Rectangular microwave resonators with magnetic anisotropy. Mapping onto pseudointegrable rhombus

To cite this article: E. N. Bulgakov and A. F. Sadreev 2002 *EPL* **57** 198

EUROPHYSICS LETTERS 15 January 2002

Europhys. Lett., **57** (2), pp. 198–204 (2002)

Rectangular microwave resonators with magnetic anisotropy. Mapping onto pseudointegrable rhombus

E. N. Bulgakov[1] and A. F. Sadreev[1,2](*)

[1] *Kirensky Institute of Physics - 660036, Krasnoyarsk, Russia*

[2] *Department of Physics and Measurement Technology, Linköping University S-581 83 Linköping, Sweden*

(received 4 May 2001; accepted in final form 26 October 2001)

PACS. 47.52.+j – Chaos.
PACS. 42.60.Da – Resonators, cavities, amplifiers, arrays, and rings.
PACS. 75.50.Gg – Ferrimagnetics.

Abstract. – A rectangular microwave resonator filled with ferrite with uniaxial magnetic anisotropy is considered. It is shown that this task can be reduced to an empty rhombus resonator with the vertex angle defined by an external magnetic field, provided that the magnetic anisotropy of the ferrite is strong. Therefore, the statistics of eigenfrequencies for TM modes is described by the Brody or semi-Poisson distribution with some exceptional cases.

One of the main research lines in quantum chaos is to investigate the statistics of energy levels and eigenfunctions of quantum systems whose classical counterpart is chaotic. A very popular class of systems are the two-dimensional Euclidean billiards, which are described by the Helmholtz equation

$$-\nabla^2\psi(x,y) = \lambda\psi(x,y), \tag{1}$$

with Dirichlet boundary conditions $\psi(x,y) = 0$ for (x,y) at the boundary of billiard. It was shown that the eigenvalue statistics of λ obey the statistics of random matrix ensemble [1,2]. The distribution function for the amplitudes of the eigenfunctions ψ is perfectly well described by a Gaussian distribution [3,4]. Correspondingly, the square $\rho = |\psi|^2$ is described by the well-known Porter-Thomas (P-T) distribution [4]

$$P(\rho) = \frac{1}{\sqrt{2\pi\rho}}\exp[-\rho/2]. \tag{2}$$

For integrable billiards, for example, a rectangular one, the eigenvalue statistics is described by the Poisson distribution.

The distribution function of ρ is given by the formula [5]

$$g(\rho) = \frac{1}{\pi^2\sqrt{\rho}}K(1-\rho), \quad 0 < \rho < 1, \tag{3}$$

(*) E-mail: almsa@ifm.liu.se, almas@tnp.krasn.ru

where $K(m)$ is an elliptic integral of the first kind and the eigenstate is $\psi(x, y) = \sin(k_x x)\sin(k_y y)$. The distribution (3) has a stepwise behavior at $\rho = 1$ with a step value equal to $K(0)/\pi^2$ resulting from the fact that $\rho(x, y)$ has the same maxima.

In the last two decades the subject of these distributions was dominated by theory and computer simulations. About ten years ago experimentalists found effective techniques to study billiards with the help of similarly shaped electromagnetic resonators [6–10]. In most of the experiments resonators with a cylindrical geometry and different cross-sections have been used. Taking the z-axis parallel to the axis of the cylinder, the boundary conditions reduce to

$$E_z|_S = 0, \qquad \nabla_\perp B_S = 0, \tag{4}$$

for electric and magnetic fields, respectively. Writing the electric field as [4]

$$E_z(x, y, z) = \psi(x, y)\cos\left(\frac{\pi n z}{d}\right), \quad n = 0, 1, 2, \dots,$$
$$B_z(x, y, z) = 0,$$

one can easily obtain that the function $\psi(x, y)$ satisfies eq. (1) with $\lambda = (\omega/c)^2 - (\pi n/d)^2$ and the Dirichlet boundary condition, as follows from (4). Here ω is the angular frequency, c is the light velocity, and d is the thickness of the resonator. This way defines the transverse magnetic (TM) modes of the electromagnetic field in the resonator. In what follows we do not consider the transverse electric (TE) modes.

We consider the more general case of the Helmhotz equation

$$\left(\mu_{xx}\frac{\partial^2}{\partial x^2} + \mu_{yy}\frac{\partial^2}{\partial y^2} + (\mu_{xy} + \mu_{yx})\frac{\partial^2}{\partial x \partial y} + \lambda\right)\psi(x, y) = 0, \tag{5}$$

with the Dirichlet boundary conditions, where the components of the tensor $\mu_{\alpha,\beta}$ will be defined below. Because of the term with mixed partial derivatives in (5), even for the rectangular billiard the solution of this equation cannot be presented as $\psi(x, y) = \psi(x)\psi(y)$. Below we will show that eq. (5) for the rectangular billiard can be transformed into the isotropic Helmholtz equation (1) but with the Dirichlet boundary conditions at the boundaries of the rhombic billiard. Equation (5) describes a microwave resonator filled with an anisotropic magnet. The use of magnets in microwave resonators was already described as a way to violate time-reversal symmetry [11,12]. Here we consider a complete filling of the resonator to have a homogeneous case for which time-reversal symmetry takes place [4].

Let us write the Maxwell equations for the TE modes of the electromagnetic fields in two-dimensional resonators:

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{n}E_z &= -ik\boldsymbol{B}, \\ \nabla \times \boldsymbol{H} &= ik\boldsymbol{n}E_z, \\ \boldsymbol{B} &= \hat{\mu}\boldsymbol{H}, \end{aligned} \tag{6}$$

where $\boldsymbol{n}$ is unit vector parallel to the electric field direction, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{B}$ is the magnetic induction, $k = \omega/c$, and ω is an eigenfrequency with wave number k. In what follows we imply the following magnetic anisotropic permeability:

$$\begin{aligned} \hat{\mu} &= 1 + \hat{\chi}, \\ \hat{\chi} &= \begin{pmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}. \end{aligned} \tag{7}$$

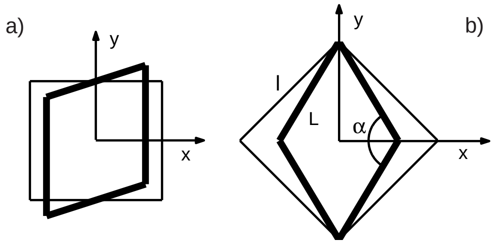

Fig. 1 – Mapping of a rectangular billiard described by the anisotropic Helmholtz equation (5) onto a polygon (a) (transformation (9)) and onto a rhombus (b) (transformation (15)).

Following [4], we consider the TM modes of the Maxwell equations (6). Substituting the permeability (7) into the last Maxwell equation we have

$$\begin{pmatrix} H_x \\ H_y \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \mu_{yy} & \mu_{xy} \\ \mu_{yx} & \mu_{xx} \end{pmatrix} \begin{pmatrix} \dfrac{i}{k} & \dfrac{\partial E_z}{\partial y} \\ \dfrac{-i}{k} & \dfrac{\partial E_z}{\partial x} \end{pmatrix}, \tag{8}$$

where

$$D = \mu_{xx}\mu_{yy} - \mu_{xy}\mu_{yx} .$$

From this equation one can find, from the Maxwell equations (6), the wave equation (5) for the electric field with eigenvalues equal to $\lambda = Dk^2$.

By the coordinate transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\dfrac{\sqrt{\mu_{xx}\mu_{yy} - (\mu_{xy} + \mu_{yx})^2/4}}{\mu_{xx}} & 0 \\ -\dfrac{\mu_{xy} + \mu_{yx}}{2\mu_{xx}} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

we can eliminate the cross-derivatives in eq. (5) and reduce the Helmholtz equation to the following one:

$$\nabla^2 E_z + \mu k^2 E_z = 0, \tag{10}$$

where

$$\mu = \frac{D\mu_{xx}}{\mu_{xx}\mu_{yy} - (\mu_{xy} + \mu_{yx})^2/4}. \tag{11}$$

Transformation (9) transforms the rectangle into a particular case of polygon, *i.e.* a parallelogram, as shown in fig. 1(a).

Next, we consider ferrite with easy plane anisotropy, *i.e.* the magnetization $\boldsymbol{M}$ in equilibrium state is perpendicular to the anisotropy axis $\boldsymbol{N}$. As shown in fig. 2, we direct the z-axis along the magnetization vector and the x-axis along the anisotropy axis. Thereby the (y, z)-plane is the easy plane for the magnetization vector. Assuming that the resonator filled with ferrite can be considered as a thin slab, we have the following demagnetization factors: $N_x = N_y = 0$, $N_z = 1$. Moreover, following Kittel [13] and Lax and Button [14], we introduce

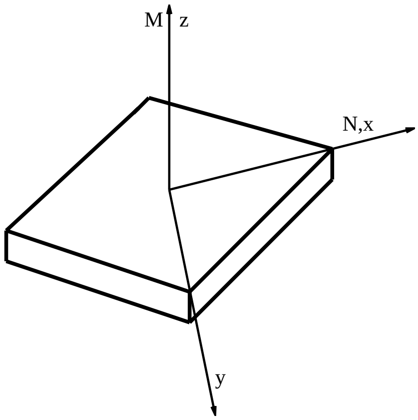

Fig. 2 – Schematical view of the rectangular resonator filled with ferrite, where $\boldsymbol{M}$ is the magnetization of ferrite and $\boldsymbol{N}$ is the anisotropy field.

an effective demagnetizing factor as follows: $N_{\rm a} = 2K_{\rm a}/M^2$, which is directed along the x-axis. Then in this specific Cartesian system of coordinates the susceptibility has the following components:

$$\begin{aligned} \chi_{xx}(\omega) &= \frac{\omega_0\omega_M}{\omega_r^2-\omega^2}, \\ \chi_{yy}(\omega) &= \frac{\omega_M(\omega_0+\omega_a-\omega_M)}{\omega_r^2-\omega^2}, \\ \chi_{xy}(\omega) &= \frac{-i\omega\omega_M}{\omega_r^2-\omega^2}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \omega_0 &= \gamma(H_0-4\pi M), \quad \omega_M = 4\pi\gamma M, \\ \omega_r^2 &= \omega_0(\omega_0+\omega_a), \\ \omega_a &= \gamma H_{\rm a}, \quad H_{\rm a} = 8\pi K_{\rm a}/M. \end{aligned} \tag{13}$$

Here H_0 is the external constant magnetic field applied along the z-axis, *i.e.* along the direction of magnetization, and $H_{\rm a}$ is the effective anisotropy field directed along the x-axis.

Since for this case of the susceptibility $\mu_{xy}+\mu_{yx}=0$, expression for (11) and transformation (9) are simplified as follows:

$$\mu = \mu_{xx} + \frac{\mu_{xy}^2}{\mu_{yy}}, \tag{14}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\sqrt{\dfrac{\mu_{yy}}{\mu_{xx}}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{15}$$

If a ferrite has the anisotropy axis directed parallel to the lateral sides of the rectangular resonator, transformation (15) maps the rectangle onto a rectangle. However, if the anisotropy axis of ferrite is directed differently, the rectangle is mapped onto a parallelogram, as shown in fig. 1(a). In the particular case of a squared resonator, transformation (15) maps a square onto a rhombus as shown in fig. 1(b), with the vertex angle of the rhombus

$$\alpha = 2\arctan\sqrt{\frac{\mu_{yy}}{\mu_{xx}}}. \tag{16}$$

Because of the frequency dependence of the susceptibility components, the expressions for eigenvalues of the wave equation (10) and the vertex angle (16) are complicated. So we consider a case for which we can neglect the frequency ω in the denominator of the susceptibility (12). Therefore we are to imply the inequality $\omega_r^2 \gg \omega^2$. In order to fulfill this inequality, we can apply a strong magnetic field such as $\gamma H_0 \gg \omega$ or explore ferrites with a large anisotropy field $H_\mathrm{a} = 2K_\mathrm{a}/M \gg \omega$. If we take the ferrite $\mathrm{Ni}_{0.932}\mathrm{Co}_{0.068}\mathrm{Fe}_2\mathrm{O}_4$ [14], we have $4\pi M = 3475\,\mathrm{g\,s}$, $H_\mathrm{a} \approx 12566\,\mathrm{Oe}$. Therefore we can neglect the frequency dependence in the denominators of the susceptibility components, if the eigenfrequencies of the resonator do not exceed the frequency of about $10^{11}\,\mathrm{s}^{-1}$, provided that the size of the resonator is of the order of 10 cm. The anisotropy field is extremely large in the ferrite $\mathrm{Ba}_2\mathrm{Co}_2\mathrm{Fe}_{12}\mathrm{O}_{22}$ ($2K_\mathrm{a}/M = 2800\,\mathrm{Oe}$), so the upper boundary for the eigenfrequencies reaches values of order 10^{12}.

As a result, we can write the following formulas for the susceptibility components instead of (12):

$$\begin{aligned}\chi_{xx} &\approx \frac{4\pi M}{H_0 + H_\mathrm{a}},\\ \chi_{yy}(\omega) &\approx \frac{4\pi M}{H_0 - 4\pi M},\\ \chi_{xy}(\omega) &\approx -\frac{i4\pi\omega M}{\gamma(H_0 + H_\mathrm{a})(H_0 - 4\pi M)}.\end{aligned} \tag{17}$$

Correspondingly, substituting these expressions into eqs. (14) and (16), we have

$$\mu \approx 1 + \frac{4\pi M}{H_0 + H_\mathrm{a}} - \frac{16\pi^2 M^2 \omega^2}{\gamma^4 (H_0 + H_\mathrm{a})^2 (H_0 - 4\pi M)^2}, \tag{18}$$

$$\alpha \approx 2\arctan\sqrt{\frac{H_0}{(H_0 - 4\pi M)(1 + \frac{4\pi M}{H_0 + H_\mathrm{a}})}}. \tag{19}$$

One can see that, if the external magnetic field H_0 and the anisotropy field H_a are both strong, we obtain from (18) and (16) that $\mu \approx 1$, $\alpha \approx \pi/2$. It means that the strong magnetic field applied along the magnetization of ferrite (z-axis) diminishes the effect of anisotropy. Thereby we assume that only the anisotropy field is strong to obtain

$$\mu \approx 1, \quad \mu_{xx} \approx 1, \tag{20}$$

$$\alpha \approx 2\arctan\sqrt{\frac{H_0}{H_0 - 4\pi M}}. \tag{21}$$

On the one hand, formula (20) shows that the eigenfrequencies of the filled resonator coincide with those of the empty resonator. On the other hand, formula (21) gives a remarkable possibility to change the vertex angle of the rhombus by an external magnetic field.

While integrable billiards, for example, a rectangular billiard, and chaotic billiards, for example, the Sinai billiard, represent two extreme billiards, the rhombus with at least one angle in the form $m\pi/n(m \neq 1)$ belongs to the pseudointegrable systems [15–18]. Date *et al.* [19] have numerically shown that these systems have a spectral statistics which cannot be presented as a mixture of Poisson statistics (the integrable systems) and Wigner-Dyson one (chaotic systems). An integrable billiard generally leads to uncorrelated energy levels (Poisson statistics) and a chaotic billiard corresponds to the Wigner-Dyson statistics [4]. Rhombus billiards are peculiar, as they are pseudointegrable systems and for this reason their statistical properties belong to another class of universality [20]. Using the boundary element method, Shudo and Shimizu [17] have found that the data of nearest-neighbour level spacing distributions are described by the Brody distribution

$$P_\beta(s) = As^\beta \exp\left[-as^{1+\beta}\right],$$
$$A = (1+\beta)a, \quad a = \left[\Gamma\left\{\frac{2+\beta}{1+\beta}\right\}\right]. \tag{22}$$

The Brody distribution is a semiempirical interpolation between Poisson and Wigner-Dyson distributions [21]. Shudo and Shimizu compared the Brody parameter β for irrational angles with that for the rational angles and found that the differences observed are very small. However, spectral rigidity reveals that the possibility of the rationality of the vertex angle is an important signature of the level spacing distribution. Grémaud and Jain [22] and Bogomolny *et al.* [23] performed extensive numerical calculations for $\alpha = \pi/n$, $n = 5, 7, \ldots, 31$ and typically 20000 [23] and even 34000 [22] eigenvalues, and found that the nearest-neighbour level spacing distribution is described by the semi-Poisson statistics

$$P_\beta(s) = 4se^{-2s}. \tag{23}$$

Biswas and Jain [16] numerically considered the amplitude distributions $P(\psi)$ of the eigenfunctions of the $\pi 3$ rhombus. They have found that eigenfunctions even-even and even-odd relative to the x, y axes (fig. 2) for this rhombus $P(\psi)$ have a Gaussian distribution in contrast to the odd-odd and odd-even eigenfunctions which are identical to those of an equilateral triangle. From the equations $\alpha = 2\pi/3$ and (21), we obtain that this integrable case of the anisotropic square resonator takes place if

$$H_0 = 2\pi M, \tag{24}$$

otherwise the resonator is a non-integrable one.

* * *

AFS is grateful to S. R. JAIN, for encouraging discussions and critics related to the rhombus billiards. This work has been supported by the Russian Foundation for Basic Research Grant 01-02-16077 and the Royal Swedish Academy of Sciences (KVA) (AFS).

REFERENCES

[1] MCDONALD S. W. and KAUFMAN A. N., *Phys. Rev. Lett.*, **42** (1979) 1189.
[2] BOHIGAS O., GIANNONI M.-J. and SCHMIT C., *Phys. Rev. Lett.*, **52** (1984) 279.
[3] MCDONALD S. W. and KAUFMAN A. N., *Phys. Rev. A*, **37** (1988) 3067.
[4] STÖKMANN H.-J., *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge) 1999.

[5] SAICHEV A. I., private communication.
[6] STÖKMANN H.-J. and STEIN J., *Phys. Rev. Lett.*, **64** (1990) 2215; **68** (1992) 2867.
[7] SRIDHAR S., *Phys. Rev. Lett.*, **67** (1991) 785.
[8] SRIDHAR S. and HELLER E. J., *Phys. Rev. A*, **46** (1992) R1728.
[9] ALT H., GRÄF H.-D., HARNEY H. L., HOFFERBERT R., LENGELER H., RICHTER A., SCHARDT P. and WEIDENMÜLLER H. A., *Phys. Rev. Lett.*, **74** (1995) 62.
[10] ALT H., DEMBOVSKI C., GRÄF H.-D., HOFFERBERT R., REHFELD H., RICHTER A. and SCHMIDT C., *Phys. Rev. E*, **60** (1999) 2851.
[11] SO P., ANLAGE S. M., OTT E. and OERTER R. N., *Phys. Rev. Lett.*, **74** (1995) 2662.
[12] STOFFREGEN U., STEIN J., STÖKMANN H.-J., KUŚ M. and HAAKE F., *Phys. Rev. Lett.*, **74** (1995) 2666.
[13] KITTEL C., *Phys. Rev.*, **73** (1948) 155.
[14] LAX B. and BUTTON K. J., *Microwave Ferrites and Ferrimagnetics* (New York) 1962.
[15] ECKHARD B., FORD J. and VIVALDI F., *Physica D*, **13** (1984) 339.
[16] BISWAS D. and JAIN S. R., *Phys. Rev. A*, **42** (1990) 3170.
[17] SHUDO A. and SHIMIZU Y., *Phys. Rev. E*, **47** (1993) 54.
[18] JAIN S. R. and LAWANDE S. V., *Proc. Indian Natl. Sci. Acad.*, **61** (1995) 275.
[19] DATE G., JAIN S. R. and MURTHY M. V. N., *Phys. Rev. E*, **51** (1995) 198.
[20] PARAB H. D. and JAIN S. R., *J. Phys. A*, **29** (1996) 3903.
[21] BRODY T. A., *Lett. Nuovo Cimento*, **7** (1973) 482.
[22] GRÉMAUD B. and JAIN S. R., *J. Phys. A*, **31** (1998) L637.
[23] BOGOMOLNY E. B., GERLAND U. and SCHMIT C., *Phys. Rev. E*, **59** (1999) R1315.