

Effects of long-wave inhomogeneities of the exchange and elastic force constants in the framework of the coherent potential approximation

V. A. Ignatchenko and V. A. Felk

L. V. Kirensky Institute of Physics SB RAS, 660036, Krasnoyarsk, Russia

(Received 10 July 2006; revised manuscript received 14 September 2006; published 14 November 2006)

The formalism of averaged Green's functions in the continuum is developed for spin waves in ferromagnets with inhomogeneities of the exchange constant and elastic waves in media with inhomogeneities of the force constant. The integral self-consistent equation in the coherent potential approximation is developed for taking into account effects of inhomogeneities of these nonlocal characteristics of the material on the wave spectrum in it. The obtained general results are applied to the study of effects of one-dimensional inhomogeneities of the exchange constant on the wave spectrum in a ferromagnet.

DOI: [10.1103/PhysRevB.74.174415](https://doi.org/10.1103/PhysRevB.74.174415)

PACS number(s): 75.30.Ds, 76.50.+g

I. INTRODUCTION

The averaged Green's function for a ferromagnet with inhomogeneities of the magnetic anisotropy was found in our paper¹ in the framework of the coherent potential approximation (CPA) and effects of these inhomogeneities on the frequencies and linewidths of magnetic resonances in ferromagnets (ferromagnetic resonance as well as spin-wave resonance) were studied. It is well known that the CPA method, which has been proposed by Soven² and Taylor,³ is applicable immediately for calculations of the density of states or wave spectrum in materials with inhomogeneities of the local characteristics, like the density of the material for elastic waves, the value and direction of the anisotropy axis for spin waves, and the dielectric constant for electromagnetic waves. At the lattice model of the medium the inhomogeneities of these parameters correspond to the single-site approximation. At the continuum model these parameters are coefficients before corresponding variables or time derivatives of these variables in the expressions for the energy density. The consideration of inhomogeneities of the nonlocal characteristics of the material, like the force constants for elastic waves or exchange constant for spin waves, in the lattice model leads to the necessity of summation of the interactions over all atoms of the environment. In connection with this the complex problems arise of taking into account the extended defects, as well as the diagonal and off-diagonal disorder. As a result the direct transfer of the CPA ideas to the case of nonlocal characteristics of the medium becomes impossible. A number of approximate approaches were suggested to overcome the problems arising in this case (see reviews,^{4,5} books,^{6,7} and references therein). However, not one of these approaches is considered to be wholly satisfactory for the present.

In this paper we develop the CPA method for the continuum model of a medium corresponding to the approximation of the long waves as compared to the interatomic distance. It is well known that in going from the lattice model to the continuum limit the nonlocal terms in the Hamiltonian containing the summation over atoms of the environment transforms into terms containing spatial derivatives of the correspondent variables. Nonlocal characteristics describing the interaction with the environment are the coefficients be-

fore these spatial derivatives of the classical Hamiltonian variables. This gives us the possibility of developing quite another approach to deriving the CPA equation than approaches used for the lattice model.

The objectives of this paper are the development of the formalism of averaged Green's functions and the derivation of the CPA equation for inhomogeneities of the nonlocal characteristics of the continuum medium, as well as the application of the obtained results for the study of effects of 1D inhomogeneities of the exchange constant on the spin-waves spectrum in ferromagnets.

The paper is made up as follows. In Sec. II, we derive the equation for spin waves in the medium with the inhomogeneous exchange constant and the equation for elastic waves in the medium with the inhomogeneous force constant. We also develop the formalism of averaged Green's functions and the analogue of the Dyson equation for these situations. In Sec. III, we derive the integral CPA equation for inhomogeneities of nonlocal characteristics of the continuum medium. In Sec. IV, we demonstrate the application of the derived equation with an example of the calculation of the exchange constant inhomogeneity effects on the spin wave spectrum in ferromagnets. In Sec. V, we summarize and discuss the obtained results.

II. MODELS AND METHOD

First we consider the spin waves in a ferromagnet with the inhomogeneous exchange constant $\alpha(\mathbf{x})$, where $\mathbf{x}=(x,y,z)$. We describe the dynamics of a ferromagnetic medium by the classic Landau-Lifshitz equation

$$\dot{\mathbf{M}} = -g[\mathbf{M} \times \mathbf{H}_{\text{eff}}], \quad (1)$$

where \mathbf{M} is the magnetization, g is the gyromagnetic ratio, and \mathbf{H}_{eff} is the effective magnetic field determined by the expression

$$\mathbf{H}_{\text{eff}} = -\frac{\partial \mathcal{H}_m}{\partial \mathbf{M}} + \frac{\partial}{\partial \mathbf{x}} \frac{\partial \mathcal{H}_m}{\partial (\partial \mathbf{M} / \partial \mathbf{x})}. \quad (2)$$

We choose the energy density \mathcal{H}_m in the form

$$\mathcal{H}_m = \frac{1}{2}\alpha(\mathbf{x})\left(\frac{\partial\mathbf{M}}{\partial\mathbf{x}}\right)^2 - \frac{1}{2}\beta(\mathbf{M}\mathbf{I})^2 - \mathbf{M}\mathbf{H} - \frac{1}{2}\mathbf{M}\mathbf{H}_m, \quad (3)$$

where β and \mathbf{I} are the value of the magnetic anisotropy and direction of its axis, respectively, \mathbf{H} is the external magnetic field, and \mathbf{H}_m is the magnetodipole field. We assume that the direction of the anisotropy coincides with the direction \mathbf{H} which is parallel to the z axis.

We represent the exchange parameter in the form

$$\alpha(\mathbf{x}) = \alpha[1 + \gamma\rho(\mathbf{x})], \quad (4)$$

where α is the average value of the exchange, γ is its relative rms fluctuations, and $\rho(\mathbf{x})$ is a centered ($\langle\rho\rangle=0$) and normalized ($\langle\rho^2\rangle=1$) random function of coordinates. The angle brackets here and following denote averaging over the ensemble of random realizations. Stochastic properties of the function $\rho(\mathbf{x})$ are characterized by a correlation function, depending on the difference of the coordinates $\mathbf{r}=\mathbf{x}'-\mathbf{x}''$,

$$K(\mathbf{r}) = \langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle, \quad (5)$$

or by the spectral density $S(\mathbf{k})$ which is connected with $K(\mathbf{r})$ by a Fourier transformation

$$S(\mathbf{k}) = \frac{1}{(2\pi)^3} \int K(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}. \quad (6)$$

The effective magnetic field for the energy density Eq. (3) has the form

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \mathbf{H}_m + \beta\mathbf{I}(\mathbf{M}\mathbf{I}) + \frac{\partial}{\partial x_i}\left(\alpha(\mathbf{x})\frac{\partial\mathbf{M}}{\partial x_i}\right). \quad (7)$$

Here and following the summation over repetitive subscripts or superscripts is assumed.

We perform the usual linearization of Eq. (1) ($M_z \approx M; M_x, M_y \ll M$), take $M_x, M_y \propto e^{i\omega t}$, and introduce the circular projections

$$m^\pm = M_x \pm iM_y. \quad (8)$$

We assume that the sample has symmetrical form in the xy plane. In this case the equations for m^\pm get uncoupled and for the resonance projection m^+ we have the equation

$$\frac{\partial^2 m^+}{\partial x_i^2} + \nu m^+ + \gamma \frac{\partial}{\partial x_i}\left(\rho(\mathbf{x})\frac{\partial m^+}{\partial x_i}\right) = 0, \quad (9)$$

where the designation is introduced

$$\nu = \frac{\omega - \omega_0}{\alpha g M}. \quad (10)$$

Here ω_0 is the uniform FMR frequency

$$\omega_0 = g[H - (N_z - N_{xy})M + H_a], \quad (11)$$

where N_z and N_{xy} are the demagnetization factors along the z axis and in the xy plane, respectively, and $H_a = \beta M$ is the average value of the anisotropy field.

We are coming now to the problem of the elastic waves in the medium with the inhomogeneous force constant. The dynamics of the elastic medium are described by the equation

$$P\ddot{\mathbf{u}} = \frac{\partial}{\partial \mathbf{x}} \frac{\partial \mathcal{H}_u}{\partial (\partial \mathbf{u} / \partial \mathbf{x})}, \quad (12)$$

where P is the density of the material, $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is the elastic displacement vector, and \mathcal{H}_u is the density of the elastic energy.

To emphasize the fundamental sides of the problem considered we restrict ourselves here to the simplest model of inhomogeneous elastic medium where the interaction with environment is described by the one scalar constant $C(\mathbf{x})$, that is the density of the energy has the form

$$\mathcal{H}_u = \frac{1}{2}C(\mathbf{x})\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^2. \quad (13)$$

We represent $C(\mathbf{x})$ in the form analogous to Eq. (4):

$$C(\mathbf{x}) = C[1 + \gamma\rho(\mathbf{x})], \quad (14)$$

where C is the average value of the force constant and γ is its relative rms fluctuation. Stochastic properties of the centered and normalized random function $\rho(\mathbf{x})$ are characterized by the correlation function $K(\mathbf{r})$ and the spectral density $S(\mathbf{k})$ described by Eqs. (5) and (6), respectively. Substituting Eq. (13) for Eq. (12) we obtain the equation of motion for the vector $\mathbf{u} \propto e^{i\omega t}$ in the form

$$\frac{\partial^2 \mathbf{u}}{\partial x_i^2} + \nu \mathbf{u} + \gamma \frac{\partial}{\partial x_i}\left(\rho(\mathbf{x})\frac{\partial \mathbf{u}}{\partial x_i}\right) = 0. \quad (15)$$

Here ν is determined by the formula

$$\nu = \left(\frac{\omega}{s}\right)^2, \quad (16)$$

where $s=(C/P)^{1/2}$ is the elastic wave velocity.

So, the forms of the equations of motion for the spin waves in the medium with the inhomogeneous exchange constant Eq. (9) and for the elastic waves in the medium with the inhomogeneous force constant Eq. (15) are the same in the approximation considered. Because of this the equation for the Green's function $G(\mathbf{x},\mathbf{x}_0)$ has the same form for both these cases

$$\frac{\partial^2 G(\mathbf{x},\mathbf{x}_0)}{\partial x_i^2} + \nu G(\mathbf{x},\mathbf{x}_0) + \gamma \frac{\partial}{\partial x_i}\left(\rho(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial x_i}\right) = \delta(\mathbf{x}-\mathbf{x}_0). \quad (17)$$

We recall that in the cases of inhomogeneities of local characteristics (the magnetic anisotropy for spin waves, the density of the material for elastic waves, and the dielectric constant for electromagnetic waves) the equation for the Green's function in the scalar approximation can be represented in the form

$$\frac{\partial^2 G(\mathbf{x},\mathbf{x}_0)}{\partial x_i^2} + \nu G(\mathbf{x},\mathbf{x}_0) + \eta\rho(\mathbf{x})G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0), \quad (18)$$

where parameters ν and η are expressed by various means for each type of wave in terms of the frequency, average values of material parameters, and their rms fluctuations.¹

It is well known for the Green's function satisfying Eq. (18) and averaging over fluctuations of the material parameter the representation in the form of the integral series in the correlators of the random function $\rho(\mathbf{x})$, as well as the Dyson equation

$$\begin{aligned} \bar{G}(\mathbf{x}, \mathbf{x}_0) &= G_0(\mathbf{x}, \mathbf{x}_0) \\ &+ \int \int G_0(\mathbf{x}, \mathbf{x}_1) Q(\mathbf{x}_1, \mathbf{x}_2) \bar{G}(\mathbf{x}_2, \mathbf{x}_0) d\mathbf{x}_1 d\mathbf{x}_2. \end{aligned} \quad (19)$$

Here \bar{G} is the averaged Green's function, G_0 is the initial Green's function for the uniform media, and Q is the mass operator, the expansion of which in terms of the correlators is also well known (see, for example, Ref. 8).

The inhomogeneous term in Eq. (17) differs radically from the corresponding term in Eq. (18): if only a product of the random function and Green's function ρG is in Eq. (18), then Eq. (17) contains in fact two terms containing derivatives of these functions, $\rho(\partial^2 G / \partial x_i^2)$ and $(\partial \rho / \partial x_i)(\partial G / \partial x_i)$. Because of that, our first task is representing the averaged solution of Eq. (17) in the form of the perturbation expansion and deriving the analogue of the Dyson equation, corresponding to Eq. (17).

Representing the Green's function in the form of $G = G_0 + G'$ we obtain the equation for G' from Eq. (17) in the form

$$\frac{\partial^2 G'(\mathbf{x}, \mathbf{x}_0)}{\partial x_i^2} + \nu G'(\mathbf{x}, \mathbf{x}_0) = -\gamma \frac{\partial}{\partial x_i} \left(\rho(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial x_i} \right). \quad (20)$$

According to the general rules the solution of Eq. (20) can be represented as the integral of a product of its right-hand side and the initial Green's function. From this it follows that the generating integral equation for the expansion of the Green's function has the form

$$\begin{aligned} \bar{G}(\mathbf{x}, \mathbf{x}_0) &= G_0(\mathbf{x}, \mathbf{x}_0) + \gamma^2 \int \int \frac{\partial G_0(\mathbf{x}, \mathbf{x}_1)}{\partial x_1^i} \frac{\partial^2 G_0(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_1^i \partial x_2^j} \frac{\partial G_0(\mathbf{x}_2, \mathbf{x}_0)}{\partial x_2^j} K(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \\ &+ \gamma^4 \int \int \int \int \frac{\partial G_0(\mathbf{x}, \mathbf{x}_1)}{\partial x_1^i} \frac{\partial^2 G_0(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_1^i \partial x_2^j} \frac{\partial^2 G_0(\mathbf{x}_2, \mathbf{x}_3)}{\partial x_2^j \partial x_3^k} \frac{\partial^2 G_0(\mathbf{x}_3, \mathbf{x}_4)}{\partial x_3^k \partial x_4^l} \frac{\partial G_0(\mathbf{x}_4, \mathbf{x}_0)}{\partial x_4^l} [K(\mathbf{x}_1, \mathbf{x}_2)K(\mathbf{x}_3, \mathbf{x}_4) + K(\mathbf{x}_1, \mathbf{x}_3)K(\mathbf{x}_2, \mathbf{x}_4) \\ &+ K(\mathbf{x}_1, \mathbf{x}_4)K(\mathbf{x}_2, \mathbf{x}_3)] d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 + \dots \end{aligned} \quad (25)$$

This series differs radically from the usual series corresponding to Eq. (18): (i) not the Green's functions but its derivatives are in the integral terms of this expansion, the first derivatives of the functions corresponding to the initial and final points of the length and the second derivatives of the

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}_0) &= G_0(\mathbf{x}, \mathbf{x}_0) \\ &- \gamma \int G_0(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial x_i'} \left(\rho(\mathbf{x}') \frac{\partial G(\mathbf{x}', \mathbf{x}_0)}{\partial x_i'} \right) d\mathbf{x}'. \end{aligned} \quad (21)$$

However, this equation and the expansion of the Green's function following from it are inconvenient for use because they contain derivatives of the random function $\rho(\mathbf{x})$. Because of that we transform Eq. (21) by applying the integration by parts:

$$\begin{aligned} \int_V G_0(\mathbf{x}, \mathbf{x}') \frac{\partial P_i(\mathbf{x}', \mathbf{x}_0)}{\partial x_i'} d\mathbf{x}' &= \int_S G_0(\mathbf{x}, \mathbf{x}') P_i(\mathbf{x}', \mathbf{x}_0) n_i(\mathbf{x}') ds \\ &- \int_V \frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial x_i'} P_i(\mathbf{x}', \mathbf{x}_0) d\mathbf{x}', \end{aligned} \quad (22)$$

where $\mathbf{n}(\mathbf{x}')$ is the outer normal in a point \mathbf{x}' on the surface S and

$$P_i(\mathbf{x}', \mathbf{x}_0) = \rho(\mathbf{x}') \frac{\partial G(\mathbf{x}', \mathbf{x}_0)}{\partial x_i'}. \quad (23)$$

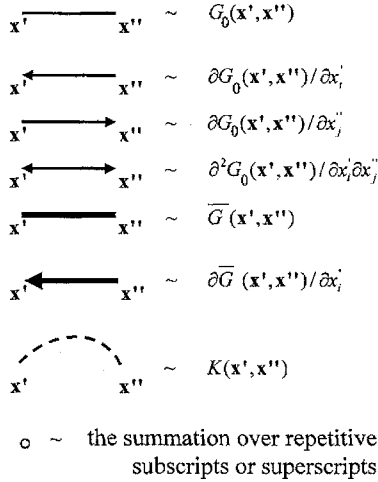
Moving off the surface S to infinity where the Green's function vanishes we obtain the generation equation in the form

$$G(\mathbf{x}, \mathbf{x}_0) = G_0(\mathbf{x}, \mathbf{x}_0) + \gamma \int \frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial x_i'} \rho(\mathbf{x}') \frac{\partial G(\mathbf{x}', \mathbf{x}_0)}{\partial x_i'} d\mathbf{x}'. \quad (24)$$

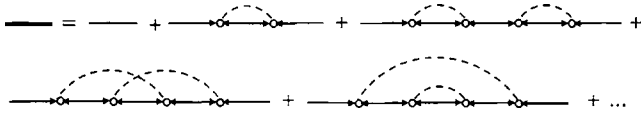
By employing the usual procedure of sequential substitutes of the left-hand side of this equation to the right-hand side we obtain an infinite series. By averaging this series over the ensemble of the random functions $\rho(\mathbf{x})$ and uncoupling the correlators with the Gauss formula, we obtain the perturbation expansion of the averaged Green's function in the form

functions corresponding to the intermediate points, (ii) the summation over repetitive superscripts are performed in all these products.

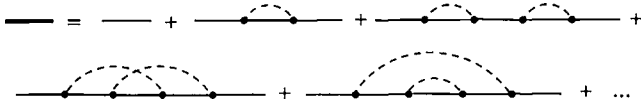
We introduce the following diagram designations as in the graphic below.



Equation (25) in the diagram representation has the form shown below.



For comparison we recollect the diagram form of the \overline{G} representation for Eq. (18) in the graphic below.



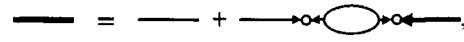
The analogue of the Dyson equation for the function \overline{G} corresponding to Eq. (17) is derived from Eq. (25) in the usual way and has the form

$$\overline{G}(\mathbf{x}, \mathbf{x}_0) = G_0(\mathbf{x}, \mathbf{x}_0) + \int \int \frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial x_i'} Q_{i'j''}(\mathbf{x}', \mathbf{x}'') \frac{\partial \overline{G}(\mathbf{x}'', \mathbf{x}_0)}{\partial x_j''} d\mathbf{x}' d\mathbf{x}'' \quad (26)$$

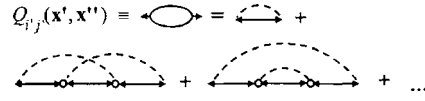
where the mass operator is the matrix whose components have the form

$$Q_{i'j''}(\mathbf{x}', \mathbf{x}'') = \gamma^2 \frac{\partial^2 G_0(\mathbf{x}', \mathbf{x}'')}{\partial x_i' \partial x_j''} K(\mathbf{x}', \mathbf{x}'') + \gamma^4 \int \int \frac{\partial^2 G_0(\mathbf{x}', \mathbf{x}_1)}{\partial x_i' \partial x_1^k} \times \frac{\partial^2 G_0(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_1^k \partial x_2^l} \frac{\partial^2 G_0(\mathbf{x}_2, \mathbf{x}'')}{\partial x_2^l \partial x_j''} [K(\mathbf{x}', \mathbf{x}_2) K(\mathbf{x}_1, \mathbf{x}'') + K(\mathbf{x}', \mathbf{x}'') K(\mathbf{x}_1, \mathbf{x}_2)] d\mathbf{x}_1 d\mathbf{x}_2 + \dots \quad (27)$$

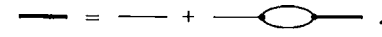
By immediate substitution of Eqs. (25) and (27) to the right-hand side of Eq. (26) one can convince himself of the validity of the latter. The analogue of the Dyson equation in the diagram representation has the form as shown below,



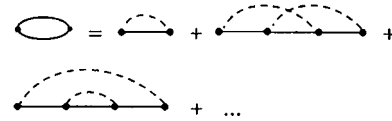
where the components of the mass operator matrix have the form as follows.



We recall that the diagram form of the Dyson equation for the function \overline{G} corresponding to the wave equation (18) and the scalar mass operator are described by the graphs shown below.



where



III. DERIVATION OF THE CPA EQUATION

We must derive the self-consistent CPA equation for the wave equation (17) describing effects of inhomogeneities of the nonlocal characteristics of the material. We use for this the method suggested in our paper¹ for the simpler situation of the local characteristics described by the wave equation (18). We repeat at first briefly the main points of this derivation which will be necessary when we consider further the more complex case of deriving the CPA equation for the nonlocal terms of the Hamiltonian. In doing so we will also correct misprints in the numerical coefficients which have been committed in Eqs. (16)-(20) of Ref. 1. In the uniformly random media, where all variables depend only on the difference of the coordinates $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, the Dyson equation for the function \overline{G} has the form

$$\overline{G}(\mathbf{x} - \mathbf{x}_0) = G_0(\mathbf{x} - \mathbf{x}_0) + \int \int G_0(\mathbf{x} - \mathbf{x}_1) Q(\mathbf{x}_1 - \mathbf{x}_2) \times \overline{G}(\mathbf{x}_2 - \mathbf{x}_0) d\mathbf{x}_1 d\mathbf{x}_2. \quad (28)$$

Using the integral Fourier transformation

$$\overline{G}(\mathbf{r}) = \int \overline{G}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}, \quad G_0(\mathbf{r}) = \int G_0(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}, \quad Q(\mathbf{r}) = \int Q(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}, \quad (29)$$

we obtain the Dyson equation for the Fourier transformations $\overline{G}(\mathbf{k})$, $G_0(\mathbf{k})$, and $Q(\mathbf{k})$ and find $\overline{G}(\mathbf{k})$ from it in the form

$$\overline{G}(\mathbf{k}) = \frac{1}{G_0^{-1}(\mathbf{k}) - (2\pi)^{2d} Q(\mathbf{k})}, \quad (30)$$

where d is the space dimensionality. For the case of the infinitive medium $G_0^{-1} = (2\pi)^d (\nu - k^2)$ and Eq. (30) takes the form

$$\bar{G}(\mathbf{k}) = \frac{1}{(2\pi)^d} \frac{1}{\nu - k^2 - (2\pi)^d Q(\mathbf{k})}. \quad (31)$$

Changing \mathbf{k} to \mathbf{k}_1 in Eq. (31), multiplying this equation by $\eta^2 S(\mathbf{k} - \mathbf{k}_1)$, and integrating it over \mathbf{k}_1 we obtain

$$\eta^2 \int \bar{G}(\mathbf{k}) S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1 = \frac{\eta^2}{(2\pi)^d} \int \frac{S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - (2\pi)^d Q(\mathbf{k}_1)}. \quad (32)$$

For obtaining the self-consistent equation for $Q(\mathbf{k})$ we require that the left-hand side of Eq. (32) is approximately equal to $Q(\mathbf{k})$:

$$Q(\mathbf{k}) \approx \eta^2 \int \bar{G}(\mathbf{k}_1) S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1. \quad (33)$$

To justify this requirement let us carry out the inverse Fourier transformation of Eq. (33):

$$Q(\mathbf{x} - \mathbf{x}_0) \approx \eta^2 \bar{G}(\mathbf{x} - \mathbf{x}_0) K(\mathbf{x} - \mathbf{x}_0). \quad (34)$$

The more detailed form of approximate equality (34) is given by the formula (21) of Ref. 1. It follows from Eq. (34) that the approximate equation for $Q(\mathbf{k})$, which comes from Eq. (32) after substituting Eq. (33) for it, takes into account in each order of the value η only those diagrams that have correlations between the initial point x_0 and the final point x along with all other allowed correlations. Therefore, the approximate equality (33) is satisfied by the conditions of the CPA.

It is convenient to renormalize the Fourier transformation of the mass operator

$$M_k = (2\pi)^d Q(k) \quad (35)$$

and to rewrite the system of the CPA equations for Eq. (18) in the form

$$\bar{G}_k = \frac{1}{(2\pi)^d} \frac{1}{\nu - k^2 - M_k}, \quad M_k = \eta^2 \int \frac{S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - M_{\mathbf{k}_1}}. \quad (36)$$

It is the case of the standard CPA corresponding to the single-site approximation in the lattice model.

We are coming now to the more complex case of the nonlocal interaction in a medium. The analogue of the Dyson equation has the form of Eq. (26) in this case. Rewriting this equation for the uniformly random media, where all variables depend only on the difference of coordinates, and using the integral Fourier transformations (29) we obtain the analogue of the Dyson equation in the form

$$\bar{G}(\mathbf{k}) = \frac{1}{(2\pi)^d} \frac{1}{\nu - k^2 - (2\pi)^d k^i k^j Q_{ij}(\mathbf{k})}. \quad (37)$$

Changing \mathbf{k} to \mathbf{k}_1 in Eq. (37), multiplying this equation by $\gamma^2 k_1^i k_1^j S(\mathbf{k} - \mathbf{k}_1)$, and integrating it over \mathbf{k}_1 we obtain

$$\eta^2 \int k_1^i k_1^j \bar{G}(\mathbf{k}_1) S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1 = \frac{\gamma^2}{(2\pi)^d} \int \frac{k_1^i k_1^j S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - (2\pi)^d k_1^i k_1^j Q_{ij}(\mathbf{k}_1)}. \quad (38)$$

The CPA in this case consists of equating the left-hand side of this equation by $Q_{ij}(\mathbf{k})$. This replacement corresponds to the fulfillment of the requirement that

$$Q_{i'j''}(\mathbf{x}' - \mathbf{x}'') \approx \gamma^2 \frac{\partial^2 \bar{G}(\mathbf{x}' - \mathbf{x}'')}{\partial x_{i'} \partial x_{j''}} K(\mathbf{x}' - \mathbf{x}''). \quad (39)$$

The latter approximate requirement means that we take into account in the graph equation only those diagrams that have correlations between the initial point x_0 and final point x along with all other allowed correlations. After the change of the left-hand side of Eq. (38) by $Q_{ij}(\mathbf{k})$ we have the system of the nine bound equations for the components of the mass operator matrix

$$Q_{ij}(\mathbf{k}) \approx \frac{\gamma^2}{(2\pi)^d} \int \frac{k_1^i k_1^j S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - (2\pi)^d k_1^i k_1^j Q_{ij}(\mathbf{k}_1)}. \quad (40)$$

We multiply Eq. (40) by $(2\pi)^d k^i k^j$ and sum over twice the repeated subscripts and superscripts

$$(2\pi)^d k^i k^j Q_{ij}(\mathbf{k}) \approx \gamma^2 \int \frac{k^i k^j k_1^i k_1^j S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k^2 - (2\pi)^d k_1^i k_1^j Q_{ij}(\mathbf{k}_1)}. \quad (41)$$

Then we represent the product of the \mathbf{k} components in the vector form

$$k^i k_1^i k^j k_1^j = (\mathbf{k} \mathbf{k}_1)^2 \quad (42)$$

and introduce to Eqs. (37) and (41) the scalar characteristic of the mass operator

$$T_{\mathbf{k}} = (2\pi)^d k^i k^j Q_{ij}(\mathbf{k}). \quad (43)$$

The final system of the CPA equations for the case of the long-wave inhomogeneities of the nonlocal parameters of the Hamiltonian has the form

$$\bar{G}_{\mathbf{k}} = \frac{1}{(2\pi)^d} \frac{1}{\nu - k^2 - T_{\mathbf{k}}}, \quad T_{\mathbf{k}} = \gamma^2 \int \frac{(\mathbf{k} \mathbf{k}_1)^2 S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - T_{\mathbf{k}_1}}. \quad (44)$$

These equations take into account the diagonal as well as the off-diagonal disorder because the total scalar potential of interactions $T_{\mathbf{k}}$ is the sum of all components of the matrix $k^i k^j Q_{ij}$.

IV. SPIN WAVES IN THE MEDIUM WITH 1D INHOMOGENEITIES OF THE EXCHANGE CONSTANT

We restrict ourselves in this work to the application of the general CPA equations obtained in the previous section to the case of one-dimensional (1D) inhomogeneities of the ex-

change constant in a ferromagnet. For this case the system of equations (44) takes the form

$$\bar{G}_k = \frac{1}{2\pi} \frac{1}{\nu - k^2 - T_k}, \quad T_k = \gamma^2 k^2 \int_{-\infty}^{+\infty} \frac{k_1^2 S(k - k_1) dk_1}{\nu - k_1^2 - T_{k_1}}. \quad (45)$$

We model the correlation properties of inhomogeneities by the exponential function $K(\mathbf{r})$ and the corresponding spectral density $S(\mathbf{k})$:

$$K(r_z) = e^{-k_c |r_z|}, \quad S(k) = \frac{1}{\pi} \frac{k_c}{k^2 + k_c^2}. \quad (46)$$

Here k_c is the correlation wave number of the inhomogeneities ($r_c = k_c^{-1}$ is the correlation radius; for the case of poly- or nanocrystal $2r_c$ is equal to the size of the grain).

The integral equation (45) can be represented in the form of the infinite chain fractions of the integral expressions which are proportional to γ^2 ,

$$T_k \approx \int \frac{\gamma^2 k_1^2 S(k - k_1) dk_1}{\nu - k_1^2 - \gamma^2 k_1^2 \int \frac{k_2^2 S(k_1 - k_2) dk_2}{\nu - k_2^2 - \gamma^2 k_2^2 \int \frac{k_3^2 S(k_2 - k_3) dk_3}{\nu - k_3^2 - \gamma^2 k_3^2 \int \dots}}. \quad (47)$$

For the numerical analysis of these chain fractions it is convenient to represent them in the form of the recurrent formula

$$T_k^{(n+1)} = \gamma^2 k^2 \int \frac{k_1^2 S(k - k_1) dk_1}{\nu - k^2 - T_{k_1}^{(n)}}. \quad (48)$$

The initial form of the mass term $T_k^{(0)}$ must be prescribed from some physical assumption. We use two approaches to this problem.

A. Standard approach

In this approach we assume that $T_k^{(0)} = 0$. The first integral in Eq. (48) in this case can be taken with the help of the residue theory, and we obtain the first approximation in the form

$$T_k^{(1)} = \gamma^2 k^2 \frac{k^2 + ik_c \sqrt{\nu} + k_c^2}{(\sqrt{\nu} - ik_c)^2 - k^2}. \quad (49)$$

In principle, the second approximation also can be found with the help of the residue theory because a biquadratic form in k_1 is obtained in the denominator of Eq. (48) after substitution of Eq. (49) in it. However, the resulting expression is so cumbersome that the second approximation, as well as all subsequent, are found by the numerical integration.

B. Method of algebraic equations

This method was proposed by us in Ref. 1. It consists of the following. Let us consider the integral equation for T_k

(45) at $k_c \rightarrow 0$. Then $S(k - k_1) \rightarrow \delta(k - k_1)$, the integral in Eq. (45) can be performed exactly, and this equation transforms into an algebraic equation of the second power in T_k . The main assumption of the method proposed in Ref. 1 is that for obtaining the first approximation at $k_c \neq 0$ we can set $T_{k_1} \approx T_k$ keeping at the same time the spectral density $S(k - k_1)$ in the exact form. Under this assumption the integral in Eq. (45) is performed exactly and we obtain for the first approximation $T_k^{(1)}$ the following transcendental equation:

$$T_k^{(1)} = \gamma^2 k^2 \frac{k^2 + ik_c \sqrt{\nu - T_k^{(1)}} + k_c^2}{(\sqrt{\nu - T_k^{(1)}} - ik_c)^2 - k^2}. \quad (50)$$

Introducing the new variable $z = (\nu - T_k^{(1)})^{1/2}$ we reduce this equation to the algebraic equation of the fourth power

$$(z^2 - \nu)[(z - ik_c)^2 - k^2] + \gamma^2 k^2 (k^2 + k_c^2 + ik_c z) = 0. \quad (51)$$

The solution of Eq. (51) was found numerically. As in the case considered in Ref. 1, the dependence of the mass term $T_k^{(1)}$ on the variables ν and k cannot be described by only one root of Eq. (51). It is necessary to use different roots z_i in the different intervals of ν and k . For the selection of the roots the following criteria are applied. First, the imaginary part of the mass term $T_k^{(1)}$ must be positive for physical reasons. Second, the solution of Eq. (51) must satisfy Eq. (50). These two criteria that were suggested in Ref. 1 permit us uniquely to construct the function $T_k^{(1)}$ which substitutes for the recurrent formula (48).

The method of algebraic equations has several advantages over the standard approach, which have been analyzed in Ref. 1. The main is a very quick convergence of the iteration process when we use $T_k^{(1)}$ obtained from Eqs. (50) and (51). The reason is that the method of algebraic equations gives the single maximum in the imaginary part of the Green's function G'' vs ν even in the first approximation, whereas in the standard approach at small k_c the first approximation as well as a number of the following approximations leads to the physically senseless many-mode dependencies of $G''(\nu)$. Because of that, we need only several steps to complete the iteration process of Eq. (48) using $T_k^{(1)}$ obtained from Eqs. (50) and (51). Contrary to that, using $T_k^{(1)}$ obtained by the standard approach one must perform several tenths or even hundreds of successive integrations for obtaining the satisfactory result. On the other hand, the method of algebraic equations in some intervals of ν and k can lead to difficulties with the selection of the roots z_i and, correspondingly, with obtaining the unique dependence of $T_k^{(1)}$ on ν and k . That is why we use both of these methods for obtaining the Green's function in the present work.

The typical calculated dependence of the imaginary part of the Green's function on ν is shown in Fig. 1 for $(k/k_c)^2 = 1$ and $\gamma^2 = 0.5$. Such dependencies were calculated for different values of k/k_c and γ . It gave us the possibility to plot the dependencies of the point ν_m at which G'' vs ν has a maximum and the linewidth $\Delta\nu$ on k^2 , as well as the dependencies of $\Delta\nu$ on γ^2 . In Figs. 2(a) and 2(b) the dependence of the maximum point ν_m on k^2 is shown at $\gamma^2 = 0.5$ for two different variation intervals of k^2 (circles). For comparison

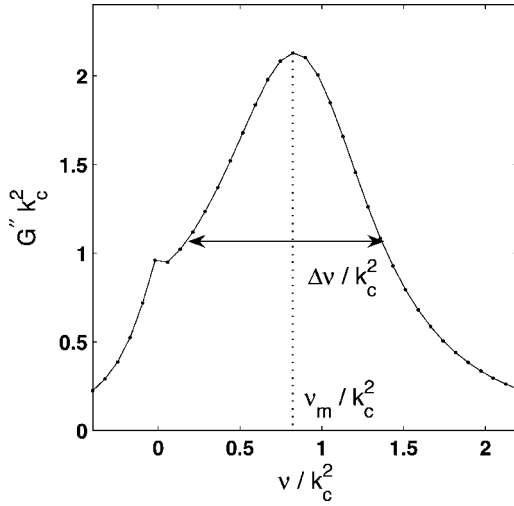


FIG. 1. The imaginary part of the Green's function $G''_k(\nu)$ for $\gamma^2=0.5$ and $(k/k_c)^2=1$.

the unperturbed dependence $\nu_m=k^2$ corresponding to $\gamma^2=0$ (dotted lines), as well as the dependence of the eigenfrequency ν' on k^2 obtained in Ref. 9 in the first order of the perturbation theory, are shown in these figures too (dashed curves). The latter dependence is described by the formula

$$\nu' = k^2 \left(1 - \gamma^2 \frac{k_c^2 + 3k^2}{k_c^2 + 4k^2} \right). \quad (52)$$

It is seen from Fig. 2(a), that Eq. (52) at small $(k/k_c)^2$ satisfactorily describes the dependence $\nu_m(k^2)$ calculated by the CPA method. According to this formula the curve $\nu'(k^2)$ has a bend in a vicinity of the point $(k/k_c)^2=0.25$ [this bend is not very noticeable in the scale of Fig. 2(a)]. This bend plays the essential role in experimental investigations of amorphous and nanocrystalline alloys. We will dwell on this problem in greater detail in the conclusion section of this paper. Here we only note that a similar but more sharply defined bend is also in the curve $\nu_m(k^2)$ in a vicinity of the same point. With the increase of k the curve $\nu_m(k^2)$ comes apart with the curve $\nu'(k^2)$, and has the second bend at $(k/k_c)^2 \approx 2.7$. Then this curve approaches the unperturbed curve $\nu=k^2$ and crosses it [Fig. 2(b)].

The dependence of the linewidth $\Delta\nu$ on k^2 for two different intervals of k^2 is shown in Figs. 3(a) and 3(b) (circles). The doubled value of the spin-wave damping ν'' , that was calculated in Ref. 9 in the first order of the perturbation theory, is shown in these figures too (dashed curves); it is determined by the formula

$$\nu'' = \gamma^2 \frac{k^3 k_c^2 + 2k^2}{k_c k_c^2 + 4k^2}. \quad (53)$$

One can see that for $(k/k_c)^2 \lesssim 1$ the difference between the curves $\Delta\nu(k^2)$ and $2\nu''(k^2)$ is quite large [Fig. 3(a)], whereas for $(k/k_c)^2 > 1$ these curves are brought close together [Fig. 3(b)]. However, these curves differ significantly qualitatively in the whole variation interval of k . As is seen from Eq. (53) the damping of the eigenfrequency is propor-

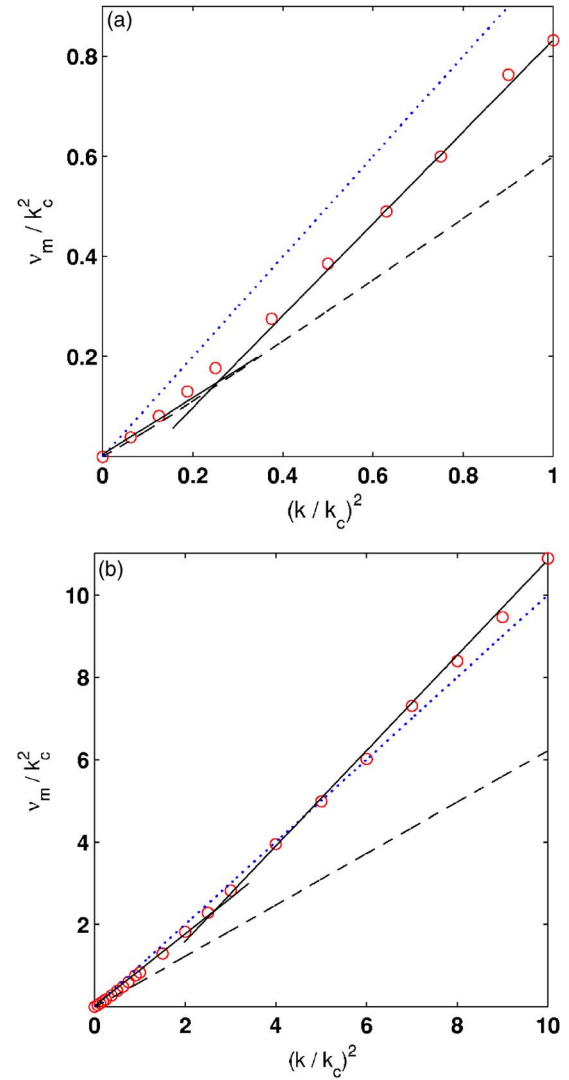


FIG. 2. (Color online) The dependence of the frequency of the resonance maximum ν_m of the Green's function $G''_k(\nu)$ on k^2 (circles), the dispersion curve $\nu'(k^2)$ corresponding to Eq. (52) (dashed curve), and unperturbed curve $\nu=k^2$ (dotted curve). The solid straight lines are tangents to the curve $\nu_m(k^2)$ at small and large k in comparison with (a) $(k/k_c)^2=0.25$ and (b) $(k/k_c)^2=2.7$.

tional to k^3 for $(k/k_c)^2 \ll 0.25$ as well as for $(k/k_c)^2 \gg 0.25$. In the vicinity of the point $(k/k_c)^2=0.25$ smooth change of the numerical coefficient before k^3 occurs. Contrary to that, the linewidth $\Delta\nu$ calculated in this paper is proportional to k^2 in the whole variation interval of k and only the numerical coefficient before k^2 changes sharply in two critical points: $(k/k_c)^2=0.25$ and 2.7 , that is in the same points where the bends of the curve $\nu_m(k^2)$ take place.

In Fig. 4 the dependence of the linewidth $\Delta\nu$ on the relative rms fluctuation γ is shown for two values of the wave number: $(k/k_c)^2=1$ (black circles) and 10 (white circles). One can see that for both values of k^2 the curves $\Delta\nu(\gamma)$ have the qualitatively similar form. The dependences of the doubled value of the damping ν'' on γ [Eq. (53)] are also shown in Fig. 4 for $(k/k_c)^2=1$ (dashed curve) and $(k/k_c)^2=10$ (dotted curve). It is seen that for $(k/k_c)^2=1$ the line-

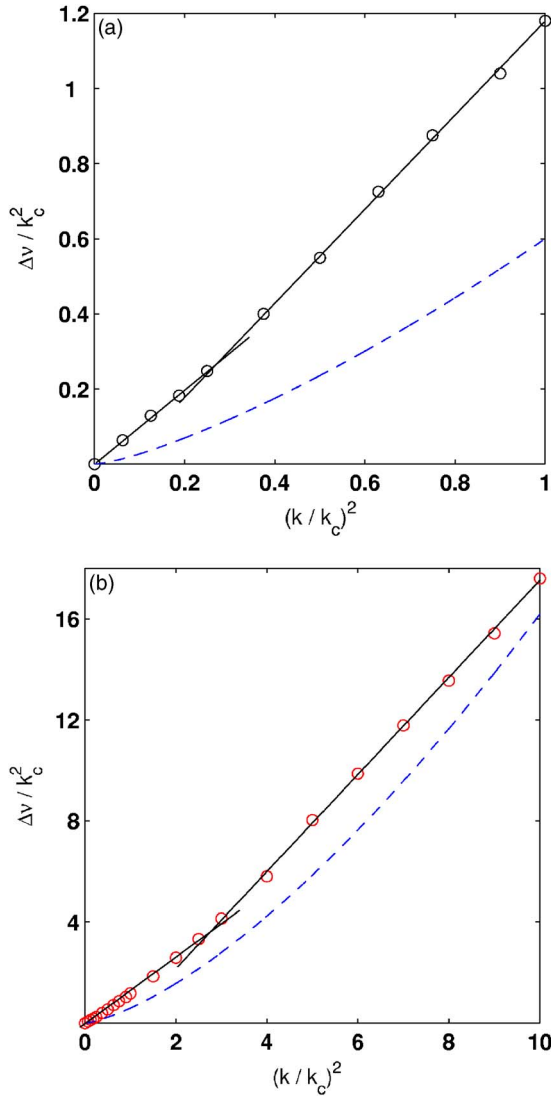


FIG. 3. (Color online) The dependence of the linewidth $\Delta\nu$ of the resonance curve $G_k''(\nu)$ on k^2 (circles) and the doubled value of the damping $\nu''(k^2)$ corresponding to Eq. (53) (dashed curves). The solid straight lines are tangents to the curve $\Delta\nu(k^2)$ at small and large values of k^2 in comparison with (a) $(k/k_c)^2=0.25$ and (b) $(k/k_c)^2=2.7$.

width to a large degree is determined by stochastic distribution of the resonance frequencies but not by the damping. For $(k/k_c)^2=10$ the calculated curve $\Delta\nu(\gamma)$ differs little from the curve $2\nu''(\gamma)$ if $\gamma<0.8$.

V. CONCLUSION

In the first part of the paper (Secs. II and III) we develop the Green's function formalism and derive the CPA integral equation for the case of long-wave inhomogeneities of the nonlocal characteristics of the continuum medium such as the exchange constant in a ferromagnet or the force constant in an elastic medium. For emphasizing the fundamental sides of the problem we consider the simplest models of spin and elastic waves for which the wave equations have the identi-

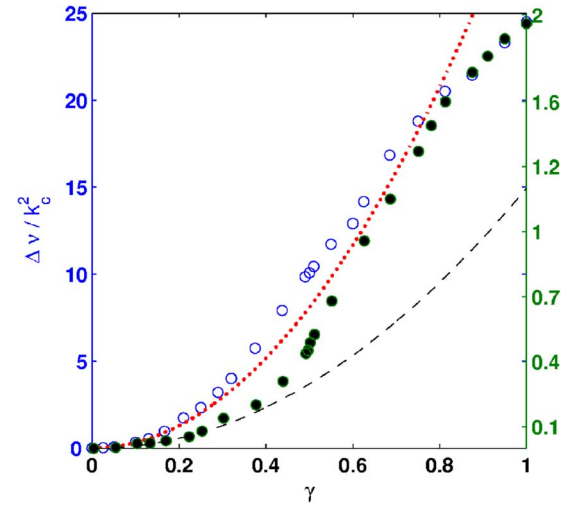


FIG. 4. (Color online) The dependencies of the linewidth $\Delta\nu$ of the resonance curve $G_k''(\nu)$ on the rms fluctuation γ for $(k/k_c)^2=1$ (black circles, right scale) and $(k/k_c)^2=10$ (white circles, left scale) and the dependencies of $2\nu''(\gamma)$ for $(k/k_c)^2=1$ (dashed curve, right scale) and $(k/k_c)^2=10$ (dotted curve, left scale).

cal form differing only by redesignation of the parameters. Because of that the equation for the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ has the same form for both situations under consideration. Let us note the principal distinctive features of this equation, as well as the following expressions for the averaged Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_0)$ from the correspondent expressions describing the inhomogeneities of the local characteristics.

The equation for the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ instead of the term with a product of the random function ρ and Green's function G contains two terms with products of these functions derivatives: $\rho(\partial^2 G/\partial x_i^2)$ and $(\partial\rho/\partial x_i)(\partial G/\partial x_i)$. The representation of the averaged Green's function \bar{G} as a series in correlators instead of the products of the initial Green's functions G_0 contains the products of their derivatives, which are summed over repetitive subscripts or superscripts. The analogue of the Dyson equation contains the derivatives of the functions G_0 and \bar{G} , and the matrix of the mass operators $Q_{i'j'}(\mathbf{x}', \mathbf{x}'')$. The diagram designations are introduced in the paper which allow us to represent the expansion of the function \bar{G} , the component of the mass operator $Q_{i'j'}$, and the analogue of the Dyson equation in the graph form. The self-consistent integral equation of the CPA for the case of non-local characteristics of the medium is derived by the method suggested by us earlier.¹ This equation takes into account both diagonal and off-diagonal disorder because the total scalar potential of interactions $T_{\mathbf{k}}$ is the sum of all components of the matrix $k^i k^j Q_{ij}$. Hence, for obtaining $T_{\mathbf{k}}$ and correspondingly the averaged Green's function by the method CPA the knowledge of the components of the matrix $Q_{i'j'}$ is not necessary.

In the second part of the paper (Sec. IV) we apply the general equation of the CPA obtained in the first part to the study of effects of the 1D inhomogeneities of the exchange constant on the spectrum of spin waves in a ferromagnet. In

the process of a numerical solution of the CPA equation by the method of successive approximations we use two approaches for obtaining the first approximation: the standard approach when we assume that $T_k^{(0)}=0$ and the method of algebraic equations suggested in Ref. 1. The method of algebraic equations has very quick convergence. However, when using it the difficulties with the sewing together of the algebraic equation roots appear in some intervals of the parameters. In these cases we use the standard approach which leads to the slower convergence of the iteration process.

There are bends at $(k/k_c)^2=0.25$ and 2.7 in the curve of the resonance frequency ν_m of the imaginary part of the Green's function $G_k''(\nu)$ vs k^2 calculated here. The point of the first bend coincides with the point of the bend of the dependence of the eigenfrequency ν' on k^2 , which has been found for the first time in Ref. 9 in the framework of the perturbation theory and described by Eq. (52). It should be pointed out that since this effect was found it plays an important role in experimental investigations of amorphous and nanocrystalline alloys. The matter is that, as it was shown in Ref. 9, on the curve $\nu'(k^2)$ in the vicinity of the point $(k/k_c)^2=0.25$ the peculiarity must be observed for the inhomogeneity of any one of the parameters of the ferromagnet: exchange, anisotropy, magnetization, and so on. The form of this peculiarity depends on the physical nature and dimensionality of the inhomogeneities. For instance, 1D inhomogeneities of the exchange parameter lead to the bend of the dispersion curve $\nu'(k^2)$ up, 3D inhomogeneities of the exchange parameter leads to the bend of the curve $\nu'(k^2)$ down, and 3D inhomogeneities of the magnetization modulus lead to the inflection of this curve. This effect was found experimentally later in thin films of amorphous alloys by the spin-wave resonance technique. The method of the correlation spin-wave spectroscopy was developed on the basis of the comparison of the theory⁹ with the experiments which permitted one to determine the physical nature of inhomogeneous parameters and to measure the correlation radii of their fluctuations. The correlation radii of a number of amorphous and nanocrystalline alloys were measured by this method

and their temperature and concentration dependencies were investigated (see review¹⁰ and references therein). The theoretical results obtained in Ref. 9 were repeatedly confirmed in subsequent theoretical papers also in the framework of the perturbation theory.^{11,12} The only one peculiarity in the point $(k/k_c)^2=0.25$ was found on the curve $\nu'(k^2)$ in these papers. The second peculiarity in the point $(k/k_c)^2=2.7$ is found in the present paper.

Two bends in the vicinity of the same points $(k/k_c)^2=0.25$ and 2.7 are also on the calculated dependency of the linewidth $\Delta\nu$ on k^2 . The point of the first bend coincides with the analogous point on the curve $\nu'(k^2)$ calculated earlier⁹ in the first order of the perturbation theory [see Eq. (53)]; the second bend is found in the present paper. The specific interest has the law $\Delta\nu \propto k^2$ obtained in the present work in the intervals between the bend points, as well as at $k \gg k_c$. The damping ν'' calculated earlier⁹ for inhomogeneities of the exchange parameter or the magnetization modulus has the law $\nu'' \propto k^3$ for $(k/k_c)^2 \gg 0.25$ [dashed curve in Fig. 3(b)]. The experimental investigations of the spin-wave resonance linewidth gave for the dependence $\Delta\nu(k)$ the law $\Delta\nu \propto k^p$ where the power $p \approx 1$ for the amorphous and $p \approx 2$ for nanocrystalline alloys. The law $\Delta\nu \propto k^2$ obtained in the present work can explain the experimental results for the nanocrystalline alloys in that the main contribution in the linewidth in this case gives the stochastic distribution of frequencies taken into account here but not the damping. The experimental law $\Delta\nu \propto k$ for the amorphous alloys cannot be explained in the framework of this theory.

It must be emphasized that the comparison of the calculations made in Sec. IV of this paper with existing experimental data has only the qualitative character because 3D inhomogeneities have been investigated experimentally.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, Grant No. 04-02-16174 and Grant of the Russian Federation President, SS-6612.2006.3.

¹V. A. Ignatchenko and V. A. Felk, Phys. Rev. B **71**, 094417 (2005).

²P. Soven, Phys. Rev. **156**, 809 (1967).

³D. W. Taylor, Phys. Rev. **156**, 1017 (1967).

⁴F. Yonezawa and K. Morigaki, Suppl. Prog. Theor. Phys. **53**, 1017 (1973).

⁵R. J. Elliott, J. A. Krumhansl, and P. L. Leath, Rev. Mod. Phys. **46**, 465 (1974).

⁶J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, UK, 1979).

⁷E. N. Economou, *Green's Functions in Quantum Physics* (Springer-Verlag, Berlin, 1982).

⁸S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarsky, *Introduction to*

Statistical Radiophysics (Part II: Random Fields) (Nauka, Moscow, 1978).

⁹V. A. Ignatchenko and R. S. Iskhakov, Zh. Eksp. Teor. Fiz. **72**, 1005 (1977) [Sov. Phys. JETP **45**, 526 (1977)]; **74**, 1386 (1978) [Sov. Phys. JETP **47**, 725 (1978)]; **75**, 1438 (1978) [Sov. Phys. JETP **48**, 726 (1978)].

¹⁰V. A. Ignatchenko and R. S. Iskhakov, in *Magnetic Properties of Crystalline and Amorphous Mediums*, edited by V. A. Ignatchenko (Nauka, Novosibirsk, 1989).

¹¹M. V. Medvedev, Fiz. Tverd. Tela (Leningrad) **22**, 1944 (1980).

¹²K. Handrich and R. Ötting, Phys. Status Solidi B **216**, 1073 (1999).