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Characteristic invariants and Darboux's method

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Abstract
We develop a method that allows us to derive reductions and solutions to hyperbolic systems of partial differential equations. The method is based on using functions that are constant in the direction of characteristics of the system. These functions generalize well-known Riemann invariants. As applications we consider the gas dynamics system and ideal magnetohydrodynamics equations. In special cases, we find solutions of these equations depending on some arbitrary functions.

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1. Introduction

One of the first methods for finding solutions to the nonlinear partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1}$$

was proposed by Monge and was then further improved by Ampere. The method was described in detail in the classic books of Goursate [1] and Forsyth [2]. To apply this method, one must find an equation of the first order

$$f(x, y, u, u_x, u_y) = c, \qquad c \in R \tag{2}$$

such that every solution of (2) satisfies equation (1) for arbitrary c. In this case the function f is called a first integral of equation (1). To find first integrals, we need to look for functions which are constant in the direction of characteristics of equation (1). If there are two first integrals f_1 and f_2 for a given family of characteristics, then integration of equation (1) reduces to solving the first-order equation

$$G(f_1, f_2) = 0,$$

with G being an arbitrary function.

In 1870, Darboux [3] announced a generalization of the Monge–Ampere method. He proposed to seek an additional partial differential equation of second order (or higher)

$$g(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = c \tag{3}$$

such that the system of equations (1) and (3) is in involution for all c. The function g turns out to be constant along a family of characteristics of equation (1). In this case, the function g is called a characteristic invariant of (1). When partial differential equation (1) has a sufficient number of characteristic invariants, it can be reduced to a system of ordinary differential equations. The detailed description of the Darboux method is also given in the above-mentioned books [1, 2].

Although equations that are integrable by Darboux's method arise rarely, they are of great interest. Vessiot [4, 5] classified all equations of type

$$u_{xy} = w(x, y, u, u_x, u_y)$$

integrable by this method and found a general solution for every equation obtained. Recently, there was a renewed interest in the method of Darboux that was studied in [6–11].

It is possible to consider the left-hand side of (3) as a differential invariant of some formal vector field generalized by characteristics of equation (1). The differential invariants of vector fields have their origins in the works of Lie. The detailed description of the classical theory of the differential invariants can be found in [12]. Fels and Olver have discovered new applications of the differential invariants in the moving frame theory [13, 14].

In this paper, we consider systems of partial differential equations in two and n independent variables. In section 2, we introduce an operator of differentiation in the direction of characteristics of the system and corresponding invariants of characteristics of order k. We prove that if a function h defined on the kth order jet space $J^{(k)}$ is constant along a vector field on the solutions of the system of partial differential equations, then this function is an invariant of characteristics. In section 3, we describe the Darboux method for systems of partial differential equations in two independent variables and give its applications to the gas dynamics system and magnetohydrodynamics equations. We use invariants of characteristics to reduce these systems and find solutions depending on some arbitrary functions.

2. Invariants of characteristics

Let us begin with a system of first-order partial differential equations in two independent and m dependent variables

$$u_t + F(t, x, u, u_x) = 0, \tag{4}$$

where $u = (u^1, \ldots, u^m)$, $u_t = \left(u_t^1, \ldots, u_t^m\right)$, $u_x = \left(u_x^1, \ldots, u_x^m\right)$ and $F = (F^1, \ldots, F^m)$. One may emphasize that the function F can be nonlinear in u_x.

We denote by D_t and D_x the total derivatives with respect to t and x. Consider the differential operator

$$D_t + \lambda D_x, \tag{5}$$

whose coefficient λ can depend on t, x, u and u_x. The operator (5) is an operator of differentiation in the direction of characteristics of system (4), if the coefficient λ satisfies the equation

$$\det\left(\frac{\partial(F)}{\partial(u_x)} - \lambda E\right) = 0, \tag{6}$$

where $\frac{\partial(F)}{\partial(u_x)} = \frac{\partial(F^1,\ldots,F^m)}{\partial(u_x^1,\ldots,u_x^m)}$ is the Jacobi matrix and E is the identity matrix.

System (4) is hyperbolic if all eigenvalues of the matrix $\frac{\partial(F)}{\partial(u_x)}$ are real and there are m corresponding eigenvectors. The characteristic curves $\phi(t, x) = \text{const}$ are defined as solutions of equation $\phi_t + \lambda\phi_x = 0$. The theory of nonlinear hyperbolic systems is described in [15].

Suppose u_k^i is the partial derivative of order k of the function u^i with respect to x, then $u_k = \left(u_k^1, \ldots, u_k^m\right)$ stands for the vector composed of these derivatives. Let L be an operator of the differentiation in the direction of characteristics of system (4). According to [11], a function $h(t, x, u, \ldots, u_k)$ defined on the kth order jet space $J^{(k)}$ is called an invariant of characteristics of order k of system (4) corresponding to the operator L, if h is a solution of the equation

$$L(h)|_{[S]} = 0. \tag{7}$$

Here [S] means system (4) and its differential consequences with respect to x. When system (4) has the Riemann invariants, they are zero-order invariants of characteristics.

Some systems have invariants of characteristics of arbitrary order. For example, consider a one-dimensional system of gas dynamics equations [16]:

$$u_t + uu_x + p_x/\rho = 0, \qquad \rho_t + (\rho u)_x = 0, \qquad s_t + us_x = 0, \tag{8}$$

where u, ρ, p and s are the velocity, the density, the pressure and the entropy. The equation of state is given by the function $p = p(\rho, s)$. The operators of the differentiation in the direction of characteristics of system (8) are

$$L_1 = D_t + uD_x, \qquad L_2 = D_t + (u + c)D_x, \qquad L_3 = D_t + (u - c)D_x, \tag{9}$$

with $c = \sqrt{\partial p/\partial\rho}$ being the speed of sound. Obviously, the entropy s is an invariant of characteristics corresponding to the operator L_1. It is easy to check that the operator $\frac{1}{\rho}D_x$ commutes with L_1 by virtue of the second equation of system (8). This implies that the recurrent formula

$$I_{n+1} = \frac{1}{\rho}D_x(I_n), \qquad n = 0, 1, \ldots$$

gives the invariants of characteristics corresponding to the operator L_1. We will show in section 3 that invariants of characteristics corresponding to the operators L_2 and L_3 exist only for the special equations of state.

It can be proved [11] that if h_1 and h_2 are invariants of characteristics of system (4) corresponding to the operator L, then both an arbitrary function $f(h_1, h_2)$ and $h = \frac{D_x h_1}{D_x h_2}$ are invariants of characteristics.

Lemma 1. *Let L be an operator of form (5). Suppose that a function $h(t, x, u, u_1, \ldots, u_n)$, with $n \geqslant 1$, satisfies (7), then L is the operator of the differentiation in the direction of characteristics of system (4) and h is an invariant of characteristics corresponding to the operator L.*

Proof. According to the condition of the theorem, h is a solution of the equation

$$D_t h + \lambda D_x h|_{[S]} = 0. \tag{10}$$

Note that

$$D_x h \simeq \sum_{i=1}^{m} u_{n+1}^i h_{u_n^i},$$

where the symbol $\simeq$ means that the difference between left- and right-hand sides contains no derivatives of order greater than n. It is easy to see that the formula

$$D_t h \simeq -\sum_{1\leqslant i,j\leqslant m} u_{n+1}^j F_{u_1^j}^i h_{u_n^i}$$

is correct because of system (4). From equation (10) we have

$$-\sum_{1\leqslant i,j\leqslant m} u^j_{n+1} F^i_{u^j_1} h_{u^i_n} + \lambda \sum_{i=1}^{m} u^i_{n+1} h_{u^i_n} \simeq 0.$$

This yields m equations

$$\sum_{j=1}^{m} \left(F^i_{u^j_1} - \delta^i_j \lambda\right) h_{u^i_n} = 0, \qquad i = 1, \ldots, m,$$

with δ^i_j the Kronecker symbol. Rewriting the above equation in the matrix form

$$\left(\frac{\partial(F)}{\partial(u_x)} - \lambda E\right)\left(h_{u^1_n}, \ldots, h_{u^m_n}\right)^t = 0,$$

where $\left(h_{u^1_n}, \ldots, h_{u^m_n}\right)^t$ is the transposed vector, we conclude that λ is a solution of equation (6). □

We now consider the system of first-order partial differential equations in $n+1$ independent and m dependent variables

$$u_t + F\left(t, x, u, u_{x_1}, \ldots, u_{x_n}\right) = 0, \tag{11}$$

with $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$, $u_{x_i} = \left(u^1_{x_i}, \ldots, u^m_{x_i}\right)$ and $F = (F^1, \ldots, F^m)$.

Let us denote by u_k the set of kth order partial derivatives of the functions $u^1, \ldots, u^m$ with respect to $x_1, \ldots, x_n$. We say that a function $h(t, x, u, u_1, \ldots, u_k)$ defined on the kth order jet space $J^{(k)}$ is an invariant of characteristics of system (11) if h satisfies the equation

$$\det\left(E D_t h + \sum_{i=1}^{m} \frac{\partial(F)}{\partial(u_{x_i})} D_{x_i} h\right)\Bigg|_{[Sn]} = 0. \tag{12}$$

Here D_t and D_{x_i} are the total derivatives with respect to t and x_i, $[Sn]$ means system (12) and its differential consequences with respect to x_i $(i = 1, \ldots, n)$, $\frac{\partial(F)}{\partial(u_{x_i})} = \frac{\partial(F^1,\ldots,F^m)}{\partial(u^1_{x_i},\ldots,u^m_{x_i})}$ is the Jacobi matrix and E is an identity matrix.

An operator

$$L = D_t + \lambda_1 D_{x_1} + \cdots + \lambda_n D_{x_n},$$

where λ_i can depend on t, x, u, u_1, is called an operator of differentiation in the direction of the vector field $= (1, \lambda_1, \ldots, \lambda_n)$.

Theorem 1. *Suppose that there is an operator L of differentiation in the direction of the vector field* $= (1, \lambda_1, \ldots, \lambda_n)$ *and a function* $h(t, x, u, u_1, \ldots, u_k)$, *with* $k \geqslant 1$, *such that*

$$L(h)|_{[Sn]} = 0,$$

then h is an invariant of characteristics of system (11).

Proof. Since h satisfies

$$D_t h + \sum_{i=1}^{n} \lambda_i D_{x_i} h|_{[Sn]} = 0, \tag{13}$$

the coefficients of $(n+1)$th derivatives on the left-hand side of (13) must vanish. To find these coefficients, we write $D_{x_i} h$ up to kth derivatives:

$$D_{x_1} h \simeq \sum_{j=1}^{m} \sum_{|\alpha|=k} u^j_{\alpha+1_1} h_{u^j_\alpha}, \qquad \ldots, \qquad D_{x_n} h \simeq \sum_{j=1}^{m} \sum_{|\alpha|=k} u^j_{\alpha+1_n} h_{u^j_\alpha}.$$

Here u^j_α denotes the derivative

$$\frac{\partial^{|\alpha|}u^j}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

of order $k=|\alpha|=\alpha_1+\cdots+\alpha_n$, $u^j_{\alpha+1_i}$ is the derivative

$$\frac{\partial^{|\alpha|+1}u^j}{\partial x_1^{\alpha_1}\cdots\partial x_i^{\alpha_i+1}\cdots\partial x_n^{\alpha_n}},$$

and the symbol $\simeq$ means that the difference between the left- and right-hand sides contains no $(k+1)$th derivatives. This yields

$$\lambda_1 D_{x_1}h+\cdots+\lambda_n D_{x_n}h\simeq\sum_{i=1}^n\lambda_i\sum_{j=1}^m\sum_{|\alpha|=k}u^j_{\alpha+1_i}h_{u^j_\alpha}.$$

On the other hand, we have

$$D_t h|_{[Sn]}\simeq-\sum_{j=1}^m\sum_{|\alpha|=k}D^\alpha(F^j)h_{u^j_\alpha}\simeq-\sum_{j=1}^m\sum_{|\alpha|=k}\left(\sum_{i=1}^n\sum_{s=1}^m u^s_{\alpha+1_i}F^j_{u^s_{x_i}}\right)h_{u^j_\alpha}.$$

The above calculations lead to

$$D_t h+\sum_{i=1}^n\lambda_i D_{x_i}h|_{[Sn]}\simeq-\sum_{j=1}^m\sum_{|\alpha|=k}\left[\sum_{i=1}^n\left(\sum_{s=1}^m u^s_{\alpha+1_i}F^j_{u^s_{x_i}}-\lambda_i u^j_{\alpha+1_i}\right)\right]h_{u^j_\alpha}=0.$$

It is convenient to represent the last relation in a matrix form

$$\sum_{i=1}^n\sum_{|\alpha|=k}u_{\alpha+1_i}A^{x_i}h_{u_\alpha}=0,\tag{14}$$

with $u_\alpha=\left(u^1_\alpha,\ldots,u^m_\alpha\right)$, $h_{u_\alpha}=\left(h_{u^1_\alpha},\ldots,h_{u^m_\alpha}\right)$ and $A^{x_i}=\frac{\partial(F)}{\partial(u_{x_i})}-\lambda_i E$.

We need to prove that h is a solution of equation (12) which is equivalent to the following:

$$\det\left(\sum_{i=1}^n A^{x_i}D_{x_i}h\right)=0.\tag{15}$$

For this purpose, it is enough to show that the linear homogeneous system

$$\left(\sum_{i=1}^n A^{x_i}D_{x_i}h\right)r=0\tag{16}$$

has a nontrivial solution r. This solution is expressed in the form

$$r=\sum_{|\alpha|=k}(Dh)^\alpha h_{u_\alpha},\tag{17}$$

where $(Dh)^\alpha=\left(D_{x_1}h\right)^{\alpha_1}\cdots\left(D_{x_n}h\right)^{\alpha_n}$. Indeed, substituting (17) in the left-hand side of (13) leads to

$$\left(\sum_{i=1}^n A^{x_i}D_{x_i}h\right)\left(\sum_{|\alpha|=k}(Dh)^\alpha h_{u_\alpha}\right)=\sum_{i=1}^n\sum_{|\alpha|=k}A^{x_i}(Dh)^{\alpha+1_i}h_{u_\alpha}.\tag{18}$$

Note that the expressions including $u_{\alpha+1_i}$ in (14) coincide with ones including $(Dh)^{\alpha+1_i}$ in (18). Since the left-hand side of (14) is zero then (18) is equal to zero as well. Hence, (15) is valid. □

When the conditions of the theorem are satisfied, we say that a function h defined on the kth order jet space $J^{(k)}$ is constant along a vector field $v=(1,\lambda_1,\ldots,\lambda_n)$ on the solutions of system (11).

3. The Darboux method and its applications

In this section, we will describe the Darboux method for systems of partial differential equations and give relevant examples. The detailed description of applications of this approach to second order partial differential equations in two independent variables is given in [1].

The Darboux method is based on using the invariants of characteristics. Let us consider system (4) and assume that the corresponding equation (6) has m distinctive real roots $\lambda_1, \ldots, \lambda_m$. If there are two functionally independent invariants of characteristics I_i, J_i for all λ_i, then we can constitute a system of ordinary differential equations in the independent variable x

$$f_1(I_1, J_1) = 0, \qquad \ldots, \qquad f_m(I_m, J_m) = 0, \tag{19}$$

where $f_1, \ldots, f_m$ are arbitrary functions. It is necessary to look for the general solution of systems (4) and (19). If we can find a general solution of (19), then substituting this solution into (4) leads to a system of ordinary differential equations in the independent variable t. It is enough to solve the last system in order to find the general solution of (4).

Remark 1. When equation (6) has a root of multiplicity k it is desirable to obtain $k + 1$ invariants $J_1, \ldots, J_{k+1}$ corresponding to this root. In this case, system (19) includes equations

$$f_1(J_2, J_1) = 0, \qquad \ldots, \qquad f_k(J_{k+1}, J_1) = 0.$$

Such an example arises naturally in the equations of magnetohydrodynamics.

As example, consider the system of gas dynamics equations in two independent variables t and x:

$$u_t + uu_x + p_x/\rho = 0, \qquad \rho_t + (\rho u)_x = 0, \qquad p_t + up_x + \rho c^2 u_x = 0. \tag{20}$$

where ρ is the density, u is the velocity, p is the pressure and $c(\rho, p)$ is the sound speed.

One can easily deduce zero-order invariants of characteristics of system (20) corresponding to the operator L_2 (or L_3) given by (9). To do so, following [11], one must seek all solutions of the equation

$$D_t h + (u + c) D_x h|_{[G]} = 0, \tag{21}$$

where $[G]$ stands for system (20) and its differential consequences with respect to x; the function h can depend on t, x, u, ρ, p. Obviously, the left-hand side of (21 is a polynomial of the first degree in u_x, ρ_x, and p_x. Collecting similar terms of these variables leads to the following equations:

$$h_\rho = 0, \qquad h_u = \rho c h_p, \qquad h_t + (u + c) h_x = 0. \tag{22}$$

It follows from the first and second equations of (22) that a nonconstant solution h exists only if

$$c = g(p)/\rho, \tag{23}$$

with g an arbitrary function of p. As a consequence of the third equation, h is independent of t and x. According to the second equation of (22), h is an arbitrary function of

$$I^+ = u + \int \frac{\mathrm{d}p}{g(p)}.$$

Similarly, it is possible to check that the Riemann invariant

$$I^- = u - \int \frac{\mathrm{d}p}{g(p)}$$

corresponds to the operator L_3.

We now use the invariant I^- to derive solutions of system (20. Setting $I^- = 0$ and introducing a new function $F = \int \frac{\mathrm{d}p}{g(p)}$, we get the following representation:

$$u = F(p).$$

In this case, system (20) reduces to

$$\rho_t + (F\rho)_x = 0, \qquad p_t + Fp_x + \frac{p_x}{F'\rho} = 0. \tag{24}$$

The previous system admits the Riemann invariants

$$p, r = \int (F')^2 \, \mathrm{d}p + \frac{1}{\rho}.$$

One then can rewrite system (24 in the form

$$p_t + \left(F(p) - \frac{G(p) - r}{F'(p)} \right) p_x = 0, \qquad r_t + F(p) r_x = 0,$$

where $G(p) = \int (F'(p))^2 \, \mathrm{d}p$. Using the hodograph transformation leads to the linear system

$$x_p - F(p) t_p = 0, \qquad (F(p)F'(p) - G(p) + r)t_r - F'(p)x_r = 0.$$

From this system it is easy to obtain equation

$$F'(p) t_{pr} - F''(p) t_r = 0.$$

The general solution of the above system is

$$t = P + R'F', \qquad x = R'(FF' + 1/\rho) - R + \int FP' \, \mathrm{d}p, \tag{25}$$

where $P = P(p)$ and $R = R(r)$ are arbitrary functions. As a consequence of (25), we obtain a solution of (20) in the implicit form.

The equation of state which corresponds to (23) has the form

$$\frac{1}{\rho} = H(s) + M(p),$$

where $M' = -\frac{1}{g(p)^2}$ and H is an arbitrary function. Using Martin's variables [17] and one intermediate integral, Zavyalov [18] found some solutions of one-dimensional gas dynamics system with the previous equation of state. However, the corresponding solutions include Martin's variables. Note that Zavyalov's solutions also depend on two arbitrary functions.

Let us try to find solutions of system (20) which depend on three arbitrary functions. It was shown in [11] that the first-order invariants corresponding to the operators L_2 and L_3 exist only if the speed sound is given by

$$c = (a + bp)^{(2/3)}/\rho, \qquad a, b \in R.$$

The corresponding equation of state is

$$p(\rho, s) = -\frac{1}{b} \left[a + \left(\frac{3\rho}{b\,(A(s)\rho - 1)} \right)^3 \right],$$

the Riemann invariants are

$$I_2 = bu + 3(a + bp)^{1/3}, \qquad I_3 = bu - 3(a + bp)^{1/3},$$

and the first-order invariants have the form

$$J_2 = \frac{\rho(a + bp)^{1/3}}{u_x(a + bp)^{2/3} + p_x} - \frac{bt}{3}, \qquad J_3 = \frac{\rho(a + bp)^{1/3}}{u_x(a + bp)^{2/3} - p_x} - \frac{bt}{3}.$$

The invariants corresponding to the operator $L_1 = D_t + uD_x$ (mentioned in section 2) are

$$I_1 = b/\rho - 3(a + bp)^{-1/3}, \qquad J_1 = 1/\rho D_x(I_1).$$

The gas dynamics system is conveniently written in terms of the Riemann invariants

$$\begin{aligned}(I_1)_t &= -\frac{I_2 + I_3}{2b}(I_1)_x,\\ (I_2)_t &= \frac{I_1\left(I_2^2 - I_3^2\right) - 2I_2M}{36b}(I_2)_x,\\ (I_3)_t &= \frac{I_1\left(I_2^2 - I_3^2\right) - 2I_3M}{36b}(I_3)_x,\end{aligned} \tag{26}$$

with $M = I_1(I_2 - I_3) + 18$. The first-order invariants J_1, J_2 and J_3 can be represented as

$$J_1 = \frac{M}{I_2 - I_3}(I_1)_x, \qquad J_2 = t - \frac{18b}{M(I_2)_x}, \qquad J_3 = t - \frac{18b}{M(I_3)_x}. \tag{27}$$

We now apply the Darboux method to reduce system (26) to some ordinary differential equations. The corresponding system (19) is equivalent to

$$J_1 = F_1(I_1), \qquad J_2 = F_2(I_2), \qquad J_3 = F_3(I_3), \tag{28}$$

where F_1, F_1 and F_3 are arbitrary functions. From (26), (27) and (28) we get two systems of ordinary differential equations:

$$(I_1)_x = \frac{F_1(I_1)(I_2 - I_3)}{M}, \qquad (I_2)_x = \frac{18b}{M\,(t - F_2(I_2))}, \qquad (I_3)_x = \frac{18b}{M\,(t - F_3(I_3))}, \tag{29}$$

and

$$\begin{aligned}(I_1)_t &= -\frac{F_1(I_1)\left(I_2^2 - I_3^2\right)}{2Mb},\\ (I_2)_t &= \frac{I_1\left(I_2^2 - I_3^2\right) - 2I_2M}{2M(t - F_2(I_2))},\\ (I_3)_t &= \frac{I_1\left(I_2^2 - I_3^2\right) - 2I_3M}{2M(t - F_3(I_3))}.\end{aligned} \tag{30}$$

Introducing new functions

$$\Psi(I_1) = \int \frac{bI_1}{F_1(I_1)}\,\mathrm{d}I_1, \qquad G_i(I_i) = \int F_i(I_i)\,\mathrm{d}I_i, \qquad i = 2, 3,$$

one may write equations (29) in the following way:

$$[bx - \Psi(I_1)]_x = \frac{18b}{M}, \qquad [tI_2 - G_2(I_2)]_x = \frac{18b}{M}, \qquad [tI_3 - G_3(I_3)]_x = \frac{18b}{M}.$$

Hence, system (29) has the first integrals

$$tI_2 - G_2(I_2) - bx + \Psi(I_1) = c_2(t), \qquad tI_3 - G_3(I_3) - bx + \Psi(I_1) = c_3(t),$$

with $c_2(t)$ and $c_3(t)$ being the arbitrary functions. Differentiating the previous relations with respect to t and using system (30), we deduce that the functions $c_2(t)$ and $c_3(t)$ are constants.

Therefore, system (26) can be reduced to a couple of differential equations

$$(I_1)_x = \frac{bI_1(I_2 - I_3)}{\Psi'(I_1)\,(I_1(I_2 - I_3) + 18)}, \qquad (I_1)_t = -\frac{I_1\left(I_2^2 - I_3^2\right)}{2\Psi'(I_1)(I_1(I_2 - I_3) + 18)}, \tag{31}$$

where I_2 and I_3 must be expressed from relations

$$tI_2 - G_2(I_2) - bx + \Psi(I_1) = 0, \qquad tI_3 - G_3(I_3) - bx + \Psi(I_1) = 0.$$

It is possible to find solutions of the gas dynamics equations by integrating equations (31) with partial functions G_2, G_3 and Ψ.

It is interesting to note that there are the second-order invariants of characteristics

$$I^{\pm}_{(4/5)} = \frac{3}{5}t + \frac{p^{1/5}\left(5\rho p p_{xx} \pm 5p^{9/5}u_x x - 5p\rho_x p_x \mp 5p^{9/5} - p^{8/5}\rho u_x^2 - 3\rho p_x^2\right)}{(p^{4/5}u_x \pm p_x)^3},$$

$$I^{\pm}_{(2)} = \frac{Gp^3(\rho u_x G \pm G_x)}{\rho},$$

with $G = \frac{\rho}{p^2 u_x \pm p_x}$, corresponding to the operators $L^{\pm} = D_t + (u \pm c)D_x$, when the speed sound is given by one of the following formulae:

$$c = p^{4/5}/\rho, \qquad c = p^2/\rho.$$

We now consider the one-dimensional magnetohydrodynamics equations [19]

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&u_t + uu_x + \frac{p_x}{\rho} + \frac{\left(B_2{}^2 + B_3{}^2\right)_x}{8\pi\rho} = 0,\\
&v_t + uv_x = 0,\\
&w_t + uw_x = 0,\\
&(B_2)_t + (uB_2)_x = 0,\\
&(B_3)_t + (uB_3)_x = 0,\\
&s_t + us_x = 0.
\end{aligned} \tag{32}$$

Here ρ is the density, p is the pressure and s is the entropy; (u, v, w) and (B_1, B_2, B_3) denote the velocity and the magnetic fields, respectively. We assume that $B_1 = 0$ and p is a function of ρ and s. In this case, there are the following invariants of characteristics:

$$I_1 = s, \qquad I_2 = v, \qquad I_3 = w, \qquad I_4 = \frac{B_2}{\rho}, \qquad I_5 = \frac{B_3}{\rho}, \qquad J = \frac{s_x}{\rho},$$

corresponding to the operator $L_1 = D_t + uD_x$.

Using these invariants we will reduce system (32) to the second-order equation for the entropy. According to the general scheme of the Darboux method, we write

$$v = V(s), \qquad w = W(s), \qquad s_x/\rho = f_1(s), \qquad B_2/\rho = f_2(s), \qquad B_3/\rho = f_3(s),$$

where V, W, f_1, f_2 and f_3 are arbitrary functions. These relations are equivalent to the following representation:

$$v = V(s), \qquad w = W(s), \qquad \rho = s_x\Pi(s), \qquad B_2 = s_x\Phi(s), \qquad B_3 = s_xF(s), \tag{33}$$

with $\Pi(s) = 1/f_1(s)$, $\Phi(s) = f_2(s)/f_1(s)$ and $F(s) = f_3(s)/f_1(s)$.

From the last equation of (32) we get

$$u = -\frac{s_t}{s_x}. \tag{34}$$

Substituting (33) and (34) into system (32) one can check that six equations of (32) are satisfied identically and only the second equation leads to

$$\begin{aligned}
&-4\pi s_x{}^2\Pi(s)s_{tt} + 8\pi s_t s_x\Pi(s)s_{tx} + \left(4\pi s_x{}^2\Pi(s)p'_\rho + s_x{}^3F(s)^2 - 4\pi s_t{}^2\Pi(s) + s_x{}^3\Phi(s)^2\right)s_{xx}\\
&\qquad + 4\pi s_x{}^3 p'_s + 4\pi s_x{}^4 p'_\rho\Pi'(s) + s_x{}^5\Phi(s)\Phi'(s) + s_x{}^5F(s)F'(s) = 0.
\end{aligned} \tag{35}$$

Suppose that the pressure has the following form:

$$p = G(s) - \frac{\Phi(s)^2 + F(s)^2}{8\pi\Pi(s)^2}\rho^2,$$

where G is an arbitrary function, then equation (35) reduces to

$$s_x{}^2 s_{tt} - 2s_t s_x s_{tx} + s_t{}^2 s_{xx} - \frac{G'(s)}{\Pi(s)} s_x^3 = 0. \tag{36}$$

It is possible to find two intermediate integrals for equation (36)

$$\frac{s_t}{s_x} - \frac{G'(s)t}{\Pi(s)} = \phi(s), \qquad x - \frac{s_t}{s_x}t + \frac{G'(s)}{2\Pi(s)}t^2 = \psi(s), \tag{37}$$

with ϕ and ψ being arbitrary functions. Eliminating s_t/s_x from (37) gives the implicit solution of (36)

$$x + \phi(s)t + \frac{G'(s)}{2\Pi(s)}t^2 = \psi(s). \tag{38}$$

In the special cases one can express s from (38) and find the explicit solutions of the one-dimensional magnetohydrodynamics equations (32). Note that other authors usually suppose that the modified pressure $p^* = p + \left(B_2^2 + B_3^2\right)/8\pi$ is constant [20]. This means that the function G is equal to zero. The general solution of (36) with $G = 0$ can be found in [1].

Now consider the system of two equations

$$u_t + v_x = 0, \qquad v_t + f^2(x, v)u_x = 0. \tag{39}$$

This system arises in many areas of science and engineering relating to fluid mechanics and nonlinear elasticity (see, for example, [15, 21]). We want to find characteristic invariants and derive some solutions of system (39).

The operators of differentiation in the direction of characteristics of system (39) are

$$D_t \pm f(v, x)D_x.$$

It can be shown that zero-order characteristic invariants exist if and only if $f(v, x) = 1/F'(av + bx)$ with F an arbitrary function. These invariants take the form

$$I^{\pm} = au - tb \pm F(av + bx), \qquad a, b \in R.$$

Introducing new functions $r = I^+$ and $s = I^-$, we rewrite system (39) as follows:

$$r_x + G(r - s)r_t = 0, \qquad s_x - G(r - s)s_t = 0,$$

with $G(\theta) = F'(F^{-1}(\theta/2))$. Using the hodograph transformation leads to the linear system

$$t_r + G(r - s)x_r = 0, \qquad t_s - G(r - s)x_s = 0.$$

From this system, it is easy to obtain the Euler–Darboux equation

$$x_{rs} + g(r - s)(x_r + x_s) = 0,$$

where $g(\theta) = G'(\theta)/2G(\theta)$. It is possible to find general solution of this equation for particular functions g [22]. For example, suppose that $f(v, x) = (av + bx)^2$ then the general implicit solution of system (39) is given by

$$x = \frac{R' - S'}{r - s}, \qquad t = \frac{(R' + S')(r - s) - 2(R - S)}{4},$$

where $R = R(r)$ and $S = S(s)$ are arbitrary functions.

If we set $f(v, x) = x^n v^m$ then the first-order characteristic invariants exist if and only if $n = 2/3$ or $n = 4/3$, with $m = 2 - n$. These invariants have the form

$$J^{\pm}_{(2/3)} = \pm\frac{u}{3} - (x^2 v)^{-1/3} - \left(\frac{v}{x}\right)^{2/3} \frac{1}{x v_x \pm x^{5/3} v^{4/3} u_x - v},$$

$$J^{\pm}_{(4/3)} = \pm\frac{t}{3} - \left(\frac{v}{x}\right)^{1/3} \frac{1}{x v_x \pm x^{7/3} v^{2/3} u_x - v}.$$

The second-order characteristic invariants exist only when $n = 2/3, n = 4/3, n = 4/5$ and $n = 6/5$. The explicit expressions of these invariants are very unwieldy.

4. Conclusion

In this paper, we have developed the method of integrating a system of first-order partial differential equations in two independent variables. This method can be extended to the hyperbolic system of high order equations. It is important that corresponding invariants should exist for every family of characteristics of the system.

When the system includes equations in n ($n \geqslant 3$) independent variables then we face a difficult task. As usual these systems have few invariants of characteristics. For example, the two-dimensional unsteady gas dynamics equations admit only one invariant of characteristics, namely the entropy. In three-dimensional case, Ertel's integral is an additional invariant of characteristics. Note that in the case of the two-dimensional steady gas dynamics equations, Kaptsov [11] have founded a first-order invariant

$$J_0 = \frac{D_y(I_B)}{s_y},$$

where I_B is Bernoulli's integral

$$I_B = \frac{u^2 + v^2}{2} + \int \frac{1}{\rho} p'_\rho \, \mathrm{d}\rho.$$

Here, as usual, u and v are the components of velocity, p is the pressure, ρ is the density and s is the entropy.

There are problems that we have not yet studied. For example, it is interesting to look for new solutions of the one-dimensional gas dynamics equations (20) using the second-order invariants ($I^{\pm}_{(4/5)}$ and $I^{\pm}_{(2)}$), give interpretation to some founded solutions, and apply the Darboux method to other hyperbolic systems. The other important problem we have not yet considered deals with nonlocal characteristic invariants. One can check that the functions

$$I^{\pm} = \xi - vt - xu \mp F\left(\frac{v}{x}\right)$$

are nonlocal characteristic invariants of system (39), with $f(v, x) = x^2/F'\left(\frac{v}{x}\right)$. Here, the variable $\xi(v, x)$ satisfies the following relations:

$$\xi_v = t, \qquad \xi_x = u.$$

To simplify calculations, we have implemented the package of analytical computations which derives characteristics and their invariants for the given system.

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References

[1] Goursat E 1898 *Leçons sur intégration des équations aux dérivées partielles du second order a deux variables indépendantes* vol 2 (Paris: Hermann)
[2] Forsyth A R 1959 *Theory of Differential Equations* vol 6 (New York: Dover)
[3] Darboux G 1870 Sur la théorie des équations aux dérivées partielles *Comptes Rendus* **70** 746–9
[4] Vessiot E 1939 Sur les équations aux dérivées partielles du second order $F(x, y, u, p, q, r, s, t) = 0$ intérgable par la méthode de Darboux *J. Math. Pure Appl.* **18** 1–61
[5] Vessiot E 1941 Sur les équations aux dérivées partielles du second order $F(x, y, u, p, q, r, s, t) = 0$ intérgable par la méthode de Darboux *J. Math. Pure Appl.* **21** 1–66
[6] Bryant R, Griffiths P and Hsu L 1995 Hyperbolic exterior differential systems and their conservation laws, II *Sel. Math. New Ser.* **1** 265–323

[7] Anderson I M and Kamran N 1997 The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane *Duke Math. J.* **87** 265–319
[8] Anderson I M and Juráš M 1997 Generalized Laplace invariants and the method of Darboux *Duke Math. J.* **89** 351–75
[9] Vassiliou P J 2001 Tangential characteristic symmetries and first order hyperbolic systems *AAECC* **11** 377–95
[10] Sokolov V V and Zhiber A V 2001 Exactly integrable equations of Liouville type *Russ. Math. Survey* **56** 61–101
[11] Kaptsov O V 2004 Invariants of characteristics of some systems of partial differential equations *Siberian Math. J.* **45** 475–87
[12] Ovsyannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[13] Fels M and Olver P J 1998 Moving coframes: I. A practical algorithm *Acta Appl. Math.* **51** 161–213
[14] Fels M and Olver P J 1999 Moving coframes: II. Regularization and theoretical foundations *Acta Appl. Math.* **55** 127–208
[15] Rozhdestvenskii B L and Yanenko N N 1983 *Systems of Quasilinear Equations and Their Applications to Gas Dynamics* (Providence, RI: American Mathematical Society)
[16] Courant R and Friedrichs K O 1999 *Supersonic Flow and Shock Waves (Series: Applied Mathematical Sciences)* vol 21 (New York: Springer) p 464
[17] Martin M H 1953 The propagation of a plane shock into a quiet atmosphere *Can. J. Math.* **5** 37–9
[18] Zavyalov Y S 1955 On some integrals of one-dimensional gas dynamics *Dokl. Akad. Nauk USSR* **103** 781–2
[19] Pai S I 1962 *Magnetogasdynamics and Plasma Dynamics* (Vienna: Springer)
[20] Kulikovskii A G and Lyubimov G A 1962 *Magnetohydrodynamics* (Moscow: Fizmatgiz)
[21] Kulikovskii A G and Sveshnikova E I 1998 *Nonlinear Waves in Elastic Medium* (Moscow: Moscow's Lyceum)
[22] von Mises R 1958 *Mathematical Theory of Compressible Fluid Flow* (New York: Academic)