the 1D case [Fig. 7(a)] strongly depends on d/l as well as on γ_u^2. That is why the surface $\Delta\nu_n(d/l,\gamma_u^2)$ for the 1D case has a more complicated form for $n=3$ than for $n=1$ (see Fig. 3). The closing of the gap $\Delta\nu_3$ occurs at a much smaller γ_u^2 than the closing of the gap $\Delta\nu_1$, in our case $(\eta=4)$, for instance, by two orders of magnitude. The gap $\Delta\nu_3$ for the 3D case [Fig. 7(b)] depends on d/l as strongly as in the 1D case [Fig. 7(a)]. But its dependence on γ_u^2 is not so strong, and the surface $\Delta\nu_3(d/l,\gamma_u^2)$ has a simpler form in the 3D case than in the 1D case. In Fig. 8 the damping ξ_3 in the one-dimensional case is shown as a function of d/l and γ_u^2. The surface $\xi_3(d/l,\gamma_u^2)$ has a much more complicated form than the surface $\xi_1(d/l,\gamma_u^2)$ (see Fig. 5) because of the strong dependence of ξ_1 on d/l. In Fig. 9 the dependencies of ξ_3 on γ_u^2 are shown at $d/l=0$ for both the 1D and 3D cases. The dependence in the 1D case has a sharp peak just at the same value of γ_u^2 at which the corresponding gap closes. Note that the scales for the 1D and 3D cases on the graph are different, and the difference between the values of the damping for the cases of one- and three-dimensional inhomogeneities is two orders of magnitude. All graphs in Figs. 3–9 correspond to the value of the dimensionless correlation wave number $\eta=4$. In Fig. 10 the phase diagram for the existence of an open gap $\Delta\nu_3$ in the 1D case is shown in the coordinates d/l and γ_u^2 for $\eta=2$, 4, and 6 (the region of existence of an open

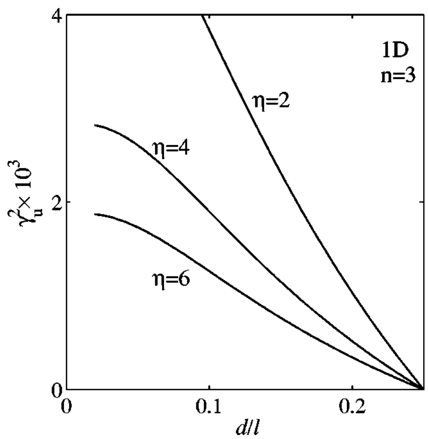

FIG. 10. The phase diagram for the existence of an open gap $\Delta\nu_3$ in the spectrum in the coordinates d/l and γ_u^2 for $\eta=2$, 4, and 6.

gap is situated under the corresponding curve). This diagram is the solution of the equation $L_n=0$ where L_n is defined by Eq. (41). The curve with $\eta=4$ corresponds to the cross section of the surface $\Delta\nu_3$ in Fig. 7(a) with the plane $d/l,\gamma_u^2$ in this figure. The other curves in Fig. 10 show how this cross section changes as η changes.

As noted above the solution of Eq. (44) for the 3D case cannot be represented in an explicit form, and this equation has been solved numerically. But the following approximate analytical solution of Eq. (44) can be used for small values of $n^2\gamma_u^2 \ll 1$:

$$\frac{\Delta\nu_n}{\Lambda}\approx B_n\left[1-\frac{3}{2}\,\frac{(n\gamma_u)^2(2n^3\gamma_u^2\eta/B_n)^2}{1+(2n^3\gamma_u^2\eta/B_n)^2}\right], \qquad (47)$$

$$\frac{\xi_n}{\Lambda}\approx\frac{3}{2}\,\frac{n^5\gamma_u^4\eta}{1+(2n^3\gamma_u^2\eta/B_n)^2}. \qquad (48)$$

Equation (47) describes well the dependencies of $\Delta\nu_n$ on n, d/l, and γ_u^2 which were shown in Figs. 4 and 7(b) for the 3D case. Equation (48) describes the dependencies of ξ_n on n, d/l, and γ_u^2 that were shown in Figs. 6 and 9, but only in the regions of γ_u^2 where ξ_n increases with increasing γ_u^2.

Let us summarize the results obtained. The widths of the gaps $\Delta\nu_n$ at the boundaries of Brillouin zones are the most sensitive points of the spectrum of a superlattice in relation to the influence of inhomogeneities (to the increase of their rms deviations γ_u). This sensitivity depends on the number of the Brillouin zone n and on the relative thickness of the interfaces d/l. The width $\Delta\nu_1$ of the first zone practically does not depend on d/l, and has the least sensitivity to the influence of inhomogeneities. For $n>1$ the sensitivity is higher the larger n and the smaller d/l. That is why with increasing disorder the successive closing of the gaps in the spectrum takes place beginning with large values of n down to $n=1$. The effects of inhomogeneities on the wave spectrum depend on their dimensionality: the one-dimensional inhomogeneities affect the spectrum more strongly than do the three-dimensional ones. A gap in the spectrum decreases much more slowly with increasing γ_u for the case of three-dimensional inhomogeneities than it does for one-dimensional inhomogeneities with the same correlation wave number η. A gap in the 3D case still has enough large value when the gap corresponding in the 1D case is already closed. The damping induced by three-dimensional inhomogeneities can be smaller by several orders of magnitude than the damping induced by one-dimensional inhomogeneities with the same values of γ_u and η.

From this entire analysis we can conclude that a detailed investigation of $\Delta\nu_n$ and ξ_n permits, in principle, determining from these spectral characteristics all the parameters of the superlattice, Λ, l, and d, as well as the parameters of the inhomogeneities, γ_u and η.

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