# Interaction-Induced Decoherence of Atomic Bloch Oscillations

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We show that the energy spectrum of the Bose-Hubbard model amended by a static field exhibits Wigner-Dyson level statistics. In itself a characteristic signature of quantum chaos, this induces the irreversible decay of Bloch oscillations of cold, interacting atoms loaded into an optical lattice, and provides a Hamiltonian model for interaction-induced decoherence.

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The Bose-Hubbard Hamiltonian serves as a paradigm in the field of quantum phase transitions [1]. Recently, this model was realized in experiments on ultracold atoms loaded into a three-dimensional optical lattice [2], opening new perspectives for the laboratory study of correlated bosonic systems. Consequently, new theoretical work on the Bose-Hubbard model was stimulated, which, in particular, addresses the response to a static field [3–5]—a question which shifts the focus from the Bose-Hubbard ground state (which is mostly studied in the literature) to dynamical and spectral properties of the system. In single-particle quantum mechanics, these are associated with Bloch oscillations in the time domain and, with the emergence of a Wannier-Stark ladder, in the energy domain [6].

This Letter is devoted to the spectral properties of the Bose-Hubbard Hamiltonian under the additional action of a static field or, equivalently, to the Wannier-Stark problem for interacting bosons. Our analysis is formulated in a spirit close to ongoing experiments on cold atoms in optical lattices [2,7], and we assume that the atoms are in the "superfluid phase"; i.e., they are delocalized over the lattice in the absence of any external perturbation. This latter assumption distinguishes the present work from previous contributions [3,4] devoted to the Mott insulator phase and restricts the values of the hopping matrix element $J$ and of the on-site interaction energy $W$ to the range $W/J < 5.8$ (see [2] and references therein). To be specific, we fix $J = 0.038$ and $W = 0.032$—the experimental values (in units of photon recoil energies) for rubidium atoms in optical lattices with a potential well depth of approximately ten photon recoils—throughout the sequel of this Letter. With $\hat{a}_l^\dagger$, $\hat{a}_l$, and $\hat{n}_l$ the particle creation, the particle annihilation, and the number operator at site $l$ of the lattice, the total Hamiltonian reads

$$\hat{H} = -\frac{J}{2}\left(\sum_l \hat{a}_{l+1}^\dagger \hat{a}_l + \mathrm{H.c.}\right) + \frac{W}{2}\sum_l \hat{n}_l(\hat{n}_l - 1) + F\sum_l l\hat{n}_l, \tag{1}$$

where the strength of the static field $F$ or, more precisely, the Stark energy (the lattice period is set to unity) will be our free parameter.

Let us first address the issue of boundary conditions. It is well known that, for an infinite lattice, there is no smooth transition between the spectrum at $F = 0$ and $F \neq 0$. Formally, this is due to the fact that for any nonvanishing value of $F$ the Hamiltonian (1) is an unbounded operator, whereas it is bounded for $F = 0$. However, for a lattice of finite size, $-L \leq l \leq L$, the operator (1) is always bounded and, hence, the spectrum of the system changes continuously as a function of $F$, as illustrated by the numerically generated level dynamics in the top left panel of Fig. 1, for $N = 1$ particle and Dirichlet (i.e., vanishing) boundary conditions. As $F$ is increased, the spectrum evolves from a Bloch spectrum with energies $E(k) = -J\sin[\pi k/(2L + 2)]$, $k = -L, \ldots, L$, into a Wannier-Stark ladder $E_l \simeq Fl$, $l = -L, \ldots, L$. The other panels in Fig. 1 show the evolution of the field-free $k = 0$

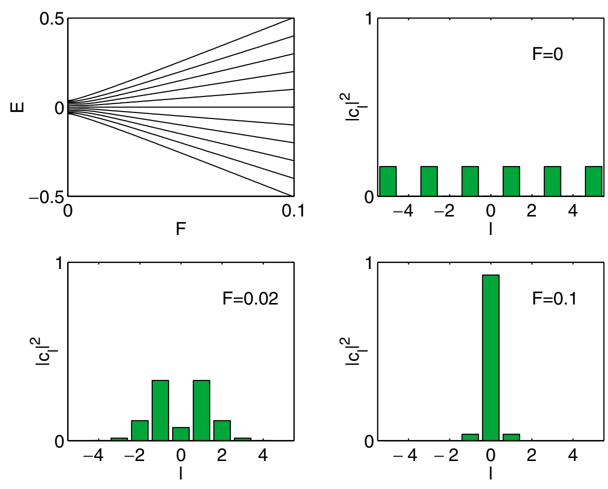


FIG. 1 (color online). Energy spectrum (top left) together with Wannier state projections $|c_l|^2$ of the field-free $k = 0$ eigenstate of (1), for increasing values of the static field $0 \leq F \leq 0.1$. The $N = 1$ particle is loaded into a lattice with 11 sites ($L = 5$). The level dynamics illustrate the transition from a Bloch band to a Wannier ladder, associated with the progressive localization of the wave function in $l$.

  

eigenstate in the basis of the Wannier states, with increasing $F$ [8]. The progressive localization of the atomic wave function in $l$ with $F$ is known as Stark localization.

When discussing the time evolution of a wave function governed by (1), it is preferable to use periodic boundary conditions instead of Dirichlet. To do so, one first eliminates the static term in (1) by transforming to the interaction representation, where the hopping and the on-site term in (1) define the unperturbed Hamiltonian, hence

$$\hat{H} \to \tilde{H}(t) = -\frac{J}{2}\left[\exp\left(-i\frac{F}{\hbar}t\right)\sum_l \hat{a}_{l+1}^\dagger \hat{a}_l + \text{H.c.}\right] + \frac{W}{2}\sum_l \hat{n}_l(\hat{n}_l - 1), \tag{2}$$

and then identifies the site $l = L + 1$ of the lattice with $l = -L$. This choice has the advantage that the time evolution operator of a system of noninteracting atoms over one Bloch period $T_B = 2\pi\hbar/F$ coincides with the unit matrix, independently of the size of the system. That facilitates the analysis of the dynamics in the thermodynamic limit $\bar{n} = N/L = \text{const}$, $N, L \to \infty$ (see below, Fig. 4) [5]. In what follows, we shall use Dirichlet boundary conditions when calculating eigenvalues, and periodic boundary conditions when simulating the dynamics [9]. Note that a restriction of Eq. (2) to two sites, with *vanishing* boundary conditions, describes the modulated two-mode model of a Bose-Einstein condensate in a double well [10] rather than Bloch dynamics in a periodic lattice [6]. While the latter has no classical counterpart, the former has a driven top, with possibly chaotic classical dynamics [11]. In strong contrast, the characteristic features of quantum chaos described below are inherited from many-particle interactions and not from classical chaos. Also, a mean field approach is justified in [10], for large $\bar{n} \to \infty$, but becomes invalid in our present problem with moderate $\bar{n} \sim 1$.

Our analysis of the spectrum of the multiparticle system (1) follows the one for the single-particle problem. Let us assume for the moment that there are no atom-atom interactions, i.e., $W = 0$. As already mentioned, for large values of $F$ the single-atom energy levels form a Wannier ladder, and the energies of an $N$-atom system are consequently given by

$$E_{\mathbf{m}} = F\sum_{l=-L}^{L} l m_l \equiv F l_{\text{tot}}, \qquad |l_{\text{tot}}| \leq LN, \tag{3}$$

where the $m_l$ ($\mathbf{m} = m_{-L}, \ldots, m_L$, $\sum_{-L}^{L} m_l = N$) are the occupation numbers of the Wannier-Stark states. Note that, in general, many different sets $\mathbf{m}$ correspond to the same total energy, and the $N$-particle Wannier ladder levels $E_{\mathbf{m}} = F l_{\text{tot}}$ are, thus, typically degenerate. The $N$-particle wave function associated with a given level $E_{\mathbf{m}}$ can be constructed from single-particle Wannier-Stark states $|\psi_l\rangle$ by an appropriate symmetrization procedure. In the basis of Fock states (symmetrized products of Wannier functions $|\phi_n\rangle$), an arbitrary Wannier-Stark state, at finite $F$, is given by the sum

$$|\Psi_{\mathbf{m}}\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}}^{(\mathbf{m})}|\mathbf{n}\rangle, \qquad |\mathbf{n}\rangle = |n_{-L}, \ldots, n_L\rangle, \tag{4}$$

and in the limit $F \to \infty$ only one coefficient $c_{\mathbf{n}}^{(\mathbf{m})}$ with $\mathbf{n} = \mathbf{m}$ differs from zero in Eq. (4). On the contrary, in the opposite limit $F \to 0$, almost all expansion coefficients are nonzero and the Wannier-Stark states approach $N$-particle Bloch states with (once again, degenerate) energies

$$E(\mathbf{k}) = -J\sum_{k=-L}^{L} \sin\left(\frac{\pi k}{2L+2}\right) n_k, \tag{5}$$

the straightforward $N$-particle generalization of the above one-particle result.

Let us now include the effect of atom-atom interactions. Figure 2 shows the energy levels of the Hamiltonian (1) as a function of $F$, for $N = 3$ atoms loaded into a lattice with 11 sites (i.e., $L = 5$). As expected, the atom-atom interactions remove the above-mentioned degeneracy—for small $F$ the spectrum appears dense (almost continuous), and for large $F$ the degenerate levels of the Wannier ladder split into "Wannier-ladder energy bands" [see Eq. (6) below]. In this latter limit, the spectrum and the associated Wannier-Stark states can still be found analytically. Indeed, since the hopping term in Eq. (1) couples only those Fock states separated by one single quantum in the Stark excitation, one has

$$E_{\mathbf{m}} \simeq F\sum_{l=-L}^{L} l m_l + \frac{W}{2}\sum_{l=-L}^{L} m_l(m_l - 1), \tag{6}$$

and

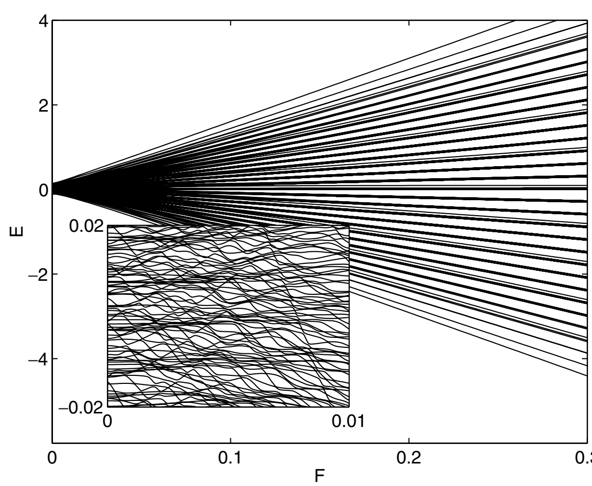


FIG. 2. Energy spectrum of the Hamiltonian (1) as a function of the static field $F$, for particle number $N = 3$ and lattice size $2L + 1 = 11$. Particle interaction strength $W = 0.032$, and hopping matrix element $J = 0.038$. The inset zooms into the central part of the spectrum, in the range $0 \leq F \leq 0.01$.

$$|\Psi_{\mathbf{m}}\rangle \simeq |\mathbf{m}\rangle - \frac{J}{2}\left(\sum_{l=-L}^{L-1}\sum_{\mathbf{m}'}\frac{\langle \mathbf{m}'|\hat{a}_{l+1}^{\dagger}\hat{a}_l|\mathbf{m}\rangle}{E_{\mathbf{m}'}-E_{\mathbf{m}}}|\mathbf{m}'\rangle + \text{H.c.}\right), \tag{7}$$

where $|E_{\mathbf{m}'} - E_{\mathbf{m}}| \sim F$.

The perturbative results (6) and (7) cannot hold when $F < J$. Moreover, the complex level dynamics which are borne out for small $F$ in the inset of Fig. 2 indicate that any attempt to assign a set of quantum numbers to individual levels is bound to fail for $F < J$. Instead, a statistical analysis of the spectrum is appropriate in this situation. For that purpose, the upper part of Fig. 3 presents the cumulative distribution of the spacings between adjacent energy levels, for $N = 4$ atoms loaded into a lattice with 11 sites, at $F = 0.01$ [12]. Clearly, the normalized energy intervals $s = \Delta E/\overline{\Delta E}$ exhibit GOE (Gaussian orthogonal ensemble) statistics, $P(s) = (\pi^2/6)s \times \exp(-\pi s^2/4)$, a hallmark of quantum chaos [11,13]. Thus, for weak static fields, the system (1) can be regarded as a quantum chaotic system. The origin of "quantum chaos," i.e., of the strongly $F$-dependent, nonperturbative mixing of energy levels can be understood here as a consequence of the interaction-induced lifting of the degeneracy of the multiparticle Wannier-Stark levels in the crossover regime from Bloch to Wannier spectra, making nearby levels strongly interact, for comparable magnitudes of hopping matrix elements and Stark shifts. In contrast, for large $F$ (and in the limit of large $L, N$), the nearest neighbor distribution tends towards Poissonian statistics, $P(s) = \exp(-s)$, as evident from the lower part of Fig. 3.

Which are the physical, that is, experimentally observable manifestations of the irregular spectrum of (1), at small field strengths? To answer this question, we consider the Bloch oscillations of the mean atomic momentum, which can be observed rather easily in state-of-the-art experiments [7,14]. In the absence of atom-atom interactions, the average momentum $p(t) = \mathrm{tr}[\hat{p}\,\hat{\rho}(t)]$ of the atoms oscillates with the Bloch frequency $\omega_B = F/\hbar$. Here $\hat{\rho}(t)$ is the single-particle density matrix with elements (in the Wannier state basis)

$$\rho_{l,l'}(t) = \langle\Psi(t)|\hat{a}_l^{\dagger}\hat{a}_{l'}|\Psi(t)\rangle. \tag{8}$$

As shown in [5], the presence of atom-atom interactions modifies the Bloch dynamics, and $p(t)$ exhibits an additional beating signal at frequency $\omega_W = W/\hbar$. Namely, after the scaling $p(t) \to p(t)/NJ$ and in the thermodynamic limit $\bar{n} = \text{const}$, $N, L \to \infty$, one has

$$p(t) = f(t)\sin(\omega_B t), \qquad f(t) = \exp(-2\bar{n}[1 - \cos(\omega_W t)]). \tag{9}$$

The appearance of the new frequency $\omega_W$ originates in the splitting of the Wannier ladder levels into "energy bands" [15]. It must be stressed that the result (9) is valid only for large values of the static field, where the spectrum is regular. Consequently, it is to be expected that for weak static fields the atomic Bloch oscillations will be qualitatively different, due to the irregular/chaotic structure of the spectrum. Indeed, numerical simulations of the dynamics indicate that, in the weak field regime, the Bloch oscillations decay irreversibly on rather short time scales. As an example, Fig. 4 shows the behavior of the scaled momentum for $F = 0.05$, $N = 7$, $L = 3$ (top), and $N = 9$, $L = 4$ (bottom) [16]. After only a few Bloch

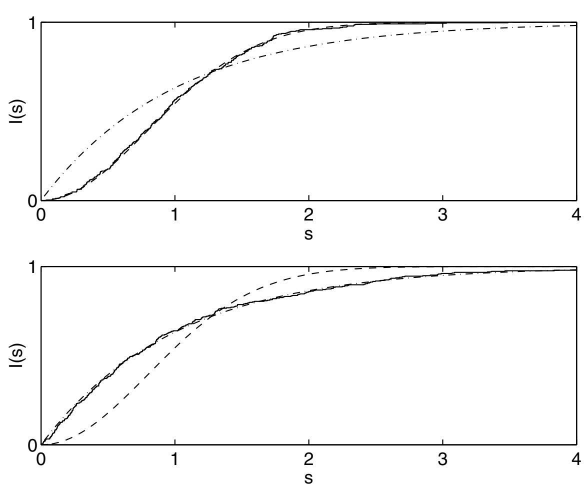


FIG. 3. Solid line: Cumulative nearest neighbor level spacing distribution $I(s) = \int_0^s P(s')ds'$ for normalized spacings $s = \Delta E/\overline{\Delta E}$ ($\overline{\Delta E}$ is the average level spacing in the central part of the spectrum), with $F = 0.01$ (top) and $F = 0.04$ (bottom). $N = 4$ atoms loaded into a lattice of size $2L + 1 = 11$. The dashed and dash-dotted lines indicate GOE and Poisson cumulative distributions, respectively.

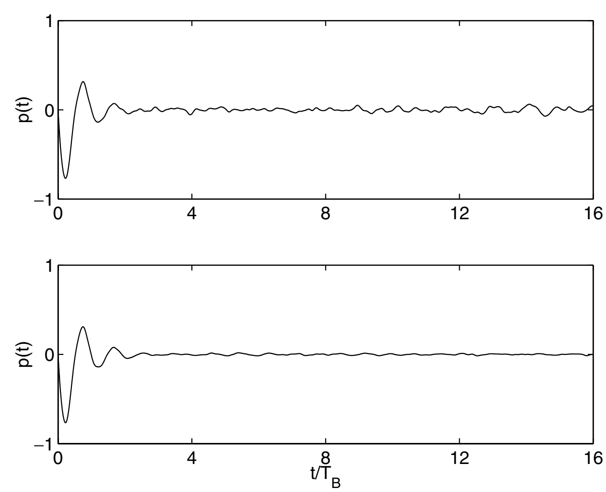


FIG. 4. Irreversible decay of the Bloch oscillations of the average atomic momentum $p(t)$ (scaled as $p \to p/NJ$) for a weak static field $F = 0.05$, a filling factor $\bar{n} = 1$, and lattice sizes $2L + 1 = 7$ (top) and $2L + 1 = 9$ (bottom). (The respective dimensions of Hilbert space are $\mathcal{N} = 1716$ and $\mathcal{N} = 24\,310$.) Comparison of both plots shows rapid convergence towards the thermodynamic limit, with a well defined characteristic decay time $\tau$.

periods, the mean momentum has almost comletely decayed. Comparison of the results for the smaller (top) and the larger (bottom) lattice sizes, at fixed $\bar{n} = 1$, clearly illustrates the rapid convergence towards the thermodynamic limit, where the residual fluctuations (present in a finite size system) die out. Thus, for a weak static field, the envelope function in Eq. (9) approaches $f(t) \sim \exp(-t/\tau)$ in the thermodynamic limit, and the characteristic decay time $\tau$ is a robust experimentally accessible quantity. Note that the decay of the Bloch oscillations is due to the decay of the off-diagonal elements of the one-particle density matrix (8) and that, hence, the time $\tau$ can equally be considered as the *decoherence time* for a system of interacting bosons. The precise dependence of $\tau$ on the system parameters $W$, $J$, and $F$ hitherto remains an open problem.

Finally, let us briefly discuss the conditions for the observed chaos transition in system (1). Our numerical simulations of the system dynamics, performed for fixed ratio $W/J$ and different values of $N$ and $L$ ($0.2 \leq \bar{n} \leq 1.2$, $L \leq 10$, $N \leq 10$), suggest the condition

$$\delta l \sim \bar{n}^{-1}, \tag{10}$$

as a criterion of the transition to chaos, where $\delta l$ denotes the localization length of the single-particle wave function on the lattice ($\delta l \simeq J/F$ for $F < J$, and $\delta l \simeq 1$ for $F > J$), and $\bar{n}^{-1}$ has the meaning of an average particle distance. It is clear, however, that condition (10) cannot be universal, since it does not account for the on-site energy $W$. Indeed, for $W \rightarrow 0$, the particle-particle interaction vanishes, and the system is integrable for arbitrary $F$. On the other hand, when $W \rightarrow \infty$, the Bose-Hubbard model can undergo a Mott transition into the insulating phase, where its response to the static field has a very different (resonantlike) character [3,4]. It therefore remains a challenging theoretical problem to formulate general criteria for the chaos transition.

To conclude, we have shown that the spectrum of the Bose-Hubbard Hamiltonian amended by a static field (and at fixed particle-particle interaction corresponding to the "superfluid" regime in the field-free case) is either regular or irregular, depending on the relative strength of the hopping matrix element and of the external perturbation. In particular, we have seen that the irregular level structure at intermediate strengths of the static field manifests in a rapid decay of the Bloch oscillations of the mean atomic momentum, and that the time scale of this decay provides a direct measure for the decay of particle-particle coherences across the lattice. Hence, dynamics of cold, interacting atoms loaded into a one-dimensional optical lattice allow for experimental probing and control of interaction-induced decoherence.

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[9] Spectral properties determine the dynamics, and the statistical features of the spectra analyzed here are robust when we choose periodic instead of vanishing boundary conditions. However, the spectral analysis for periodic boundary conditions is more involved, since we need to introduce the Floquet-Bloch operator generated by $\tilde{H}(t)$, and is presented in a separate contribution: A. Kolovsky and A. Buchleitner, Phys. Rev. E **68**, 056213 (2003).

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