Floquet-Bloch operator for the Bose-Hubbard model with static field

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This paper deals with the spectral properties of the one-dimensional Bose-Hubbard Hamiltonian amended by an external static field—a model for cold spinless atoms loaded in a quasi-one-dimensional optical lattice and subject to an additional static (for example, gravitational) force. The analysis is performed in terms of the Floquet-Bloch operator, defined as the evolution operator of the system over one Bloch period. Depending on the particular choice of parameters, the spectrum is found to be either regular or chaotic. Moreover, in the chaotic case, the matrix of the Floquet-Bloch operator is well characterized as a random matrix of the circular orthogonal ensemble.

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I. INTRODUCTION

Recently, much attention has been paid to the dynamics of ultracold neutral atoms in optical lattices. Such systems mimic crystalline electrons and allow to study many phenomena of solid state physics with unprecedented experimental control. A beautiful example is provided by Bloch oscillations (BOs) [1], which were observed in experiments [2,3] on dilute gases of cold alkali-metal atoms. Note that a dilute gas implies the absence of atom-atom interactions and, hence, a single-particle approach suffices to describe such a system. The next generation of BO experiments involved a relatively dense gas of atoms, where the atom-atom interaction cannot be neglected any more [4,5]. (From now on we use the term Bloch oscillations in a wide sense, as the system's response to a static force.) A number of effects incompatible with the single-particle picture were observed, thus challenging theory.

Among the theoretical approaches to BOs of interacting (bosonic) atoms one can single out the Gross-Pitaevski equation [6–8] and the Bose-Hubbard model [9–11] as the most important ones. In the former case it is assumed that the N-particle wave function of the system can be expressed in terms of a single macroscopic wave function. It should be noted, however, that this (rather strong) assumption can be questioned. Indeed, as shown in a recent paper [12] (discussing the dynamics of some model system) the validity of the Gross-Pitaevskii equation requires a large filling factor (mean number of bosons per lattice site), which is definitely not the case realized in present-day experiments. The Bose-Hubbard model, on the other hand, does not involve an *a priori* assumption on the structure of the N-particle wave function, and the only limitation of this approach is the validity of the tight-binding ansatz (which, in fact, defines the Bose-Hubbard model). The present paper follows the second approach and, in this sense, extends our previous studies on BOs in correlated bosonic systems [10,11].

An important role in the single-particle theory of BOs is played by the Floquet-Bloch operator (FB operator), defined as the evolution operator of the system over one Bloch period $T_B=\hbar/dF$ (here F is the magnitude of the static force and d is the lattice period) [13]. Indeed, given the solution of the eigenvalue problem for the FB operator, one can express the dynamics of the system in terms of its eigenfunctions. It appears reasonable to study the properties of the FB operator also in the presence of atom-atom interactions. This constitutes the aim of the present work, which is an analysis of the spectrum of the FB operator for variable system parameters. Following this aim, we will confirm our earlier conjecture [11] that BOs with interacting bosons defines a problem of *quantum chaos* and, hence, requires the specific tools of this theory. In turn, the spectral problem considered here contributes an interesting example to the field of quantum chaos, which merits discussion in some detail.

The foundation of quantum chaos is usually traced back to the energy spectra of compound nuclei (see Refs. [14,15] and references therein). A celebrated postulate of the theory states that, because of the extreme complexity of the system, the Hamiltonian of a compound nucleus can be modeled by a random matrix of appropriate symmetry. This conjecture led to a number of theoretical predictions on the statistical properties of the spectrum, well supported by experimental data. The idea of the generic ''randomness'' of a complex system developed further with the systematic research on classical chaos. It was shown that the spectrum of a quantum system, which shows chaotic dynamics in the classical limit, exhibits universal statistical properties. Here, the complexity of the system stems from the underlying classically chaotic dynamics, which (for autonomous systems) may already emerge in only two degrees of freedom. In fact, the quantum analysis of few degree of freedom, classically chaotic systems was the main subject of the theory of quantum chaos during the past two decades. Most recent developments of quantum chaos (facilitated to much extent by the ever growing performance of computers) bring us back to the initial formulation of the problem, where the systems of interest consist of a large number of identical particles and, fundamentally, have no direct classical counterpart. Recent studies have shown that quantum chaos is indeed a generic phenomenon in interacting fermionic systems [one-dimensional (1D) chains of interacting spins, complex atoms, electrons in a random lattice potential, etc.] [16–20]. The results reported in the present paper indicate that quantum chaos is also generic for *bosonic* systems. In particular, we show that, for a certain region in parameter space, the FB operator of the 1D Bose-Hubbard model can be identified with a *random circular orthogonal ensemble (COE) matrix*. It should be stressed in this context that, on the level of the Hamiltonian, our system of interest does not contain any randomness. This distinguishes the present spectral problem from that of disordered many-body systems (see, e.g., Refs. [17,20]).

The structure of the paper is as follows. In Sec. II we recall some single-particle results on BOs and explain why it is of advantage to analyze the FB operator rather than the system Hamiltonian. Section III is devoted to the symmetry properties of the many-body FB operator, induced by the translational symmetry of the Bose-Hubbard model. A preliminary analysis of the spectrum of the FB operator is given in Sec. IV. Depending on the particular choice of the parameters, the spectrum of the FB operator is found to be either regular or irregular (chaotic). These two kinds of spectra are then studied in more detail in Secs. V and VI. The main results of the paper are summarized in the concluding Sec. VII.

II. FLOQUET-BLOCH STATES

In the one-dimensional case, the single-particle Hamiltonian of our system has the form

$$H=\frac{\hat{p}^2}{2}+V(x)+Fx, \tag{1}$$

where $V(x)$ is the periodic potential and F is the magnitude of the static field. (Here and below we use scaled variables, where the period of the potential is 2π, and the mass of the quantum particle as well as the Planck constant are set equal to unity.) First of all, we note that the translational symmetry of the Hamiltonian, apparently broken by the static field, can actually be recovered. Indeed, with the substitution

$$\psi(x,t)=\exp(-iFtx)\tilde{\psi}(x,t) \tag{2}$$

the Schrödinger equation reads $i\partial\tilde{\psi}/\partial t=\tilde{H}(t)\tilde{\psi}$, with $\tilde{H}(t)$ the time-dependent Hamiltonian

$$\tilde{H}(t)=\frac{(\hat{p}-Ft)^2}{2}+V(x). \tag{3}$$

It is easy to see that the Hamiltonian (3) commutes with the lattice shift operator. Then it is a matter of a few lines to show that the Schrödinger equation for the original wave function $\psi(x,t)$ has solutions in the form of translationally invariant Floquet states,

$$\psi(x+2\pi,t)=\exp(-i2\pi\kappa)\psi(x,t), \tag{4}$$

$$\psi(x,t+1/F)=\exp(-iE/F)\psi(x,t),$$

where κ is the quasimomentum ($-1/2<\kappa\leq 1/2$), and E is the quasienergy ($-\pi F<E\leq\pi F$). Analytically, one finds the Floquet-Bloch (or Wannier-Bloch) states (4) by solving the eigenvalue problem for the unitary operator $U(T_B)$ of the system evolution over one Bloch period $T_B=1/F$.

For a detailed formulation of Floquet-Bloch theory, we refer the reader to the review [13], and here restrict ourselves to the simpler tight-binding approximation. Then, the Hamiltonian (1) takes the form

$$H=E\sum_l |l\rangle\langle l|-\frac{J}{2}\left(\sum_l |l+1\rangle\langle l|+\text{H.c.}\right)+F\sum_l 2\pi l|l\rangle\langle l|, \tag{5}$$

where $|l\rangle$ denotes the Wannier state, $\langle x|l\rangle=\phi_l(x)$, associated with the lth well of the periodic potential, and J is the hopping matrix element between two neighboring Wannier states. Substitution (2) amounts to the interaction representation with respect to the static term, and the tight-binding counterpart of the Hamiltonian (3) reads as

$$\tilde{H}(t)=E\sum_l |l\rangle\langle l|-\frac{J}{2}\left(e^{-i2\pi Ft}\sum_l |l+1\rangle\langle l|+\text{H.c.}\right). \tag{6}$$

At last, the explicit form of the Floquet-Bloch solution (4) is given by

$$|\psi(t)\rangle=e^{-iEt}\exp\left(i\frac{J}{F}\sin 2\pi(\kappa+Ft)\right)\sum_l e^{i2\pi l(\kappa+Ft)}|l\rangle, \tag{7}$$

which describes the time evolution of the Bloch state $|\psi_\kappa\rangle=\sum_l\exp(i2\pi\kappa l)|l\rangle$. Thus, Eq. (7) can be rewritten in the form $|\psi_\kappa(t)\rangle\sim|\psi_{\kappa+Ft}\rangle$, which represents one of the possible formulations of the BOs phenomenon.

A remark on boundary conditions is in place here. In what follows we shall consider a system of finite size, $l=1,\ldots,L$, with periodic (cyclic) boundary conditions imposed. As easily shown analytically, in this case the Schrödinger equation with the Hamiltonian (6) has L linearly independent solutions of the same form (7), but with discrete quasimomentum $\kappa=j/L$ $(j=1,\ldots,L)$. We also note the time-reversal symmetry $(t\to -t,\kappa\to -\kappa)$ of the Floquet-Bloch solution. Because of this symmetry, the quasienergies E of the many-body problem are typically doubly degenerate (see the following section).

We conclude this section by introducing the Floquet-Bloch operator

$$U(T_B)=\widehat{\exp}\left(-i\int_0^{T_B}\tilde{H}(t)dt\right),\quad T_B=\frac{1}{F}, \tag{8}$$

where the hat over the exponential denotes time ordering. As an immediate consequence of Eq. (7), operator (8) is a diagonal matrix (in the Wannier state basis) in the single-particle approach, with elements $\exp(-iE/F)$ along the diagonal. Without loss of generality, one can set $E=0$, then $U(T_B)=\hat{1}$. (To avoid confusion with Ref. [13], we stress that the last equation only holds in the tight-binding approximation.)

III. FLOQUET-BLOCH OPERATOR FOR INTERACTING BOSONS

The generalization of the tight-binding model (5) for the multiparticle case is given by the Bose-Hubbard model, with an additional Stark term:

$$H=-\frac{J}{2}\left(\sum_{l}\hat{a}_{l+1}^{\dagger}\hat{a}_{l}+\text{H.c.}\right)+F\sum_{l}2\pi l\hat{n}_{l}+\frac{W}{2}\sum_{l}\hat{n}_{l}(\hat{n}_{l}-1). \tag{9}$$

Here, $\hat{a}_l^{\dagger}$ and $\hat{a}_l$ are the bosonic creation and annihilation operators, $\hat{n}_l=\hat{a}_l^{\dagger}\hat{a}_l$ is the occupation number operator on the lth lattice site, and the parameter W characterizes the interaction between the particles. Since the Bose-Hubbard Hamiltonian conserves the total number of particles N, the wave function of a system of finite size L can be presented in the form

$$|\Psi(t)\rangle=\sum_{\mathbf{n}}c_{\mathbf{n}}(t)|\mathbf{n}\rangle, \tag{10}$$

where the vector $\mathbf{n}$, with L integer components n_l ($\Sigma_l n_l=N$), labels the N-particle bosonic wave function constructed from N Wannier functions. The dimension of the Hilbert space spanned by the Fock states $|\mathbf{n}\rangle$ is

$$\mathcal{N}=(N+L-1)!/N!(L-1)!.$$

The key issue we address in this work is the spectral properties of the FB operator (8), generated by the Hamiltonian $\widetilde{H}(t)$ now given by

$$\widetilde{H}(t)=-\frac{J}{2}\left(e^{-i2\pi Ft}\sum_{l=1}^{L}\hat{a}_{l+1}^{\dagger}\hat{a}_{l}+\text{H.c.}\right)+\frac{W}{2}\sum_{l}\hat{n}_{l}(\hat{n}_{l}-1). \tag{11}$$

The advantage of the FB operator over the Hamiltonian is that it recovers the translational symmetry of the system (see below). Simultaneously, knowledge of $U(T_B)$ suffices to describe the system dynamics by a discrete one-cycle map: $|\Psi(t+T_B)\rangle=U(T_B)|\Psi(t)\rangle$.

Let us investigate the translational symmetry properties of the FB operator $U(T_B)$ [which are actually inherited from the Hamiltonian (11)] in more detail. For $|\mathbf{n}'\rangle=|n_1,n_2,\ldots,n_L\rangle$ an arbitrary Fock state, and the shift operator S defined by

$$S|n_1,n_2,\ldots,n_L\rangle=|n_L,n_1,\ldots,n_{L-1}\rangle, \tag{12}$$

there can be at most L different Fock states $|\mathbf{n}\rangle$ related to each other by the shift operator. Using these states, we construct an alternative basis of the Hilbert space:

$$|\kappa,\mathbf{n}'\rangle=\frac{1}{\sqrt{M}}\sum_{l=1}^{M}\exp(i2\pi\kappa l)S^{l}|\mathbf{n}'\rangle. \tag{13}$$

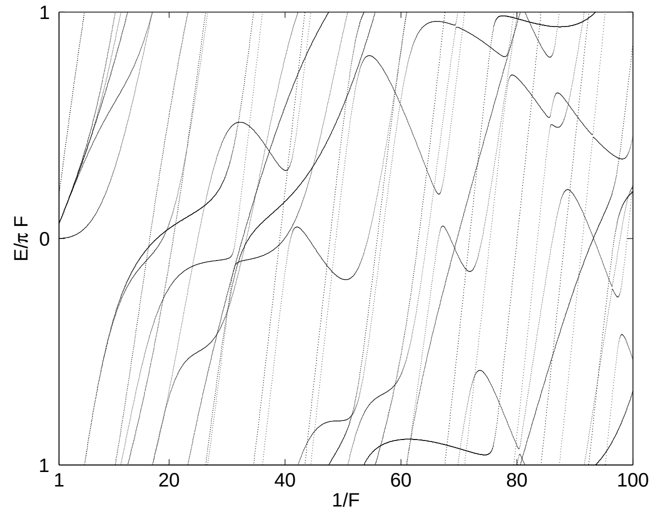

FIG. 1. Evolution of the quasienergies of the Floquet-Bloch operator (14) under changes of the inverse static field $1/F$. Particle number $N=3$, lattice size $L=3$, size of Hilbert space $\mathcal{N}=10$, block sizes $M=4,3,3$ [in representation (14)].

In Eq. (13), M is the number of different Fock states generated by repeated application of the shift operator on the "seed" state $|\mathbf{n}'\rangle$, and $\kappa=j/M$ ($j=1,\ldots,M$) denotes quasimomentum. In the new basis $|\kappa,\mathbf{n}'\rangle$, the matrix of the Hamiltonian (11) and, therefore, the matrix of the FB operator, factorizes into a block matrix,

$$U(T_B)=\oplus_{j=1}^{L}U^{(\kappa_j)}(T_B), \tag{14}$$

where each block $U^{(\kappa_j)}(T_B)$ corresponds to one of L possible values of quasimomentum. Furthermore, there are pairs of blocks in decomposition (14) which are related to each other by time-reversal symmetry.

Let us illustrate this factorization by a concrete example, for $N=L=3$. The dimension of the Hilbert space is $\mathcal{N}=10$, and one can choose the states $|111\rangle$, $|201\rangle$, $|210\rangle$, $|300\rangle$ as seed states. The value of M in Eq. (13) is $M=1$ for the first seed state, and $M=3$ for all others. Thus, the matrix $U(T_B)$ consists of one block of size 4×4, corresponding to $\kappa=1$ (or $\kappa=0$, if one chooses $-1/2<\kappa\leq 1/2$ as the first Brillouin zone), and of two "identical" blocks of size 3×3, corresponding to $\kappa=1/3,2/3$ (or $\kappa=\pm 1/3$). Each block can be diagonalized separately and, hence, the spectrum of the Floquet operator $U(T_B)$ is a superposition of several independent spectra. As an example, Fig. 1 shows the quasienergies E as a function of $1/F$ for $J=0.038$, and $W=0.032$. (Levels corresponding to $\kappa=\pm 1/3$ coalesce.) Note that quasienergy levels with different κ always cross, but that levels with the same value of quasimomentum may exhibit avoided crossings. (For a more detailed analysis see the subsequent sections.)

To proceed further, we need to agree on the ordering of the seed states. In what follows we use an ordering defined by the quantity

$$\mu=\frac{1}{2}\sum_{l=1}^{L} n_l'(n_l'-1), \tag{15}$$

which is proportional to the interaction energy for the given Fock seed state $|\mathbf{n}'\rangle$. The advantage of this ordering is that, in some cases (see the following section), the quantity μ can be considered as a second (in addition to quasimomentum) conserved quantum number.

IV. SPECTRUM OF THE FLOQUET-BLOCH OPERATOR

The aim of this section is the (preliminary) analysis of the spectrum of the κ-specific Floquet-Bloch operator $U^{(\kappa)}(T_B)$. If not stated otherwise, we shall assume $\kappa=0$ and omit this index in the formulas.

It is instructive to begin with the case $J\ll W$, and to follow the structural evolution of the spectrum as J is increased. Figure 2(a) shows the spectrum of the FB operator for a system of $N=3$ atoms in $L=5$ wells, as a function of $1/F$, for $W=0.032$ and $J=0.000\,76$. [For the chosen N and L, matrix (14) consists of five equal blocks of size 7×7.] For such a small value of the hopping matrix element, the quasienergy spectrum E of the system is essentially given by the quasienergy spectrum at $J=0$ which, in turn, is given by the equation

$$\exp(-iET_B)=\exp(-i\mu W/F), \tag{16}$$

where the quantity μ was defined earlier in Eq. (15). Thus, the three (different slope) straight lines in the figure correspond to quasienergy levels with quantum numbers $\mu=0$ (two levels), $\mu=1$ (four levels), and $\mu=3$ (one level), respectively. (The number of levels is obviously given by the number of different seed states with equal μ.) The nonzero value of J is reflected by the lifted degeneracy of levels with the same μ, and by the avoided crossings between levels with different μ. On the scale of the figure, one only observes avoided crossings at a static force value $F=W$ ($1/F=1/W\simeq33$). Similarly, a finite width of the μ band is detectable only for the μ manifold with the largest multiplicity, i.e., $\mu=1$.

As J is increased, the width of the μ bands becomes larger and, simultaneously, the avoided crossings increase [see Fig. 2(b)]. However, because of the cylindric topology of the quasienergy spectrum, this process cannot be continued up to arbitrary J, and above some critical value of the hopping matrix element, $J\simeq W$, the spectrum is dominated by avoided crossings [Fig. 2(c)] (except for those regions of very large and very small F, which will be analyzed separately in Sec. V).

In addition to Fig. 2, we display analogous spectra for a larger system of four atoms in seven wells in Fig. 3. (The FB operator is now represented by a matrix that consists of seven blocks, each of size 30×30.) It is seen that the quasienergy spectrum shows the same structural evolution as in Fig. 2, in spite of the dramatic increase of the number of levels.

It is worth noting an analogy between the structural changes in the quasienergy spectrum discussed above and the

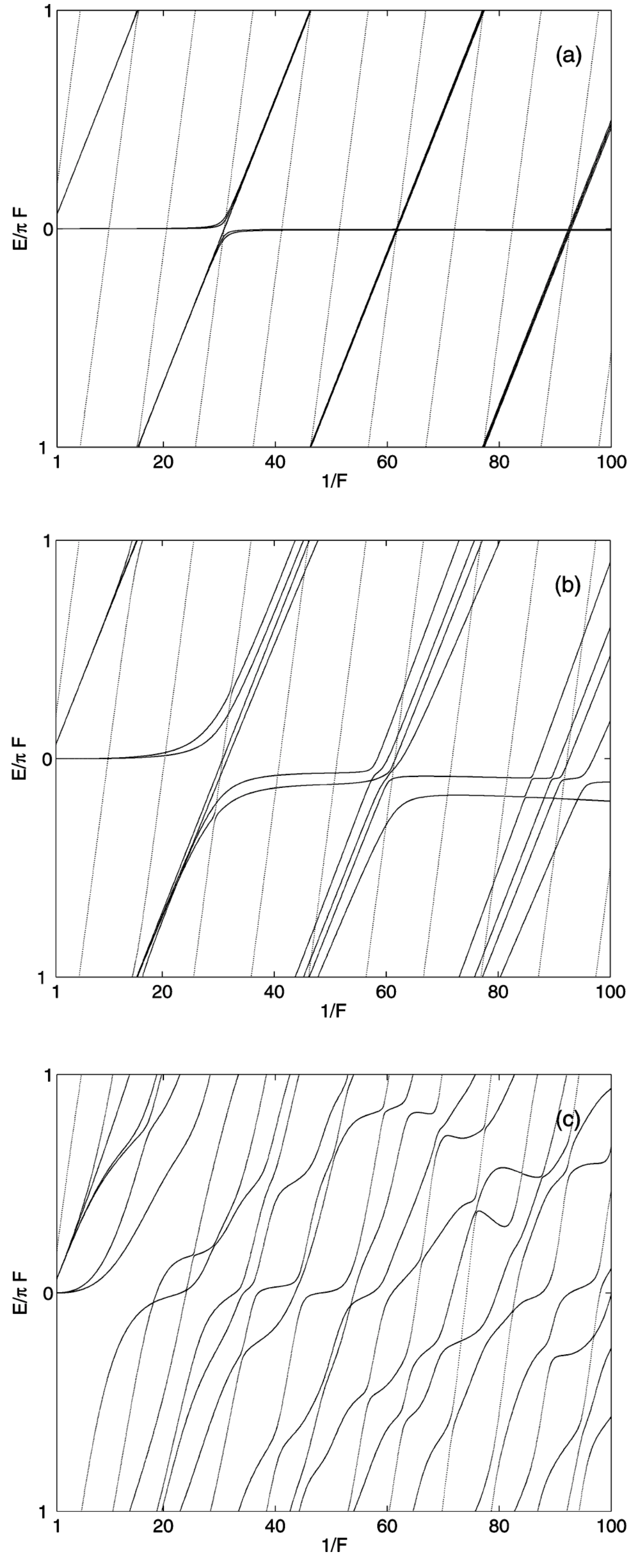

FIG. 2. Spectrum of the Floquet-Bloch operator as a function of $1/F$, for particle number $N=3$, lattice size $L=5$, quasimomentum $\kappa=0$, interaction strength $W=0.032$. The hopping matrix element between adjacent lattice sites varies from $J=0.000\,76$ (a), over $J=0.0038$ (b), to $J=0.038$ (c). Clearly, the level dynamics turn from regular (top) to chaotic (bottom) as the coupling between neighboring sites is increased.

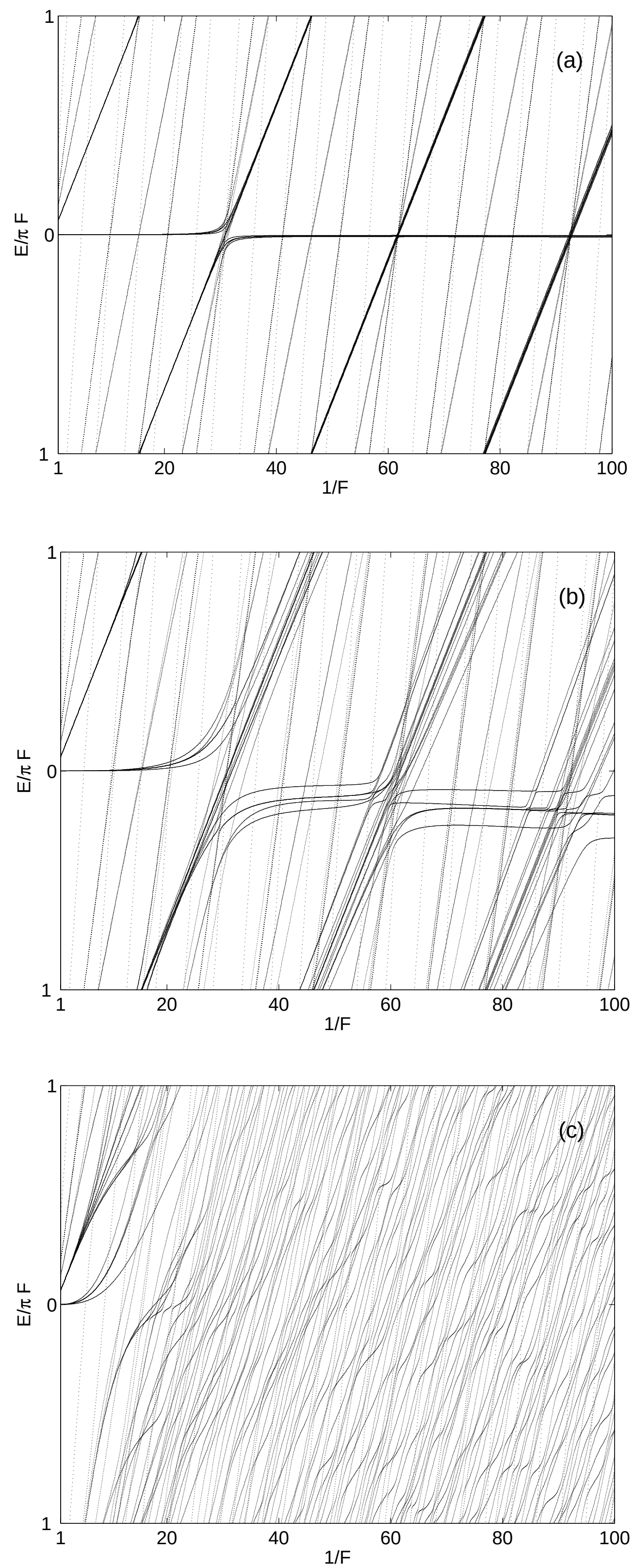

FIG. 3. Same as in Fig. 2, but for $N=4$, $L=7$. Once again, a clear transition from regular to chaotic spectral structure is observed, at significantly increased level densities.

evolution of the spectrum of a quantum chaotic system (i.e., a quantum system with classically chaotic analog), when the control parameter of the system crosses the border separating (quasi)regular and chaotic classical dynamics. This analogy can be taken further by noting that the transition to chaos in a classical system is related to the destruction of the system's integrals of motion. Quantum mechanically this means a "destruction" of good quantum numbers associated with such integrals. Although our system of interest has no classical counterpart, it has "integrals of motion" (15), which are destroyed in some regions of parameter space. This justifies the use of the terms "regular" and "chaotic" for different parameter regimes of the Bose-Hubbard model with static field. Let us get into a more detailed analysis of the quasienergy spectrum in these two different (regular and chaotic) regimes in the following sections.

V. REGULAR REGIME

In what follows we choose F as a control parameter and fix the hopping matrix element at $J\simeq W$. Then the regular regime (manifest in regular level dynamics) formally corresponds to the limits $F\to\infty$ or $F\to 0$.

Considering the limit of strong static field, it is convenient to treat the interparticle interaction as a perturbation. Then, using the interaction representation (with respect to the interaction term), the FB operator can be represented in the form

$$U(T_B)=\widehat{\exp}\left(-i\frac{W}{2}\int_0^{T_B}dt\,U_0^{\dagger}(t)\sum_l \hat{n}_l(\hat{n}_l-1)U_0(t)\right), \tag{17}$$

where $U_0(t)$ is the evolution operator of the system in the absence of particle-particle interaction. [Here we use $U_0(T_B)=\hat{1}$—see the above discussion following Eq. (8)]. Since we are interested in the case $J/F\ll 1$, we can use perturbation theory to find $U_0(t)$:

$$U_0(t)\simeq\hat{1}+\frac{J}{2F}\left((e^{-i2\pi Ft}-1)\sum_l \hat{a}_{l+1}^{\dagger}\hat{a}_l-\text{H.c.}\right). \tag{18}$$

Substituting Eq. (18) into Eq. (17), we have

$$U(T_B)=\exp\Bigg(-i\frac{WT_B}{2}\Bigg\{\sum_l \hat{n}_l(\hat{n}_l-1) -\frac{J}{2F}\left[\left(i\sum_l \hat{a}_{l+1}^{\dagger}\hat{a}_l-\text{H.c.}\right),\ \left(\sum_l \hat{n}_l(\hat{n}_l-1)\right)\right]\Bigg\}\Bigg), \tag{19}$$

where the square brackets denote the commutator. It follows from the last equation that at vanishing order in the small parameter J/F, the FB operator is a diagonal matrix (in the Fock states basis) with elements

$$\langle\mathbf{n}'|U(T_B)|\mathbf{n}'\rangle=\exp(-iW\mu/F) \tag{20}$$

[see Fig. 4(a)]. Note that, since μ is an integer number, result (20) leads to the appearance of a new fundamental period for

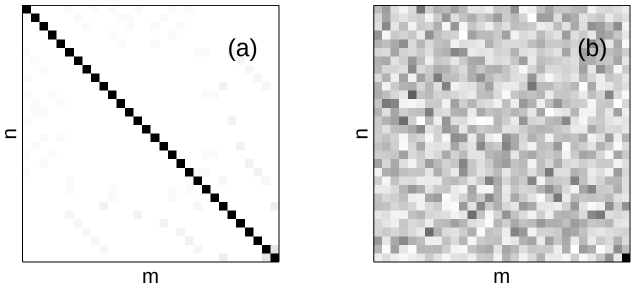

FIG. 4. Matrix of the κ-specific Floquet-Bloch operator for $N=4$, $L=7$, $\kappa=0$, $J=0.038$, $W=0.032$, $F=0.5$ (a), and $F=0.01$ (b). Absolute values of the matrix elements are coded by a (linear) gray scaling.

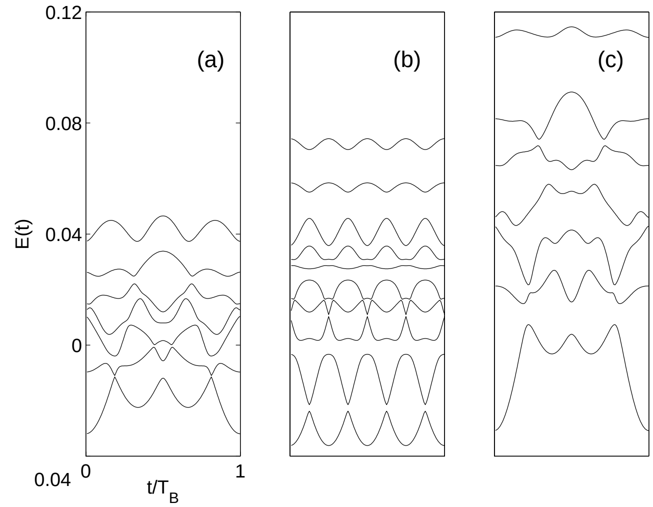

FIG. 5. Instantaneous energy spectrum of the time-dependent Hamiltonian (11). Parameters are $W=0.032$, $J=0.038$, $\kappa=0$, and $N=3$, $L=5$ (a), $N=4$, $L=4$ (b), $N=5$, $L=3$ (c). Observe the time periodicity for the case $N=L=4$.

the system dynamics [10,21]. In other words, given the condition $F\gg J$, an arbitrary observable of the system becomes a quasiperiodic function of time with $T_B=1/F$ and $T_W=2\pi/W$ induced by the external field and the particle-particle interaction, respectively.

Let us now consider the opposite limit $F\to 0$. Here it is convenient to treat the static field as a perturbation of the field-free Hamiltonian. Indeed, let $|\tilde{\Psi}(t)\rangle$ be an arbitrary instantaneous eigenfunction of the time-dependent Hamiltonian (11):

$$\tilde{H}(t)|\tilde{\Psi}(t)\rangle=\tilde{E}(t)|\tilde{\Psi}(t)\rangle. \tag{21}$$

Then, in the adiabatic limit $F\to 0$, the function $|\tilde{\Psi}(0)\rangle$ is also an eigenfunction of the FB operator, corresponding to the quasienergy

$$E=\frac{1}{T_B}\int_0^{T_B}\tilde{E}(t)dt. \tag{22}$$

Examples of the instantaneous spectrum $\tilde{E}(t)$ of the Hamiltonian $\tilde{H}(t)$ are given in Fig. 5. Note that the instantaneous spectrum reveals a hidden symmetry of the system for $N=L$. In this case, $\tilde{E}(t)$ are periodic functions of time with period T_B/L. (This also holds for any integer value of the ratio N/L.) Note that this symmetry causes a qualitative difference in the statistical properties of the chaotic quasienergy spectrum for integer and noninteger N/L (see Sec. VI).

We come back to Eq. (22). In a representation of the spectrum alike that in Fig. 2, result (22) implies that the quasienergy spectrum at $F\to 0$ consists of a number of crossing straight lines, with slopes defined by the values of E. A reminiscence of these lines is clearly seen in Fig. 6(a), where we plot the spectrum of the system for static field values around $F=0.001$. The presence of avoided crossings indicates a partial failure of the adiabatic approximation: As $F\to 0$, the gaps progressively close, and the FB matrix collapses onto the diagonal, in the eigenfunction basis of the field-free Hamiltonian.

We conclude this section by a brief remark concerning the thermodynamic limit $N,L\to\infty$, $N/L=\mathrm{const}$. It is easy to see that the above conclusion about the regular quasienergy spectrum for $F\gg J$ is ''insensitive'' to the increase of the system size. This is, however, not the case for weak static fields. As the size of the system is increased, the vicinity of the point $F=0$, where the adiabatic approximation holds (and, thus, where the level dynamics are regular), shrinks to zero. This statement is illustrated in Fig. 6(b), showing the spectrum of a larger system for the same interval of the static field as in Fig. 6(a). Only a thorough inspection of the figure reveals remnants of straight lines with fixed (negative) slope.

VI. CHAOTIC REGIME

As shown in Sec. IV, the Bose-Hubbard system with static field can be regarded as a regular or as a chaotic system, depending on the particular choice of the parameters. Regular regimes correspond to the limit of large F, where the FB matrix is diagonal in the basis of Fock states, and to the limit of small F, where the matrix is diagonal in the basis of the eigenfunctions of the field-free Hamiltonian. For intermediate values of the static field (and $J\simeq W$), the matrix of the FB operator is diagonal in neither of these basis sets. Moreover, visually it looks like a random matrix [see Fig. 4(b)]. In this section we explore this conjecture on the randomness of the FB operator in more detail.

Let us first study the statistics of the matrix elements (where we omit the quasimomentum index as in the previous sections)

$$u=\langle\mathbf{m}'|U(T_B)|\mathbf{n}'\rangle. \tag{23}$$

Note that, since $U(T_B)$ is unitary, we have the following relation for the matrix elements ($\mathcal{J}\simeq\mathcal{N}/L$ is the matrix size):

$$\sum_{j=1}^{\mathcal{J}}u_{m,j}^{*}u_{j,n}=\delta_{m,n}, \tag{24}$$

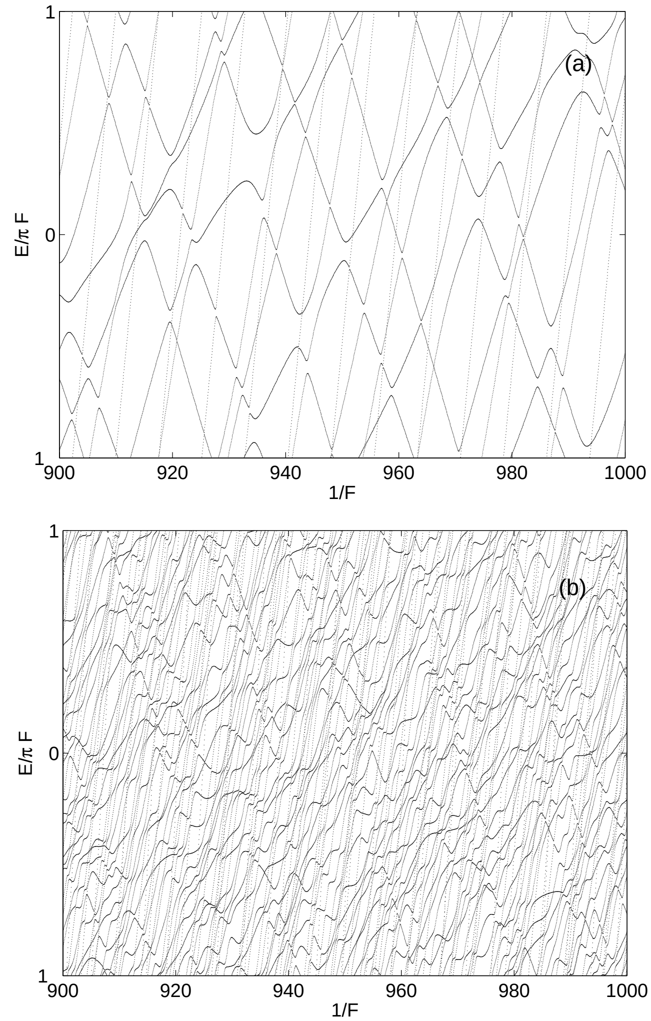

FIG. 6. Spectrum of the Floquet-Bloch operator for very weak static force. The parameters are the same as in Fig. 2(c) (upper panel) and Fig. 3(c) (lower panel).

and, hence, $|u|\sim\mathcal{J}^{1/2}$. The lower panel in Fig. 7 shows the distribution of the real part $\mathrm{Re}(u)$ for $F=0.01$, $J=0.038$, $W=0.032$, $\kappa=0$, and $N=7$, $L=9$ (matrix size $\mathcal{J}=715$). It is seen that the numerically obtained distribution is well approximated by a Gaussian with variance $\sigma=(\mathcal{J}/2)^{1/2}$. The same result is obtained for the distribution of the imaginary part $\mathrm{Im}(u)$. Finally, the distribution of the absolute values $|u|$ is found to fit the equation

$$P(|u|)\sim|u|\exp(-|u|^2/2\mathcal{J}), \tag{25}$$

which follows from the Gaussian distributions for the real and imaginary parts, provided that these are *independent* random variables. Thus we can conclude that the FB operator can be indentified with a random unitary matrix, indeed.

Next we analyze the distribution of the eigenvalues of the FB operator,

$$U(T_B)|\psi_E\rangle=\lambda|\psi_E\rangle,\quad \lambda=\exp(-iET_B). \tag{26}$$

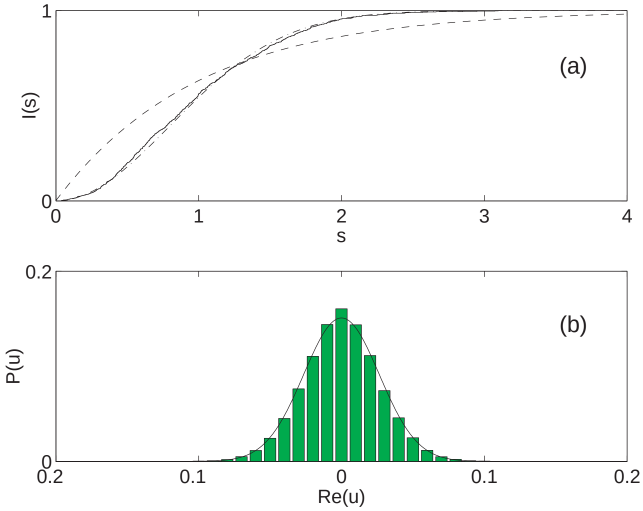

FIG. 7. Upper panel—cumulative distribution $I(s)$ for the level spacing. (The dashed and dashed-dotted lines correspond to Poisson and COE cumulative distributions, respectively.) Lower panel—distribution of the real parts of the matrix elements u of the FB operator. The parameters are $F=0.01$, $J=0.038$, $W=0.032$, $\kappa=0$, and $N=7$, $L=9$.

In this paper we restrict ourselves to the nearest-neighbor level statistics, i.e., the distribution of the quantity $s=(E_{j+1}-E_j)/\overline{\Delta E}$, where $\overline{\Delta E}=2\pi F/\mathcal{J}$ stands for the mean level spacing. Since $U(T_B)$ is found to be a random matrix, we anticipate that the spacing distribution obeys Wigner-Dyson statistics. The numerical analysis undoubtedly confirms this expectation. The upper panel in Fig. 7 shows the cumulative distribution $I(s)=\int_0^s P(s')ds'$ for system size $N=7$, $L=9$. The numerical data closely follow the Wigner-Dyson distribution for the circular orthogonal ensemble,

$$P(s)=\frac{\pi}{2}s\exp\left(-\frac{\pi}{4}s^2\right). \tag{27}$$

We have checked result (27) to hold for other values of the quasimomentum and for different N and L ($N,L\leqslant 11, 0.5<N/L<1.5$), *excluding* the case $N=L$, which requires a special approach described in the remainder of this section.

Analyzing the statistical properties of the quasienergy spectrum, the case of an integer ratio N/L might come as a surprise—the straightforward calculation of the level spacing reveals Poisson statistics,

$$P(s)=\exp(-s), \tag{28}$$

instead of the expected Wigner-Dyson distribution [see Fig. 8(a)]. The reason for this result is an additional symmetry of the system mentioned above in Sec. V. Namely, for $N=L$ (more generally, for integer ratio N/L) the instantaneous spectrum of the time-dependent Hamiltonian (11) is periodic in time with the period T_B/L. This means that the operators $\widetilde{H}(t)$ and $\widetilde{H}(t+T_B/L)$ are related to each other by a unitary transformation

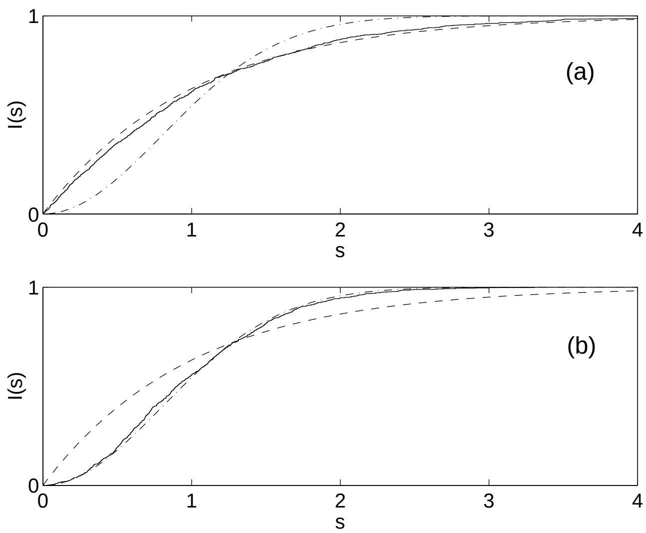

FIG. 8. Cumulative distribution $I(s)$ of the level spacing for $N=L=8$. The upper and lower panels refer to the spectrum of the operators $U(T_B)$ and $\tilde{U}$, respectively.

$$\tilde{H}(t+T_B/L)=Q^{\dagger}\tilde{H}(t)Q, \tag{29}$$

where Q is a diagonal matrix with elements $\exp(-iET_B)^{1/L}$. Thus, for $N=L$ the FB-operator has the form

$$U(T_B)=(Q^{\dagger}U(0,T_B/L))^L \tag{30}$$

(here we used the relation $Q^L=\hat{1}$). It is easy to see now that one has to analyze the spectrum $\{\tilde{\lambda}\}$ of the operator $\tilde{U}=Q^{\dagger}U(0,T_B/L)$, but *not* the spectrum $\{\lambda\}$ of the FB operator $U(T_B)$. Indeed, because of the relation $\lambda=\tilde{\lambda}^L$, the quasienergy spectrum of the FB operator approaches Poisson statistics when $L\to\infty$, as illustrated in Fig. 8, where the upper and lower panels refer to the spectrum of the operators $U(T_B)$ and $\tilde{U}$, respectively.

VII. CONCLUSION

We have studied the spectral properties of the 1D Bose-Hubbard model in an external static field, which models interacting cold spinless atoms in a quasi-one-dimensional optical lattice subject to a static force. The analysis is performed in terms of the Floquet-Bloch operator (FB operator), defined as the evolution operator of the system over one Bloch period. The advantage of the FB operator over the Hamiltonian is that one can impose periodic boundary conditions for a system of finite size L, which greatly facilitates the transition to the thermodynamic limit $L,N\to\infty$, $N/L=\text{const}$. Besides that, the FB operator allows to describe the system dynamics by a one-cycle map.

An important result of the paper consists in uncovering *all symmetries* of the 1D FB operator. These are the translational symmetry, which leads to the notion of the quasimomentum κ, and the "Mott-insulator" symmetry, which is present in the system for an integer filling factor (mean number of bosons per lattice site). In this work, we restricted ourselves to analyzing the spectrum of the 1D FB operator only for $\kappa=0$. It is also interesting to study the dependence of the spectrum on quasimomentum. This problem is reserved for the future.

For fixed κ, given the condition $J\ll W$ (i.e., hopping matrix element much smaller than the on-site interaction energy), the spectrum of the FB operator of a system of finite size consists of a large number of (quasi)energy levels arranged in a few bands labeled by the mean interaction energy. As a function of the static field F, these bands show avoided crossings at points where the Stark energy is a rational fraction of the on-site interaction energy. It is worth noting the relevance of this observation to the resonancelike response of the system to a static field, observed in experiment [5] (see also the theoretical papers [9,22], discussing this problem from different points of view).

The case $J\simeq W$ appears to be essentially more complicated. Here the spectrum of the FB operator is regular (i.e., can be characterized by a set of good quantum numbers) only in the limits of large and small static field strengths. For intermediate values of F, the spectrum of the FB operator can be characterized as *irregular*, qualitatively resembling the spectrum of a quantum chaotic system. Using the tools of random matrix theory, we prove that for $J\simeq W$, $N/L\simeq 1$, and moderate strengths of the static field, the matrix representing the FB operator is actually a *random matrix*, belonging to the Wigner-Dyson circular orthogonal ensemble. This constitutes the main result of the paper and opens a perspective for the theory of multiparticle quantum chaos. Notwithstanding our 1D analysis, we expect that our results will qualitatively prevail also in 2D and 3D configurations, since increasing the dimension cannot compensate for the interaction-induced destruction of dynamical symmetries of the 1D problem.

Let us conclude by noting that multiparticle quantum chaos implies observable consequences for state of the art experiments on Bloch oscillations [2,4,5], which are performed in the same parameter regime as our numerical computations above: Related theoretical work has shown that Bloch oscillations of the average momentum of the atoms in the lattice decay *irreversibly* for underlying chaotic (multiparticle) spectra [11], while exhibiting interaction-induced wave packet collapse and revival [10] in the strong field limit $F\gg J$ with regular spectral structure. Furthermore, the time scale of the chaos-induced, irreversible decay apparently only depends on the filling factor of the atoms in the lattice, but not on the actual lattice size. This observation bridges the gap from numerical to experimental lattice sizes—the latter being out of reach even for advanced supercomputing.

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