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Differential constraints and exact solutions of nonlinear diffusion equations

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Abstract
The differential constraints are applied to obtain explicit solutions of nonlinear diffusion equations. Certain linear determining equations with parameters are used to find such differential constraints. They generalize the determining equations used in the search for classical Lie symmetries.

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1. Introduction

Differential constraints were introduced originally in the theory of partial differential equations of the first order. In particular, Jacobi used differential constraints to find the total integral of nonlinear equation

$$F\left(x_1, \ldots, x_n, z, z_{x_1}, \ldots, z_{x_n}\right) = 0.$$

König applied them to the equation of the second order [1]. They required that the corresponding overdetermined system was compatible. The general theory of overdetermined systems was developed by Delassus, Riquier, Cartan, Ritt, Kuranishi, Spencer and others; one can find references in [2]. Now the applications of overdetermined systems include diverse fields such as differential geometry, continuum mechanics and nonlinear optics.

The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$F^1 = 0, \ldots \qquad F^m = 0 \tag{1}$$

be enlarged by appending additional differential equations (differential constraints)

$$h_1 = 0, \ldots \qquad h_p = 0 \tag{2}$$

such that the overdetermined system (1), (2) satisfies some conditions of compatibility.

One can derive many exact solutions of partial differential equation by means of differential constraints. It was particularly shown in [3] that some soliton solutions can be found using differential constraints. Olver and Rosenau [4], Olver [5], Kaptsov [3], Levi and Winternitz [6] show that many reduction methods such as non-classical symmetry groups, partial invariance, separation of variables and the Clarkson–Kruskal direct method can be included into the method of differential constraints. In practice, methods based on the Riquier–Ritt theory of overdetermined systems of partial differential equations may be difficult. The problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of the original equations.

Recently, a new method was proposed for finding differential constraints, which uses linear determining equations. These equations are more general than the classical determining equations for Lie generators [7] and depend on some parameters. Given an evolution equation

$$u_t = F(t, x, u, u_1, \ldots, u_n) \tag{3}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$, then according to [8] the linear determining equation corresponding to (3) is of the form

$$D_t(h) = \sum_{i=0}^{n} \sum_{k=0}^{i} b_{ik} D_x^{i-k} \left(F_{u_{n-k}}\right) D_x^{n-i}(h) \qquad b_{ik} \in \mathbb{R}. \tag{4}$$

Here and throughout D_t, D_x are the operators of total differentiation with respect to t and x. Equality (4) must hold for all solutions of (3). The function h may depend on $t, x, u, u_1, \ldots, u_p$. The number p is called the order of the solution of equation (4). If we have some solution h, then the corresponding differential constraint is

$$h = 0. \tag{5}$$

It was also shown in [8] that equations (4) and (5) constitute the compatible system. Thus we sketch the derivation of some solutions to the evolution equation (4):

(I) Find solutions of the linear determining equations (4).
(II) Fixing the function h, we obtain differential constrain (5).
(III) Find the general solution of (5) which includes some arbitrary functions a_i depending on t.
(IV) Substitute the general solution into (3). It leads to ordinary differential equations for functions a_i.
(V) Solve the ordinary differential equations and obtain a solution of the evolution equation (3).

In this paper, we start with determination of the solutions of linear determining equations of the second and third orders for the nonlinear diffusion equation

$$u_t = (u^k u_x)_x + f(u).$$

These solutions exist only if f belongs to the special forms. Then we use the obtained functions h to find solutions of the last equation. In the final section, we derive exact solutions of two-dimensional equation

$$u_t = \Delta \ln(u).$$

2. Linear determining equations for differential constraints

In this section, we briefly discuss the method of linear determining equations [8]. For simplicity we will consider an equation of the second order,

$$u_t = F(t, x, u, u_1, u_2). \tag{6}$$

According to [3] a differential constraint

$$h = u_n + g(t, x, u, u_1, \dots, u_{n-1}) = 0 \tag{7}$$

and equation (6) satisfy the compatibility conditions if and only if

$$D_t h|_{[E]\cap[H]} = 0. \tag{8}$$

The latter fact means that $D_t h$ vanishes on $[E] \cap [H]$. We denote by $[E]$ equation (6) and its differential consequences with respect to x. The constraint (7) and its differential consequences with respect to x are denoted by $[H]$.

A manifold H given by equation (7) is called an invariant manifold of (6) if the function h satisfies (8). It can be shown that (8) is equivalent to the following condition,

$$D_t(h)|_{[E]} = F_{u_2} D_x^2(h) + \left(F_{u_1} + nD_x\left(F_{u_2}\right)\right) D_x(h) + \Bigg(F_u + nD_x\left(F_{u_1}\right) - h_{u_{n-1}} D_x\left(F_{u_2}\right) + \frac{n(n-1)}{2} D_x^2\left(F_{u_2}\right) + F_{u_2} h h_{u_{n-1}u_{n-1}} - 2F_{u_2} D_x\left(h_{u_{n-1}}\right)\Bigg) h \tag{9}$$

with $n \geqslant 4$.

Indeed, it is easy to see that

$$D_t(h)_{|[E]} \simeq D_x^n(F) + h_{u_{n-1}} D_x^{n-1}(F) + h_{u_{n-2}} D_x^{n-2}(F). \tag{10}$$

Here and throughout we write $\alpha \simeq \beta$ to indicate that there are no terms including u_n, u_{n+1}, u_{n+2} in the difference $\alpha - \beta$. Since $n \geqslant 4$, the terms on the right-hand side of (10) can be represented as follows:

$$D_x^n(F) \simeq F_{u_2} u_{n+2} + \left[F_{u_1} + nD_x\left(F_{u_2}\right)\right] u_{n+1} + \left[F_u + nD_x\left(F_{u_1}\right) + \frac{n(n-1)}{2} D_x^2\left(F_{u_2}\right)\right] u_n$$

$$h_{u_{n-1}} D_x^{n-1}(F) \simeq h_{u_{n-1}} \left[u_n\left(F_{u_1} + (n-1)D_x\left(F_{u_2}\right)\right) + u_{n+1} F_{u_2}\right]$$

$$h_{u_{n-2}} D_x^{n-2}(F) \simeq u_n F_{u_2} h_{u_{n-2}}.$$

Hence (10) can be written as

$$D_t(h)_{|[E]} \simeq F_{u_2} u_{n+2} + u_{n+1}\left[F_{u_1} + nD_x\left(F_{u_2}\right) + F_{u_2} h_{u_{n-1}}\right] + \Bigg[F_u + \frac{n(n-1)}{2} D_x^2\left(F_{u_2}\right) + nD_x\left(F_{u_1}\right) + h_{u_{n-1}}\left(F_{u_1} + nD_x\left(F_{u_2}\right)\right) + F_{u_2} h_{u_{n-2}}\Bigg] u_n.$$

It is easy to see that

$$D_x(h) \simeq u_{n+1} + u_n h_{u_{n-1}}$$

$$D_x^2(h) \simeq u_{n+2} + u_{n+1} h_{u_{n-1}} + u_n\left[h_{u_{n-2}} + 2D_x\left(h_{u_{n-1}}\right) - u_n h_{u_{n-1}u_{n-1}}\right].$$

Hence the difference

$$D_t(h)_{|[E]} - F_{u_2} D_x^2(h) - \left[F_{u_1} + nD_x\left(F_{u_2}\right)\right] D_x(h)$$

contains no terms with u_{n+2} and u_{n+1}. A direct calculation shows that there are no terms containing u_n in the expression for the function

$$\gamma = D_t(h)_{|[E]} - M(h)$$

where

$$M(h) = F_{u_2} D_x^2(h) + \left[F_{u_1} + n D_x\left(F_{u_2}\right)\right] D_x(h) + \Bigg[F_u + n D_x\left(F_{u_1}\right) - h_{u_{n-1}} D_x\left(F_{u_2}\right) + \frac{n(n-1)}{2} D_x^2\left(F_{u_2}\right) + F_{u_2} h h_{u_{n-1}u_{n-1}} - 2F_{u_2} D_x\left(h_{u_{n-1}}\right)\Bigg] h$$

that is, $\gamma \simeq 0$. We claim that γ is equal to zero. Since H is an invariant manifold, it follows that

$$M(h) + \gamma = 0$$

on the set $[E] \cap [H]$. Clearly, $M(h)$ vanishes there, therefore

$$\gamma_{|[E]\cap[H]} = 0.$$

Since γ is independent of $u_t, u_{tx}, \ldots$, we can rewrite the last equality as follows:

$$\gamma_{|[H]} = 0.$$

As shown above, $\gamma \simeq 0$, that is γ can depend only on $u_{n-1}, u_{n-2}, \ldots$. On the other hand, h depends on u_n. Hence γ is equal to zero.

It is clear that equation (9) is difficult to solve, therefore, in place of the nonlinear equation, we propose to use linear equations of a similar kind in the search for invariant manifolds. This leads to the following definition.

The equation of the form

$$D_t(h)_{|[E]} = F_{u_2} D_x^2(h) + \left(c_1 F_{u_1} + c_2 D_x\left(F_{u_2}\right)\right) D_x(h) + \left(c_3 F_u + c_4 D_x\left(F_{u_1}\right) + c_5 D_x^2\left(F_{u_2}\right)\right) h \tag{11}$$

is called the linear determining equation corresponding to (6). Here $c_1, \ldots, c_5$ are some constants.

Obviously, if h satisfies (11), then $h = 0$ is an invariant manifold of (6). The classical determining equations used in the search for infinitesimal operators [7] are special cases of (11), with $c_1 = c_3 = 1, c_2 = c_4 = c_5 = 0$. However, the method of conditional symmetries [9] of Bluman and Cole can give other invariant manifolds. In fact, they require that the function

$$h = \xi(t, x, u)u_t + \tau(t, x, u)u_x + \eta(t, x, u)$$

be a solution of equation (8). Although it is difficult to seek the general solution of (8), some authors found interesting solutions [6, 9, 10].

King, Galaktionov and Posashkov derived many solutions of equation (7) constructing finite-dimensional functional linear spaces invariant under a given differential operator [11–16]. They seek solutions of the form

$$u(t, x) = \sum_{i=0}^{N} f_i(t) g_i(x).$$

In fact, the functions g_i satisfy linear ordinary differential equations [17]. Moreover, it is necessary to rewrite equation (6) in the form which includes linear and quadratic terms or cubic nonlinearities [14]. This approach will be called invariant subspace method. When we apply the linear determining equations we can use the original equation (6) without restrictions on linear differential constraints.

3. Solutions of linear determining equations

The nonlinear diffusion equation

$$u_t = (Q(u)u_x)_x + f(u) \tag{12}$$

often arises in the description of various physical processes. The group classification of the equation has been carried out in [18]. In physical applications Q is usually taken to be a power function. In this section, we consider the equation

$$u_t = (u^q u_x)_x + f(u) \tag{13}$$

where f is a differentiable function, $q \neq 0$. If $q = -2$, $f = u$ or $f = \text{const}$ then equation (8) can be linearized. We shall not discuss this case here.

The main goal of this section is to find solutions of (11). The linear determining equation, which corresponds to (13), is

$$D_t h|_{[E]} = u^q D_x^2 h + b_1 q u_x u^{q-1} D_x h + \left(b_3 q u^{q-1} u_{xx} + b_2 q(q-1) u^{q-2} u_x^2 + b_4 f_u\right) h \tag{14}$$

where $b_1, \ldots, b_4 \in \mathbb{R}$. We shall seek solutions to (14) in the form

$$h = u_n + g(t, x, u, \ldots, u_{n-1})$$

where $n \geqslant 2$, $u_k = \frac{\partial^k u}{\partial x^k}$. The method for finding solutions is very similar to the standard procedure applied in the group analysis of differential equations [19] and only one of all possibilities is described here for the sake of brevity.

We set $n = 2$. First, let us express all t-derivatives in (14) using (13). As a result, the left-hand side of (14) becomes a polynomial with respect to u_3, u_2. The polynomial must identically vanish. Collecting similar terms we obtain the following relations for the coefficients of u_3 and u_2^2

$$q(b_1 - 4) = 0 \qquad u g_{u_1 u_1} + q(b_3 - 3) = 0.$$

Thus $b_1 = 4$ and g can be represented as follows,

$$g = \frac{(3 - b_3)q}{2u} u_1^2 + a(u, t, x) u_1 + g_1(u, t, x).$$

Here a and g_1 must be functions of u, t and x alone. Collecting the coefficients of $u_2 u_1^2$ and $u_2 u_1$, we have the equations

$$\begin{aligned} &2b_2 q - 2b_2 - b_3^2 q + b_3 q + 4b_3 - 6q = 0 \\ &2u a_u + q(b_3 + 1)a = 0. \end{aligned} \tag{15}$$

It follows from the last equation that

$$a = a_1(t, x) u^{-\frac{(1+b_3)}{2} q}$$

where a_1 is a function of t and x. Next we consider the coefficient u_1^3 and obtain equation

$$4b_2 q - 4b_2 + b_3^2 q - 4b_3 q + 2b_3 - 9q + 6 = 0. \tag{16}$$

From (15) and (16) it follows that $b_3 = 1$ or $b_3 = \frac{q+2}{q}$.

Assuming $b_3 = 1$, we obtain $b_2 = \frac{3q-2}{q-1}$. The coefficient of u_2 gives us equation

$$u^{q+2}\left(2a_{1_x} + f_u(b_4 - 1)\right) + u^{2q+1} q g_1 = 0.$$

The equation enables us to express

$$g_1 = \frac{1}{q} u^{1-q} \left(f_u(1 - b_4) - 2a_{1_x}\right).$$

The coefficient of u_1^2 yields Euler equation

$$u^3(1-b_4)f_{uuu} + u^2(2-qb_4-2b_4)f_{uu} - uq^2 f_u + q^2 f = 0.$$

Consider for simplicity the case $b_4 = 1$. It is easy to see that the last equation has two types of solutions,

$$f = ku + nu^{-q} \qquad q \neq -1$$

or

$$f = ku + nu \ln u \qquad q = -1$$

where k, n are arbitrary constants. Let us focus on $f = ku + nu^{-q}$. It follows from above calculations and equation (14) that

$$u^{q+1}\left(-qa_{1_t} - 3u^q q a_{1_{xx}} - 4u^q a_{1_{xx}} + kq^2 a_1\right)u_1 + nq^2 a_1 u_1 + 2u^{q+2}\left(a_{1_{tx}} - u^q a_{1_{xxx}} - kqa_{1_x}\right) + 2una_{1_x} = 0. \tag{17}$$

From (17) we have $na_1 = 0$.

If $a_1 = 0$, then the solution of (14) is $h = u_2 + qu_1^2/u$. If $n = 0$ and $q \neq -\frac{4}{3}$, then one easily computes

$$a_1 = (rx+s)\,\mathrm{e}^{kqt}$$
$$h = u_2 + q\frac{u_1^2}{u} + \left((rx+s)u_1 - \frac{2}{q}ur\right)u^{-q}\,\mathrm{e}^{kqt}.$$

In the case $q = -\frac{4}{3}$, we obtain

$$a_1 = (rx^2+sx+p)\,\mathrm{e}^{-\frac{4}{3}kt}$$
$$h = u_2 - \frac{4u_1^2}{3u} + \left((rx^2+sx+p)u_1 + \frac{3}{2}u(2rx+s)\right)u^{4/3}\,\mathrm{e}^{-\frac{4}{3}kt}.$$

It can be shown that the functions h lead to invariant solutions of the corresponding equation (13).

We omit here for the sake of brevity intermediate calculations and give the list of solutions to equation (14):

(1) If $q = -1$ and $f = su + ru\ln(u)$, then

$$h = u_2 - \frac{u_1^2}{u}.$$

(2) If $q \neq -1$ and $f = su + ru^{-q}$, then

$$h = u_2 + \frac{qu_1^2}{u}.$$

(3) If $q = -2$ and $f = su + ru^3$, then

$$h = u_2 - \frac{3u_1^2}{2u}.$$

(4) If $q = 1$ and $f = ru$, then

$$h = u_2 + s\,\mathrm{e}^{rt}u^{-2}u_1 + r/3.$$

(5) If q is an arbitrary constant and $f = su + ru^{1-q}$, then

$$h = u_2 - \frac{(q-1)u_1^2}{u}.$$

Here $r, s \in \mathbb{R}$.

We did not include functions h that correspond to invariant solutions of equation (13).

If we look for solutions to equation (14), which depend on third derivative, then we will obtain the following list:

(1) If q is an arbitrary constant and $f = su + ru^{1-q} + \frac{n(q+1)}{q^2}u^{q+1}$, then

$$h = u_3 + \frac{3(q-1)}{u}u_1u_2 + (q^2 - 3q + 2)\frac{u_1^3}{u^2} + nu_1.$$

(2a) If $q \neq 1$ and $f = nu + \frac{r}{q}u^{q+1}$, then

$$h = u_3 + \frac{(3q-1)}{u}u_1u_2 + q(q-2)\frac{u_1^3}{u^2} + ru_1. \tag{18}$$

(2b) If $q = -2$ or $q = -\frac{4}{3}$ and $f = nu + \frac{r}{q}u^{q+1} + mu^{q+3}$ then h is also given by (13).

(3) If $q = -\frac{1}{2}$ and $f = mu$, then

$$h = u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + r\,\mathrm{e}^{-3mt/2}u^{5/2} + s\,\mathrm{e}^{mt/2}u^{1/2}.$$

(4) If $q = -\frac{3}{2}$ and $f = nu + mu^{5/2}$, then

$$h = u_3 - \frac{15u_1u_2}{2u} + \frac{35u_1^3}{4u^2} + r\,\mathrm{e}^{-3nt/2}u^{5/2}.$$

(5) If $q = -\frac{1}{2}$ and $f = mu - 2ku^{1/2}$, then

$$h = u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + ku_1 + s\,\mathrm{e}^{mt/2}u^{1/2}.$$

(6) If $q = -\frac{3}{2}$ and $f = nu$, then

$$h = u_3 - \frac{15u_1u_2}{2u} + \frac{35u_1^3}{4u^2} + s\,\mathrm{e}^{-7n/2t}u^{9/2} + r\,\mathrm{e}^{-3nt/2}u^{5/2}.$$

(7) If $q = -1$ and $f = mu$, then

$$h = u_3 - \frac{4u_1u_2}{u} + \frac{3u_1^3}{u^2} + s\,\mathrm{e}^{-2mt}u^2u_1.$$

Here $r, s, m, n \in \mathbb{R}$. We also did not include functions h leading to invariant solutions of (13).

We have found solutions of (14) which correspond to special functions f. Most of these functions lead to the diffusion equations (13) that have been studied in [11–16]. However, we did not transform equation (13). Moreover, the solutions of (14) generate new nonlinear differential constraints.

4. Solutions of diffusion equations

In this section, we shall use the functions obtained above to construct solutions of diffusion equations (13). One can apply the method described in the introduction.

We first take the function $h = u_2 + qu_1^2/u$, where $q \in \mathbb{R}$, corresponding to some cases mentioned above. Simply by equating this function to zero, we obtain the differential constraint

$$u_2 + qu_1^2/u = 0. \tag{19}$$

Equation (19) has two types of solutions,

$$u = (c_1x + c_2)^{\frac{1}{q+1}} \qquad q \neq -1 \tag{20}$$

$$u = c_1\,\mathrm{e}^{c_2x} \qquad q = -1 \tag{21}$$

where c_1, c_2 are functions of t.

If we substitute the representation (21) into equation

$$u_t = (u_x/u)_x + ku \ln u \tag{22}$$

then this leads us to differential equations for c_1, c_2. From these equations it is easy to find c_1, c_2 and obtain the following solution of (22),

$$u = s_1 \exp(s_2 x \, \mathrm{e}^{kt}) \qquad s_1, s_2 \in \mathbb{R}.$$

Obviously, this is a regular solution. If $s2 < 0$, then $u \longrightarrow 0$ as $x \to \infty$.

This solution can be found by means of easy generalization of the invariant subspace method. Indeed, the transformation $u = \mathrm{e}^v$ leads (22) to the form

$$v_t = \mathrm{e}^{-v} v_{xx} + kv.$$

Obviously, a linear subspace $W = \{1, x\}$ generated by functions 1 and x is invariant under nonlinear operator $A(v) = \mathrm{e}^{-v} v_{xx}$. It follows that equation (22) has above solution.

Substituting (20) into equation

$$u_t = (u^q u_x)_x + su + ru^{-q}$$

we find the solution

$$u = \mathrm{e}^{st} \left(ax + b - \frac{r}{s(q+1)} \exp(-s(q+1)t) \right)^{\frac{1}{q+1}}$$

with $a, b \in \mathbb{R}$. If q satisfies the condition $-1 < q$, then this solution is regular. The last solutions can be found by generalized version of the invariant subspace method.

Now let us consider some differential constraints of the third order. We start with the equation

$$u_t = (u^q u_x)_x + su + ru^{1-q} + n\frac{q+1}{q^2} u^{q+1} \qquad n \in \mathbb{R}. \tag{23}$$

As explained above, this equation is compatible with the differential constraint

$$u_3 + 3(q-1)\frac{u_1 u_2}{u} + (q^2 - 3q + 2)\frac{u_1^3}{u^2} + nu_1 = 0. \tag{24}$$

By a change of variable $v = u^q$ one may rewrite (23), (24) in the following way:

$$v_t = v v_{xx} + \frac{1}{q} v_x^2 + n\frac{q+1}{q} v^2 + sqv + rq \tag{25}$$

$$v_3 + nv_1 = 0. \tag{26}$$

If $n \geqslant 0$, then we have Galactionov's solutions [15] obtained by means of the invariant subspace method.

We will give one example of explicit solution of equation (25), with $n = -1, q = -1$ and $r = 0$. This solution has the representation

$$v(t, x) = a + b\,\mathrm{e}^x + c\,\mathrm{e}^{-x} \tag{27}$$

and the functions a, b, c are

$$a = a_1 \sin(pr_1 \mathrm{e}^{r_1 t} + m)\, \mathrm{e}^{r_1 t} / \cos(pr_1 \mathrm{e}^{r_1 t} + m)$$
$$b = a_2 \mathrm{e}^{r_1 t} / \cos(pr_1 \mathrm{e}^{r_1 t} + m)$$
$$c = a_3 \mathrm{e}^{r_1 t} / \cos(pr_1 \mathrm{e}^{r_1 t} + m)$$

where $a_1 = pr_1^2, a_3 = p^2 r_1^4 / 4a_2, r_1 = -s$ and p, a_2, m are arbitrary constants.

Now let us consider the special case of equation (23)

$$u_t = (u^{-1/2}u_x)_x + mu - 2k\sqrt{u} \qquad m, k \in \mathbb{R} \tag{28}$$

and the differential constraint

$$u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + ku_1 + s\,\mathrm{e}^{mt/2}\sqrt{u} = 0. \tag{29}$$

Using equation (28), one can write (29) as

$$(\ln u)_{tx} + s\,\mathrm{e}^{mt/2}u^{-1} = 0.$$

Replacing $\ln(\mathrm{e}^{mt/2}u^{-1})$ by w, the last equation is replaced by the Liouville equation

$$w_{tx} = s\,\mathrm{e}^{w}.$$

Since the general solution of the Liouville equation is

$$w = \ln \frac{2T'X'}{s(T+X)^2}$$

it gives the representation

$$u = \frac{s(T+X)^2}{2T'X'}\,\mathrm{e}^{mt/2}$$

where T and X are the arbitrary functions of t and x respectively.

It is possible to show that the last representation leads to the solution

$$u = (a_1 + a_2\,\mathrm{e}^{mt/2})^2 \tag{30}$$

where a_1, a_2 are functions of x. This yields the following system for a_1 and a_2:

$$a_{1_{xx}} + a_1^2 m\big/2 - a_1 k = 0 \tag{31}$$

$$a_{2_{xx}} + a_1 a_2 m/2 - a_2 k = 0. \tag{32}$$

In general, it is possible to express a_1 in terms of the Weierstrass function $\wp$ and a_2 in terms of Lamé's function [20] . However, if $m = 12$ and $k = 4$, then the functions

$$a_1 = \frac{1}{\cosh^2(x)}$$

$$a_2 = \frac{a}{\cosh^2(x)} + \frac{b}{\cosh^2(x)}\left(\frac{\sinh 4x}{32} + \frac{\sinh 2x}{2} + \frac{3x}{8}\right) \qquad a, b \in \mathbb{R}$$

satisfy equations (31) and (32).

King [11] found solutions of the multidimensional version of (28), with $m = k = 0$, namely

$$u_t = \nabla \cdot (u^{-1/2}\nabla u).$$

The solutions have the form

$$u = (a_1 + a_2 t)^2$$

where a_1 and a_2 are some functions of $x \in \mathbb{R}^n$. Galaktionov and Posashkov [16] explained existence of these solutions by means of the invariant subspace method. The same is true of the representation (30).

According to our results in the previous section, as $k = 0$, equation (28) is compatible with the differential constraint

$$u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + r\,\mathrm{e}^{-3mt/2}u^{5/2} + s\,\mathrm{e}^{mt/2}\sqrt{u} = 0. \tag{33}$$

Using equation (28), one can write (33) as

$$(\ln u)_{tx} + r\,\mathrm{e}^{-3mt/2}u + s\,\mathrm{e}^{mt/2}u^{-1} = 0. \tag{34}$$

Replacing $\ln(\mathrm{e}^{-3mt/2}u)$ by w in (34) yields

$$w_{tx} + \mathrm{e}^{w} + sr\,\mathrm{e}^{-w-mt} = 0.$$

If we set $s = 0$, then from the last equation we find the following representation,

$$u = -\frac{2X'T'}{(X+T)^2}\,\mathrm{e}^{3mt/2}$$

where T and X are the arbitrary functions of t and x respectively. Substituting this representation into (28) leads to equation

$$\begin{aligned}\sqrt{-2/r}\,\mathrm{e}^{3mt/4}&(-m(T')^{1/2}(X+T) - 2(T')^{-1/2}T''(X+T) + 4(T')^{3/2})\\ &= -2X'''(X')^{-3/2}(X+T)^2 + 8X''(X')^{-1/2}(X+T)\\ &\quad - 8(X')^{3/2} + (X'')^2(X')^{-5/2}(X+T)^2.\end{aligned}$$

It can be shown that the functions X and T satisfy the previous relation if and only if they are solutions of the equations

$$(X')^3 = (c_3X^3 + c_2X^2 + c_1X + c_0)^2 \tag{35}$$

$$(T')^3 = A(-c_3T^3 + c_2T^2 - c_1X + c_0)^2 \tag{36}$$

where c_3, c_2, c_1 and c_0 are arbitrary constants, $A = (-2r)^{1/3}$.

The solutions of (35) and (36) can be expressed in terms of the Weierstrass function $\wp$ [21]. Indeed, one can write (35) and (36) as

$$(X')^3 = (c_3(X-\alpha_1)(X-\alpha_2)(X-\alpha_3))^2 \tag{37}$$

$$(T')^3 = A(-c_3(T+\alpha_1)(T+\alpha_2)(T+\alpha_3))^2. \tag{38}$$

Replacing $X - \alpha_1$ by $1/Y$ in (37) yields

$$(Y')^3 + B^2(Y-b_1)^2(Y-b_2)^2 = 0$$

where $B = c_3(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)$, $b_1 = \frac{1}{\alpha_2-\alpha_1}$, $b_2 = \frac{1}{\alpha_3-\alpha_1}$. Introducing new function Z such that $Z^3 = B(Y-b_1)(Y-b_2)$, we obtain equation

$$(Z')^2 + \frac{4B}{9}Z^3 + \frac{B^2(b_1-b_2)^2}{9} = 0. \tag{39}$$

The solutions of the last equation are expressed in terms of the Weierstrass function $\wp$. Applying the above process to (36), we obtain an equation such as (39). We think that it is difficult to obtain the last solution by means of the invariant subspace method because the Weierstrass function satisfies the nonlinear equation. The invariant subspaces are generated by solutions of linear differential equations. On the other hand, we applied the differential constraints of the third order in contradistinction to the method of conditional symmetries where the constraints of the first order are considered.

If we fail in integrating the differential constraints explicitly, then we can do this numerically [3]. We shall omit here other cases of the differential constraints and discuss briefly in the next section the two-dimensional equation.

5. Two-dimensional equation

We consider here the fast diffusion equation

$$u_t = \Delta \ln(u). \tag{40}$$

Some applications of this equation can be found in [22]. If we set $u = 1/v$, we obtain

$$v_t = v^2 \Delta \ln(v). \tag{41}$$

Galaktionov [14] used the above representation to find exact solutions of (40). King [23] and Pukhnachev [24] also obtained some interesting solutions.

One of these solutions is the travelling wave given by

$$v = 1 + c \exp(mx + ny - (m^2 + n^2)t) \tag{42}$$

where c, m and n are arbitrary constants. Obviously, this is invariant solution.

We can derive other explicit solutions using invariant subspaces [14] or linear differential constraints. Galaktionov [14] obtained the following representation,

$$v = s_0 + s_1 \cos(x) + s_2 \sin(x) + s_3 \,\mathrm{e}^y + s_4 \,\mathrm{e}^{-y}$$

where the functions $s_i(t)$ satisfy ordinary differential equations

$$s_0' + s_1^2 + s_2^2 - 4s_3 s_4 = 0 \tag{43}$$
$$s_1' + s_1 s_0 = 0$$
$$s_2' + s_2 s_0 = 0 \tag{44}$$
$$s_3' - s_3 s_0 = 0 \tag{45}$$
$$s_4' - s_4 s_0 = 0.$$

Now we find explicit solution of the last system. Because of (44) and (45) we have

$$s_3'/s_2' + s_3/s_2 = 0.$$

This yields

$$s_2 = c_2/s_3 \qquad c_2 \in \mathbb{R}.$$

By arguments similar to that used above, we have

$$s_1 = c_1/s_3 \qquad s_4 = c_4 s_3 \qquad c_1, c_4 \in \mathbb{R}.$$

Substituting this into (43) leads to

$$s_0' + \left(c_1^2 + c_2^2\right) s_3^{-2} - 4c_4 s_3^2 = 0.$$

From (45) we express the function s_0 and obtain

$$(\ln s_3)'' = a s_3^2 - b s_3^{-2} \tag{46}$$

with $a = 4c_4$, $b = c_1^2 + c_2^2$.

Setting $a = b = 1$ one can derive two elementary solutions

$$s_3 = \tanh(t) \qquad s_3 = \tan(t).$$

In general, the solutions of (46) can be expressed in terms of elliptic functions. It is easy to obtain the correspondent function u.

Using differential constraints we can seek new solutions of (41). Obviously, the solution (42) satisfies the differential constraints

$$v_x - mv + m = 0 \qquad v_y - nv + n = 0.$$

It is possible to find other constraints that are linear with respect to x, y and v.

It can be shown that the differential constraints

$$v_x + \frac{c_1 - c_0 \tan(t)}{c_0^2 + c_1^2}\left(v - xc_0 - yc_1 + t\left(c_0^2 + c_1^2\right)\right) = c_0$$
$$v_y - \frac{c_0 + c_1 \tan(t)}{c_0^2 + c_1^2}\left(v - xc_0 - yc_1 + t\left(c_0^2 + c_1^2\right)\right) = c_1$$

are compatible with equation (41). Here c_0 and c_1 are arbitrary constants. The solution of (41) corresponding to these constraints is

$$v = c_0 x + c_1 y - \left(c_0^2 + c_1^2\right) t + c_2 \cos(t) \exp(m_1 x + m_2 y + m_3 t). \tag{47}$$

Here c_0, c_1, c_2 are arbitrary constants and

$$m_1 = \frac{c_0 \tan(t) - c_1}{c_0^2 + c_1^2} \qquad m_2 = \frac{c_1 \tan(t) + c_0}{c_0^2 + c_1^2} \qquad m_3 = -\tan(t).$$

Galaktionov [14] constructed the invariant subspace $\mathcal{W}$ generated by the functions 1, $\cos(2x)$, $\cosh(2y)$, $\cos(x)\cosh(y)$. Obviously, the above solution cannot be obtained by means of this invariant subspace.

It is well known that equation (40) is invariant under infinite-dimensional algebra of symmetry [25]. Some solutions of (40) were obtained by means of these symmetries in [18, 23]. It is convenient to apply the complex conjugate variables $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$. We can write equation (40) as

$$u_t = \frac{1}{4} \frac{\partial^2 \ln u}{\partial z \partial \bar{z}}. \tag{48}$$

It is easy to check that (48) is invariant under the transformation

$$z' = A(z) \qquad \bar{z}' = B(\bar{z}) \qquad u' = u/(A_z B_{\bar{z}})$$

where $A(z)$ and $B(\bar{z})$ are arbitrary functions. In other words, if the function $f(t, z, \bar{z})$ is a solution of (48), then $f(t, A(z), B(\bar{z}))A_z B_{\bar{z}}$ also satisfies (48).

For example, from (47) we can construct the following solution of equation (40),

$$u = \frac{A_z \bar{A}_{\bar{z}}}{C + a \cos t \exp(C \tan(t) + ((\bar{d} - d)(A + \bar{A}) + (\bar{d} + d)(-A + \bar{A}))\,\mathrm{i}/2)}$$

where A is an arbitrary function of z, $\bar{A}$ is the complex conjugate function, $C = dA + \bar{d}\bar{A} - 4|A|^2 t$, and $a \in \mathbb{R}$, $d \in \mathbb{C}$.

6. Conclusions

In sections 3 and 4 we have shown how the method of the linear determining equations can be applied to find explicit solutions to nonlinear diffusion equations. We have found exact solutions of these equations, using only the simplest solutions of the linear determining equations. It is interesting to find solutions of the linear determining equations depending on derivatives of higher orders. Shmidt [26, 27] applied this method to other parabolic equations and some systems; application to the elliptic equation is discussed in [8].

In section 5 we have considered the two-dimensional equation. Applications of systems of the linear determining equations to multidimensional equations are discussed in [28]. Solutions of these systems give differential constraints which are compatible with the input equations. In [28], we introduced the linear determining equations to some classes of non-evolution equations as well. However, this is the first step in application of the above method.

Using results of section 4 one can find the following representation,

$$u = (a + b\,\mathrm{e}^{mt/2})^2 \tag{49}$$

of solution of equation

$$u_t = \Delta(u^{1/2}) + mu + nu^{1/2} \qquad m, n \in \mathbb{R}$$

where the functions $a(x, y)$ and $b(x, y)$ must satisfy the system

$$\Delta a = ma^2 + na \qquad \Delta b = mab + nb.$$

It is easy to show that the differential constraint

$$u_{tt} = u_t^2 \big/ 2u + m u_t / 2$$

leads to the representation (49). The interesting reductions of some diffusion equations in several independent variables can be found in [16, 29]. It is important to explain these reductions by means of differential constraints.

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