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TREE GROWTH PARAMETER IN THE EDEN MODEL ON FACE-CENTERED HYPERCUBIC LATTICES

V. E. Zobov* and M. A. Popov†

In the Eden model, we investigate how the tree growth parameter depends on the space dimension d for face-centered hypercubic lattices. We find the first three terms of the $1/d$-expansion for this parameter directly from the generating function without calculating the number of trees because the growth parameter is the reciprocal coordinate of the singular point of the tree generating function. The same growth parameter was calculated by computer experiment where the ratios between the numbers of trees without intersections and trees without restrictions in the dimensions 3, 4, 6, 8, and 10 were estimated by the Monte Carlo method on face-centered cubic lattices. The results of the two methods agree well. Comparing with the previously performed computer experiment for simple hypercubic lattices, we observe that the values of the singular exponents for the tree generating functions are close for two different types of lattices.

Keywords: Eden model, number of lattice trees, Monte Carlo method, growth parameter, singular points of generating function, large-dimension expansion, face-centered hypercubic lattice

1. Introduction

Many papers have been devoted to studying the statistical properties of trees constructed on lattices of various dimensions d with the restriction forbidding branch intersections. The interest in such objects is aroused, on one hand, by their relative simplicity and, on the other hand, by their close relation to actual phenomena: the growth of branched molecules and their properties, percolation, etc., where different models correspond to different phenomena, and the tree properties also differ in these models. For example, in models of branched molecules or clusters in the percolation theory, each tree on a lattice contributes with unit weight [1], [2], whereas the tree weight in the Eden growth model is equal to the number of ways of constructing such a tree [3]–[5], neglecting the excluded volume effect.

The Eden model was introduced as a simple model for studying cluster growth laws and their individual statistical properties, such as the mean square radius of gyration and the fractal dimension. The exact solution was obtained only for formal Bethe lattices [4]. For actual lattices, the excluded volume effect hinders solving them by analytic methods. The method of computer modeling of clusters was used in [6]–[8], and the mean square radius of gyration was found by the $1/d$-expansion method in [3], [5].

In the growth models, we pass from analyzing properties of a separate cluster to analyzing properties of the whole ensemble of clusters based on studying their generating function [9]–[13]. Such an analysis is necessary for studying the analytic properties of time–spin correlation functions at high temperature because in the case of high-dimension lattices after passing to the imaginary time, these functions can be considered the generating functions of trees constructed following special rules. These rules follow because the order coefficient $2n$ of the power series in time can be determined in terms of the trace of the $2n$-fold

*Kirenskii Institute of Physics, Siberian Branch, RAS, Russia, e-mail: rsa@iph.krasn.ru.
†Krasnoyarsk State University, Krasnoyarsk, Russia.

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commutator of the system Hamiltonian with the operator of spin projection at a lattice site. This coefficient can be represented as a sum of weighted lattice figures composed of $2n$ bonds. The leading contribution at large d comes from trees composed of double bonds. The weight factor is determined primarily by the number of ways of constructing a given tree for which in each double bond, the first bonds are attached consecutively from the root to the end of a branch and the second bonds are attached consecutively from the end of a branch to the root. In addition, to each tree, we ascribe a small factor determined by the properties of the multiple commutator for the bond sequence determined by the tree. In passing to lower-dimensional lattices, we must add the rule forbidding branch intersections to the tree construction rules, and we must add the lattice figures with loops from bonds and with multiple bonds allowed between the neighbor sites to the exact coefficient of the time series. We observe a similarity with the Eden growth model in that we obtain a new tree upon changing the sequence of the bond attachment. Because the rules for constructing trees from elementary bonds are simpler in this model, we decided to study the effects of excluded volume in this example.

A clarification is in order here. If we neglect the volume restrictions, then the trees composed of the elementary bonds under consideration coincide exactly with the trees in the traditional Eden model. Taking the excluded volume effects into account results in substantial changes. In the traditional Eden model, a single cluster is constructed, and each newly attached bond must be chosen with equal probability from all accessible bonds, i.e., from bonds on the perimeter of the given, already constructed cluster. In contrast, we construct the whole ensemble of clusters, i.e., all clusters. We can therefore consequently choose all bonds at the perimeter of each cluster. In our case, clusters with a large perimeter are favored because they have a larger number of descendants. The difference between the approaches reflects the difference in aims. The quantity traditionally studied in growth models is the mean square radius of gyration of a cluster. In contrast, we want to estimate the increase in the number of trees when a new bond is added. This increase is characterized by the tree growth parameter, which is not defined for a separate tree.

In [11], we elaborated the procedure for finding corrections via the $1/d$-expansion for tree generating functions in the growth models. The close similarity of the properties of trees composed of single and of double bonds allows using the same technique to obtain the results for both. For simple hypercubic (SC) lattices, we obtained the leading $1/d$-corrections for the reciprocal tree growth parameters, i.e., for the coordinates of singular points of generating functions. The result for the Eden model was verified by computer experiment [12]. Other lattices are also interesting. For example, face-centered cubic (FCC) lattices are very important when analyzing the results of actual NMR experiments [13]. The FCC lattices differ from the SC lattices in the form of the dependence of the coordination number on the space dimension and in the form of lattice figures. In particular, the minimum loop from the nearest neighbors is a triangle, not a square as in the SC lattice case. Comparing the results for two types of lattices can provide additional information on the excluded volume effects in the growth models under consideration.

In this paper, we study the dependence of the tree growth parameter on the space dimension in the Eden model for FCC lattices. In Sec. 2, we consider the generating functions of trees with built-in loops. In Sec. 3, we construct the $1/d$-expansion for the coordinate of the singular point of the generating function. In Sec. 4, we use computer modeling to calculate the tree growth parameters in the Eden model for hypercubic lattices of dimensions from two to ten. Trees are constructed by the Monte Carlo method under the condition forbidding branch intersections. Finally, in Sec. 5, we compare results obtained by the $1/d$-expansion and by computer experiment as well as the results for the SC and FCC lattices.

2. Generating functions

We consider the d-dimensional lattice with the coordination number Z and find the number T_n of rooted trees composed of n elementary bonds. Given a lattice site (the tree root), we begin to construct a tree. The first bond can be attached in Z ways, i.e., $T_1 = Z$. The number T_2 already depends on the conditions imposed when constructing the tree. In the simplest case without restrictions,

$$T_2 = V_2 = 2Z^2$$

because the second bond can be attached to either of the two ends of the first bond in any of Z directions. If we impose the condition that the second bond must not coincide with the first bond, then

$$T_2 = 2Z(Z-1).$$

Consecutively attaching new bonds, we can easily find that the number of trees composed of n bonds constructed without restrictions is

$$V_n = n!\, Z^n. \tag{1}$$

If we exclude the bonds already attached at a site from the set of allowable new bonds at the site, then

$$T_n = B_n = \prod_{m=1}^{n} \big[m(Z-2)+2\big] = (Z-2)^n \frac{\Gamma(n+p+1)}{\Gamma(p+1)}, \tag{2}$$

where $p = p_{\mathrm{B}} = 2/(Z-2)$ and $\Gamma(x)$ is the gamma function. Result (2) gives the total number of trees without intersections constructed for the Bethe lattice (or for the Cayley tree) [4]. For such a construction on a hypercubic lattice, tree branches can intersect. In the limit $n \to \infty$, formula (2) admits the simple asymptotic expression

$$B_n \sim An!\,(Z_{\mathrm{C}})^n n^p, \tag{3}$$

where the tree growth parameter is $Z_{\mathrm{C}} = Z_{\mathrm{B}} = Z-2 = k$, the exponent $p = p_{\mathrm{B}} = 2/k$, and the constant $A = A_{\mathrm{B}} = 1/(p_{\mathrm{B}}+1)$ in the Bethe approximation under consideration,.

For comparison, we give the asymptotic formula for the number of standard rooted trees on the Bethe lattice [1]:

$$b_n \sim A_{\mathrm{B}} \lambda_{\mathrm{B}}^n n^{-3/2}, \quad n \to \infty,$$

where $\lambda_{\mathrm{B}} = \sigma^\sigma k^{-k}$, $A_{\mathrm{B}} = (2\pi)^{-1/2}(1+1/\sigma)(1+1/k)^{k+5/2}$, and $\sigma = Z-1$. The qualitative difference from formula (3) is the factorial factor.

The tree generating function for the Bethe lattice with the number of trees given by formula (2) has the simple form

$$E_{\mathrm{B}}(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \left(1 - \frac{x}{x_{\mathrm{B}}}\right)^{-Z/k}, \tag{4}$$

where $x_{\mathrm{B}} = 1/k = 1/(Z-2)$ is the coordinate of the singular point, which is the reciprocal tree growth parameter. In what follows, we use the generating function of the so-called trees with pending root for which only one bond is attached to the root,

$$E_1(x) = \left(1 - \frac{x}{x_{\mathrm{B}}}\right)^{-1/k}. \tag{5}$$

We now transfer trees constructed on the Bethe lattice to the d-dimensional hypercubic lattice. Some trees can be embedded only with a multiple use of the same sites and bonds, i.e., only if we allow self-intersections. The form of intersections and their number depend on the lattice type: SC or FCC. We turn to the differences between these cases.

We set the origin at a lattice site. We write the coordinates of another site measured in units of the lattice constant in the form

$$(a_1, a_2, \ldots, a_d). \tag{6}$$

The nearest neighbors on the SC lattice are placed along the coordinate axes and have only one nonzero coordinate in representation (6):

$$a_i = \pm 1, \quad i = 1, \ldots, d.$$

For the FCC lattice, two coordinates in representation (6) are nonzero for nearest neighbors:

$$a_i = \pm\frac{1}{2}, \qquad a_j = \pm\frac{1}{2}, \quad i, j = 1, \ldots, d.$$

The results in [11]–[13] for the coordination numbers Z and the numbers N_m of different embeddings of the loop composed of m bonds with the vertex at the origin for these two lattices are collected in Table 1.

Table 1

SC	FCC
$Z = 2d$	$Z = 2d(d-1)$
$N_4 = 2d(d-1)$ $N_6 = 2d(d-1)(8d-13)$	$N_3 = 4d(d-1)(d-2)$ $N_4 = 8N_3(d-2) - 3N_3 + 2d(d-1)$ $N_5 = 5N_3\big[4(d-2)\big]^2 + O(d^4)$

Coordination numbers Z and numbers N_m of different embeddings of the loop composed of m bonds with the vertex at the origin for SC and FCC lattices.

The loop composed of m bonds with the vertex at a site can be located in N_m ways, whereas the same bonds without restrictions can be located in Z^m ways. The ratios of these two quantities are of the order of $1/d^2$ for a square on the SC lattice and of the order of $1/d^3$ for a triangle on the FCC lattice, and they are therefore small quantities as $d \to \infty$. For a loop composed of a larger number of bonds, the losses are even greater. We consider trees with a loop as corrections to trees in Bethe approximation (2).

The generating function $R_m(x)$ for the number of trees with a pending root and a single branch intersection in the form of a simple loop composed of m bonds can be obtained using the auxiliary generating function

$$R(\theta, x) = \sum_{n=0}^{\infty} R_n(x)\theta^n, \tag{7}$$

for which (see the appendix) we have

$$R(\theta, x) = \left\{1 - y - 2\theta\frac{1 - y^{1+(1-\theta)/k}}{1 + (1-\theta)/k} + \theta^2\frac{1 - y^{1+2(1-\theta)/k}}{1 + 2(1-\theta)/k}\right\}\frac{1}{k(1-\theta)^2 y^{\sigma/k}}, \tag{8}$$

where $y = 1 - kx$, $k = Z - 2$, $\sigma = Z - 1$, and θ is the parameter of the vertex number. For instance, the coefficient of θ^3 in expansion (8) provides the generating function of trees with a triangle,

$$R_3(x) = 2\frac{4k^2 + 11k + 8}{kZ^2\sigma^3 y^{\sigma/k}} - \frac{4}{ky^{1/k}} - (k+3)\frac{2}{Z^2}y^{1/k} + (3k^2 + 8k + 6)\frac{2}{\sigma^3} + {} \\ {} + \frac{2}{kZ}y^{1/k}\log y - (2k+3)\frac{2}{k\sigma^2}\log y + \frac{1}{k^2\sigma}\log^2 y. \tag{9}$$

We do not write cumbersome expressions for other loops and instead give only their leading parts in the neighborhood of a singular point. These parts determine the numbers of large trees needed in what follows:

$$R_4(x) \cong 2\frac{8k^3+31k^2+42k+20}{Z^3\sigma^4 k y^{\sigma/k}},$$
$$R_5(x) \cong 2\frac{16k^4+79k^3+152k^2+136k+48}{Z^4\sigma^5 k y^{\sigma/k}}. \tag{10}$$

In the limit $d \to \infty$, we have the very simple formula

$$R_m(x) \approx 2^m k^{-(m+1)} y^{-\sigma/k}. \tag{11}$$

The above expressions for $R_m(x)$ pertain to a fixed position of the loop with the vertex at a given tree site. To obtain the complete result, we must sum over all positions, i.e., multiply by N_m.

3. The $1/d$-expansion for the singular point coordinate

We find the generating function of trees without intersections on a higher-dimensional hypercubic lattice from generating function (5) of the Bethe approximation. For this, we subtract the generating functions of trees with one loop:

$$E_1(x) = y^{-1/k} - E_{\mathrm{C}}(x), \tag{12}$$

where

$$E_{\mathrm{C}}(x) = \sum_m N_m\bigl(R_m(x) - R_{m+1}(x)\bigr). \tag{13}$$

The first term in this sum is the leading contribution of the loop composed of m bonds. The second term is the correction of the next order in $1/d$. While the loop composed of m bonds is created when two vertices belonging to different branches coincide in the first case, it is created in the second case when two bonds (four vertices) coincide [11]. The correction has the sign opposite to the leading term in (13) because it is already implicitly contained in the leading term but with a larger coefficient. As $d \to \infty$, the second term in (12) disappears, but it prevails over the first term at finite d in the limit $y \to 0$, because it contains the term $y^{\sigma/k}$ in the denominator. Of course, this does not mean that the number of trees with one loop is greater that the number of trees without restrictions. The reason is that we have several loops on a very large tree. But these loops are situated far from each other, and their contributions to corrections to the singular point coordinate x_{C} can be considered independent. The loop contribution can be found if we choose the dimension d sufficiently large and reason as follows.

In Bethe approximation (5), $x_{\mathrm{C}} = x_{\mathrm{B}} = 1/k$. Forbidding branch intersections increases the coordinate:

$$x_{\mathrm{C}} = x_{\mathrm{B}} + \delta x_{\mathrm{C}}.$$

We consider a function of two parameters x and x_{C} as the generating function and decompose it w.r.t. the small parameter δx_{C}:

$$E_1(x, x_{\mathrm{C}}) = E_1(x, x_{\mathrm{B}}) + \delta x_{\mathrm{C}} \frac{\partial E_1(x, x_{\mathrm{C}})}{\partial x_{\mathrm{C}}}\bigg|_{x_{\mathrm{C}}=x_{\mathrm{B}}} + \dots. \tag{14}$$

Comparing this formula with (12) and setting

$$\frac{\partial E_1(x, x_{\mathrm{C}})}{\partial x_{\mathrm{C}}}\bigg|_{x_{\mathrm{C}}=x_{\mathrm{B}}} = \frac{\partial E_1(x, x_{\mathrm{B}})}{\partial x_{\mathrm{B}}} = -kx(1-x)^{-\sigma/k},$$

we find

$$\delta x_{\mathrm{C}} = \frac{E_{\mathrm{C}}(x) y^{\sigma/k}}{kx}\bigg|_{x=x_{\mathrm{B}}}. \tag{15}$$

From this formula for the FCC lattice, taking triangles, quadrangles, and pentagons into account in (13), we find the first three terms of the $1/d$-expansion for the singular point coordinate,

$$\frac{\delta x_{\mathrm{C}}}{x_{\mathrm{B}}} = \frac{4}{d^3} + \frac{32}{d^4} + \frac{268}{d^5} + \ldots, \tag{16}$$

while for the SC lattice, taking squares and hexagons into account in (13), we have the first two terms of the $1/d$-expansion,

$$\frac{\delta x_{\mathrm{C}}}{x_{\mathrm{B}}} = \frac{2}{d^2} + \frac{12}{d^3} + \ldots. \tag{17}$$

To conclude this section, we note that the above procedure for expanding in $1/d$ differs from those used by other authors in [2], [3], [5], [14], [15]. Fisher and Gaunt found the $1/d$-expansion for the generating function for self-avoiding walks (linear polymers with the excluded volume) and noted that the corrections have a singularity power greater than the one for the leading term [14]. Then they again expanded the generating function in a series and estimated the critical value based on the number of walks. Harris applied a similar method to branched polymers with an excluded volume [15]. Calculating correlation functions for linear [16] and branched [17] polymers, other authors, using a formal redefinition of critical parameters, constructed strongly divergent corrections leading to an "ultraviolet divergence." But they did not address numerical estimates of these parameters. We have united these two approaches, directly finding the critical value x_{C} from $1/d$-corrections to the generating function. Eventually, Friedberg, following Parisi and Zhang [3], directly calculated the corrections to the mean square radius of gyration for trees with various loops in the Eden growth model and summed these corrections for like powers of $1/d$ [5]. He did not use generating functions.

4. Computer experiment

It is known [18] that the power series in $1/d$ are asymptotic series. To investigate the accuracy of series (16) above, we performed a computer experiment. Using the Monte Carlo method, we estimated the ratio T_n/V_n of trees composed of n bonds in the Eden model without branch intersections (their number is T_n) to the total number V_n of trees without restrictions (see formula (1)) on the d-dimensional FCC lattice. We represent the set of trees without restrictions as a set of points on the plane. From this set, T_n points correspond to trees without branch intersections, and the rest correspond to trees with intersections. If we choose points randomly, then the probability of hitting a tree without intersections is the desired ratio. To find this quantity, we proceed as follows. We take a site of the lattice as the root and consecutively construct a tree of n bonds each time randomly choosing a site of the already constructed part of the tree and a direction of the newly attached bond. Adding the current bond, we check: if an intersection occurs, we return to construct a new tree from the very beginning; if no intersection occurs, we continue building the tree. The desired ratio is the ratio of fully constructed trees to the total number of attempted trees.

The results for trees composed of different numbers n of bonds for the FCC lattices of dimensions 3, 4, 6, 8, and 10 are shown in Fig. 1. The values for $d = 3$, 4, and 6 are obtained with the total number of attempted trees equal to 10^9, and those for $d = 8$ and 10 are obtained with the total number of attempted trees equal to 10^8. We note that the calculated quantity is called the survival probability in the theory of linear polymers [19].

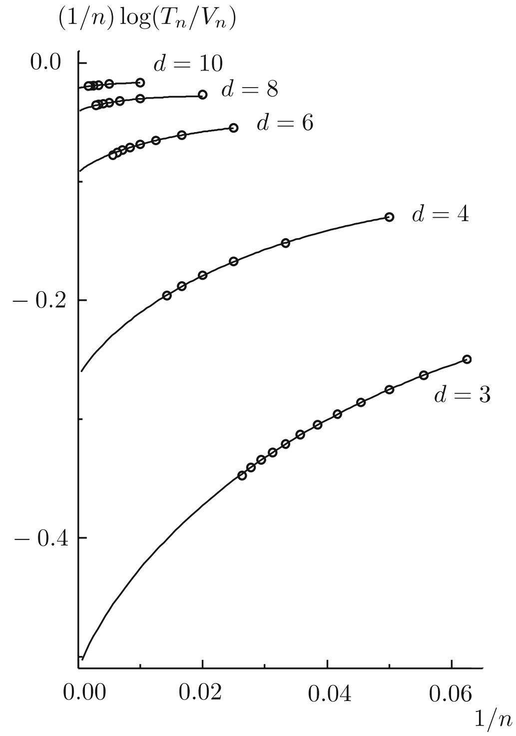

Fig. 1. The quantities $(1/n)\log(T_n/V_n)$ obtained by the Monte Carlo method (points) and extrapolation curves (19) with the parameters in Table 2 (lines) for d-dimensional FCC lattices as functions of $1/n$.

Table 2

d	Z/Z_C	A	p	Z_B/Z_C
3	1.670	0.073	2.46	1.392
4	1.304	0.079	1.76	1.196
6	1.097	0.117	0.99	1.061
8	1.042	0.155	0.64	1.023
10	1.021	0.371	0.32	1.010

Values of the parameters in formula (19) for the FCC lattices with d ranging from 3 to 10.

We previously performed the same calculation for the SC lattices. In [12], we also calculated the desired ratio for small trees in the Eden model by the exact enumeration method, thus confirming the correctness of results obtained with our realization of the Monte Carlo method. The calculation scheme was as follows. Constructing a usual lattice tree composed of n bonds without intersections, we calculated its weight equal to the number of ways of constructing the given tree. Each time, interchanging the order in which bonds are attached to a tree of a given form, we obtained a new way of constructing the tree. We thus accounted for all lattice trees composed of n bonds. The sum of their weights gave the desired number of trees composed of n bonds in the Eden model.

We return to the results for the FCC lattices. We assume that the number T_n of trees without branch intersections is also described by asymptotic expression (3) at large n but, of course, with other parameter values. For the desired ratio, we then have

$$\frac{T_n}{V_n} \sim A\left(\frac{Z_{\rm C}}{Z}\right)^n n^p. \tag{18}$$

Hence, the desired growth parameter $Z/Z_{\rm C}$ can be obtained by extrapolating the sequence

$$\frac{1}{n}\log\frac{T_n}{V_n} \approx \log\frac{Z_{\rm C}}{Z} + \frac{1}{n}\log A + \frac{p}{n}\log n, \quad n = 1, 2, \dots, \tag{19}$$

to the domain $1/n \to 0$. Such sequences for different d are shown by points in Fig. 1 as functions of $1/n$; we also show the extrapolation curves. The parameters in formula (19) for these curves obtained by the least square technique are collected in Table 2.

We estimate the error in finding $Z/Z_{\rm C}$ to be 1 to 2% and the error in finding the parameters p and A to be 20 to 30%. Such an indeterminacy presumably results from the indeterminacy of extrapolation using formula (18), not from a random distribution of data obtained by the Monte Carlo method.

5. Discussion

The desired result for correction (16) to the coordinate of the singular point of the generating function (the reciprocal tree growth parameter) in the FCC case can be obtained by subtracting unity from the quantities given in the last column of Table 2. The dependence of these results on the space dimension is shown in Fig. 2. There, we also show the corresponding dependences that follow from asymptotic formula (16) in which we keep different numbers of terms. At $d = 3$ and $d = 4$, the last term of the order of $1/d^5$ makes the matching worse, and this term must therefore be eliminated there, whereas all three terms can be retained at higher dimensions. The necessity of dropping terms of higher powers is an unpleasant property of asymptotic series [18]. For comparison, the dashed line in Fig. 2 represents the result for $\delta x_{\rm C}/x_{\rm B}$ obtained without expanding the quantities N_m and $R_m(x)$ for $m = 3, 4, 5$ in powers of $1/d$ in (13). This curve goes down at the dimensions $d = 3$ and $d = 4$ because the number N_m decreases because of the multiplier $(d-2)^{m-2}$, which we replace with d^{m-2} in the expansion.

The last term of the order of $1/d^3$ in asymptotic series (17) is redundant for all dimensions under consideration in the case of the SC lattice. Moreover, it was established in [12] that substituting $1/\sigma$, where $\sigma = 2d - 1$, for the expansion parameter $1/d$ in series (17) results in improving the match.

The comparison shows that the $1/d$-expansion leads to results matching the numerical simulations for both the FCC and the SC lattices. We hence conclude that leading corrections to the values of the singular point coordinate and the tree growth parameter are due to branch intersections with small loops, while contributions of large loops are small. We mention that these values themselves depend primarily on the coordination number of a lattice and differ for different lattices.

We now turn to the parameter p in formula (18). Its value shown in Table 2 differs from the result obtained with the Bethe approximation $p_{\rm B} = 2/(Z-2)$ by more than an order. On the other hand, the quantity p for the FCC lattice agrees within the error limits with its value for the SC lattice [12]. Based on this observation, we can assume that the singularity exponent p is determined by the dimension of the space. But it is impossible to draw a reliable conclusion based on the currently accessible tree sizes.

The above results on the number of trees show that the factorial multiplier is preserved in asymptotic formula (3) at $d = 3$. Hence, the perimeter of growing trees must be proportional to the number of bonds n. At a first glance, this conclusion contradicts the numerical data obtained by other authors (see, e.g., [8]), which state that the cluster in the Eden model has a smaller perimeter. In fact, there is no contradiction

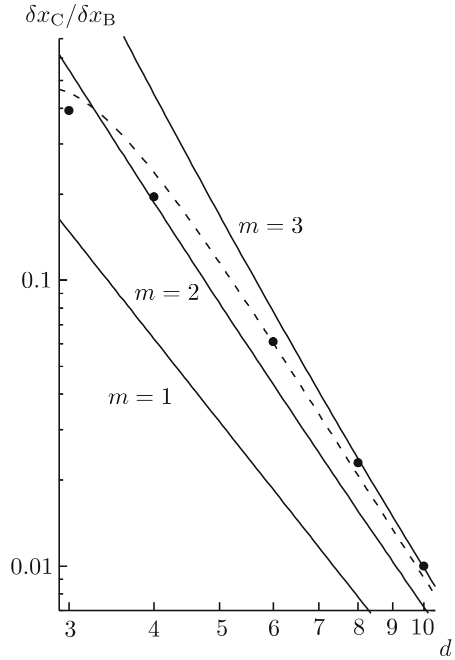

Fig. 2. The dependence of $\delta x_C/x_B$ on d in the double-logarithmic coordinates for the FCC lattices. Points correspond to data from Table 2 obtained by the Monte Carlo method. Solid lines represent expansion (16), in which we retain the first m terms. The dashed line is the result for $\delta x_C/x_B$ obtained without expanding N_m and $R_m(x)$ ($m = 3, 4, 5$) in powers of $1/d$.

because clusters are constructed following different rules. In [8], clusters grew from the beginning to the given size, one after another. The result can then be averaged over several constructed clusters. We tried to analyze the total ensemble of all possible clusters (trees) whose number grows exponentially as n increases (see formula (3)). In our case, a cluster with a larger perimeter has more descendants and therefore has a higher probability of being found in the total ensemble. In the approach in [8], the number of constructed clusters is predefined, and each of them will necessarily be constructed independently of the size of its perimeter.

In conclusion, we note that the results in the present paper concerning the excluded volume effect in the growth models are also important for trees constructed from double bonds. For the latter, the results agree well with the conclusions about the presence of singular points on the imaginary time axis, which follow because the spectra of these functions have exponential wings observed experimentally (see the references and discussions in [9]–[13]).

Appendix

For the reader's convenience, we here derive formula (8) for the generating function $R(\theta, x)$, which was presented without derivation in [11], where the main attention was given to trees composed of double bonds.

In the Bethe approximation, different branches of a tree are constructed independently, and generating function (5) therefore satisfies the self-consistent equation

$$E_1(x) = 1 + \int_0^x \left[E_1(x_1)\right]^\sigma dx_1.$$

When a loop is built into a large rooted tree, first, there exists a unique path (chain) leading to it from the tree root, and second, branches can grow from vertices of the chain and of the loop. The generating function of such trees with the chain composed of n bonds is

$$L_n(x,f)=\int_0^x dx_1\,E_k(x_1)\int_0^{x_1} dx_2\,E_k(x_2)\cdots\int_0^{x_{n-1}} dx_n\,E_k(x_n)f(x_n), \tag{A.1}$$

where

$$L_0(x,f)=f(x) \tag{A.2}$$

is the generating function of trees with the loop attached to the site next to the root and

$$E_k(x)=\bigl[E_1(x)\bigr]^k=\left(1-\frac{x}{x_{\mathrm{B}}}\right)^{-1}.$$

We introduce the auxiliary generating function

$$L(\beta,x,f)=\sum_{n=0}^{\infty}\beta^n L_n(x,f). \tag{A.3}$$

From (A.1) and (A.2), we obtain the equation for this function:

$$L(\beta,x,f)=f(x)+\beta\int_0^x E_k(x_1)L(\beta,x_1,f)\,dx_1 \tag{A.4}$$

or

$$\frac{d}{dx}L(\beta,x,f)=\frac{d}{dx}f(x)+\beta E_k(x)L(\beta,x,f). \tag{A.5}$$

For example, for the loop composed of three bonds, we have four variants of intersections and obtain

$$f(x)=2\int_0^x \bigl[E_1(x_1)\bigr]^{k-1}\bigl[L_2(x_1,F_1)L_1(x_1,F_1)+L_3(x_1,F_1)\bigr]\,dx_1, \tag{A.6}$$

where $L_n(x,E_1)$ is function (A.1) at $f(x)=E_1(x)$. Solving Eq. (A.5) with this condition imposed, we find

$$\Phi(\theta,x)=\sum_{n=0}^{\infty}\theta^n L_n(x,E_1)=\frac{y^{-1/k}-\theta y^{-\theta/k}}{1-\theta}, \tag{A.7}$$

where $y=1-kx$ and θ is the parameter of the number of vertices. Squaring the sum, we can easily find that the sum in the square brackets in (A.6) is equal to the coefficient of θ^3 in $\bigl[\Phi(\theta,x)\bigr]^2$.

We then obtain the desired expression (8) for

$$R(\theta,x)=L(Z-1,x,f)=\sum_{n=0}^{\infty}(Z-1)^n L_n\bigl(x,f(x,\theta)\bigr)$$

after solving Eq. (A.5) with $\beta=\sigma=Z-1$ and substituting

$$f(x,\theta)=\int_0^x \bigl[E_1(x_1)\bigr]^{k-1}\bigl[\Phi(\theta,x_1)\bigr]^2\,dx_1,$$

where the function $\Phi(\theta,x)$ is defined in (A.7).

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