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Pulse Sequences for Realizing the Quantum Fourier Transform on Multilevel Systems

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Sequences of selective pulses of an RF magnetic field for realizing the quantum Fourier transform by NMR methods on systems with four, six, and eight nonequidistant levels are found using the virtual spin formalism. The results can be applied to other quantum systems when laser pulses are used.

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To date, the main attention in quantum computations is focused on qubits, i.e., physical systems with two stationary states [1], e.g., nuclear spins $I = 1/2$. As achievements, we mentioned the realization of Shor's factorization algorithm on 7 qubits by the NMR methods [2] and the receipt of an effectively pure state on 12 qubits [3]. One of the causes of the achieved progress is that the quantum algorithms in binary logics are most developed [1]. However, quantum systems with numerous states are much more widespread in nature: quadrupole nuclei with spin $I > 1/2$, ions, radicals or molecular magnets with multilevel electronic states, spin clusters with strong spin–spin interaction, etc. Many authors think that the use of such d-level systems (qudits) in quantum computations will be more profitable. Although it has been already proved that any unitary transformation of qudits can be represented in the form of elementary logical operators (gates), particular schemes of basic algorithms appropriate for physical realization have not yet been developed. This is also the case for the quantum Fourier transform, which is of key importance in many quantum algorithms. The quantum Fourier transform is realized by the Walsh–Hadamard matrix [1, 4]

$$QFT_d = \frac{1}{\sqrt{d}}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \omega^{2(d-1)} & \dots & \omega^{(d-1)^2} \end{pmatrix}, \quad (1)$$

where

$$\omega = \exp\left(\frac{2\pi i}{d}\right)$$

on a d-dimension state vector.

Gates are, as a rule, experimentally realized by means of pulses of the RF magnetic field in NMR or laser pulses in optics [1]. For quantum systems with nonequidistant levels, the action of each such pulses is described by the rotation matrix for two states of the system the difference of whose energies is equal to the pulse frequency. Klimov et al. [5] found the sequence of laser pulses for realizing matrix (1) of the quantum Fourier transform on qutrits ($d = 3$). Fujii et al. [4] gave the Jarlskog's parameterization of QFT_d for $d = 3$, 4, and 5, but the sequence of pulses was not obtained. Moreover, Fujii et al. [4] concluded that qudit theory is not realistic for $d > 5$, because the formulas were too complicated. Brennen et al. [6] described the general procedure of the representation of an arbitrary unitary $d \times d$ matrix in terms of the product of selective rotation matrices. However, the procedure is laborious, because numerous such matrices (about d^2) must be calculated. In this work, we show that the results that have been already obtained for multispin systems can be applied to multilevel systems by using the virtual spin formalism proposed by Kessel et al. [7–9]. Our approach makes it possible to represent the QFT_d operators in the form of the sequences of the rotation operators by selective RF pulses for large $d = (2I_1 + 1)(2I_2 + 1)\dots(2I_n + 1)$, where $I_1, I_2\dots I_n$ are virtual spins.

For a system consisting of n qubits, the network of quantum gates for the quantum Fourier transform on $N = 2^n$ states is well known [1, 2, 10] (operators act from left to right):

$$QFT_N = H_1B_{1,2}H_2B_{1,3}B_{2,3}H_3\dots H_{n-1}B_{1,n}B_{2,n}\dots B_{n-2,n}B_{n-1,n}H_n, \quad (2)$$

where H_i is the Hadamard operator acting on the ith spin (qubit) and $B_{ij} = B_{ij}(\theta)$ is the operator of the controlled phase shift by the angle $\theta = \pi/2^{i-j}$. In the matrix form, they are represented as

$$H_j = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B_{ij}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}. \tag{3}$$

We use the sequential numeration of levels (states) beginning with the ground state: 1, 2, …, m, …, N. The corresponding states of spins (real or virtual) are determined by the binary representation of $m - 1$ for the level m, e.g., $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ for two qubits of the four-level system. The application of network (2) implying the direct (tensor) product of matrices (3) acting on different spins yields a state that is bit-to-bit inverse to that obtained after the application of operator (1). To reduce the result to the corresponding form, it is necessary to apply the state permutation operator *SWAP* [1].

Dorai and Suter [10] demonstrated how network (2) is realized for the quantum Fourier transform by means of selective RF pulses using the following formula of the transformation of operator $B_{ij}(\theta)$ (3) to the operator of the selective rotation of two last states:

$$H_j B_{ij}(\theta) H_j = \exp(i\theta/2)\{\theta\}_Z^i \{\theta\}_X^{r-s}. \tag{4}$$

Hereinafter, the superscript $r - s$ means a selective pulse at the $r \longrightarrow s$ transition, which turns by the angle represented in the parentheses, and the superscript i or i, j means a selective pulse acting on the corresponding qubit or qubits. The subscript indicates the rotation axis X, Y, or Z. Dorai and Suter [10] represented sequences of pulses for two qubits:

$$\underset{1\,4\,7\,9}{\left\{\frac{\pi}{2}\right\}_Y^{1,2}} \longrightarrow \underset{2\,5\,8\,10}{\{\pi\}_X^{1,2}} \longrightarrow \underset{10}{\left\{\frac{\pi}{2}\right\}_X^{3-4}} \longrightarrow \underset{3\,6}{\left\{\frac{\pi}{4}\right\}_Z^{1}} \longrightarrow \underset{11\,12\,13}{SWAP(2 \longleftrightarrow 3)} \tag{5}$$

and for three qubits:

$$\underset{\substack{1\,4\,7\,10\,13\,16\,19\\22\,25\,27\,29\,31}}{\left\{\frac{1}{2}\pi\right\}_Y^{1,2,3}} \longrightarrow \underset{\substack{2\,5\,8\,11\,14\,17\,20\\23\,26\,28\,30\,32}}{\{\pi\}_X^{1,2,3}} \longrightarrow \underset{23}{\left\{\frac{1}{2}\pi\right\}_X^{6-8}}$$
$$\longrightarrow \underset{20}{\left\{\frac{1}{2}\pi\right\}_X^{5-7}} \longrightarrow \underset{32}{\left\{\frac{3}{4}\pi\right\}_X^{7-8}} \longrightarrow \underset{30}{\left\{\frac{1}{4}\pi\right\}_X^{5-6}} \longrightarrow \underset{28}{\left\{\frac{1}{2}\pi\right\}_X^{3-4}} \tag{6}$$
$$\longrightarrow \underset{3\,6\,9\,12}{\left\{\frac{3}{8}\pi\right\}_Z^{1}} \longrightarrow \underset{15\,18\,21\,24}{\left\{\frac{1}{4}\pi\right\}_Z^{2}} \longrightarrow \underset{33\,34\,35\,36}{SWAP(2 \longleftrightarrow 5, 4 \longleftrightarrow 7)}.$$

Using the symmetry of sequence (2), we permutate the Hadamard operators compared to [10]. The arrows between pulses show the time direction. The ordinal numbers of selective RF pulses used in this operator are shown under each pulse in Eqs. (5) and (6).

The quantum network for the quantum Fourier transform on $N = d^n$ states of the system of n qudits with the same number of levels was considered in [11]. Its difference from network (2) on qubits is in the change of the operators H_i to the operators QFT_d and the operator $B_{ij}(\theta)$ to the operator of the controlled phase shift with a diagonal matrix in the space of two-qudit states [5, 11, 12] of the dimension d^2 with the elements

$$\langle km|P_{ij}(d, d, \theta)|km\rangle = \exp(ikm\theta),$$
$$k, m = 0, 1, 2\ldots(d-1), \quad \theta = 2\pi/d^{j-i+1}. \tag{7}$$

To expand the sequence of pulses to the multilevel system, it is necessary to assign real qubits or qudits to virtual ones. We begin with the case $d = 2^n$. The selective pulse operators from Eqs. (5) and (6) are directly transferred. For the Hadamard operators written in terms of pulses acting on individual spins and the *SWAP* operators presented in [10], we choose analogues consisting of selective pulses. Thus, we find the following sequence of 13 pulses for QFT_4:

$$\left(\underset{1}{\left\{\frac{\pi}{2}\right\}_Y} \longrightarrow \underset{2}{\{\pi\}_X} \longrightarrow \underset{3}{\left\{\frac{\pi}{4}\right\}_Z}\right)^{1-3}$$
$$\longrightarrow \left(\underset{4}{\left\{\frac{\pi}{2}\right\}_Y} \longrightarrow \underset{5}{\{\pi\}_X} \longrightarrow \underset{6}{\left\{\frac{\pi}{4}\right\}_Z}\right)^{2-4}$$
$$\longrightarrow \left(\underset{7}{\left\{\frac{\pi}{2}\right\}_Y} \longrightarrow \underset{8}{\{\pi\}_X}\right)^{1-2}$$
$$\longrightarrow \left(\underset{9}{\left\{\frac{\pi}{2}\right\}_Y} \longrightarrow \underset{10}{\left\{\frac{3}{2}\pi\right\}_X}\right)^{3-4}$$
$$\longrightarrow \underset{11}{\{\pi\}_X^{2-3}} \longrightarrow \underset{12}{\left\{\frac{\pi}{2}\right\}_Z^{1-2}} \longrightarrow \underset{13}{\left\{\frac{7}{2}\pi\right\}_Z^{3-4}},$$

and the following sequence of 36 pulses for QFT_8:

$$\left(\underset{1}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{2}{\{\pi\}_X} \to \underset{3}{\left\{\frac{3}{8}\pi\right\}_Z}\right)^{1-5}$$

$$\to\left(\underset{4}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{5}{\{\pi\}_X} \to \underset{6}{\left\{\frac{3}{8}\pi\right\}_Z}\right)^{2-6}$$

$$\to\left(\underset{7}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{8}{\{\pi\}_X} \to \underset{9}{\left\{\frac{3}{8}\pi\right\}_Z}\right)^{3-7}$$

$$\to\left(\underset{10}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{11}{\{\pi\}_X} \to \underset{12}{\left\{\frac{3}{8}\pi\right\}_Z}\right)^{4-8}$$

$$\to\left(\underset{13}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{14}{\{\pi\}_X} \to \underset{15}{\left\{\frac{1}{4}\pi\right\}_Z}\right)^{1-3}$$

$$\to\left(\underset{16}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{17}{\{\pi\}_X} \to \underset{18}{\left\{\frac{1}{4}\pi\right\}_Z}\right)^{2-4}$$

$$\to\left(\underset{19}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{20}{\left\{\frac{3}{2}\pi\right\}_X} \to \underset{21}{\left\{\frac{1}{4}\pi\right\}_Z}\right)^{5-7}$$

$$\to\left(\underset{22}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{23}{\left\{\frac{3}{2}\pi\right\}_X} \to \underset{24}{\left\{\frac{1}{4}\pi\right\}_Z}\right)^{6-8}$$

$$\to\left(\underset{25}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{26}{\{\pi\}_X}\right)^{1-2}$$

$$\to\left(\underset{27}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{28}{\left\{\frac{3}{2}\pi\right\}_X}\right)^{3-4}$$

$$\to\left(\underset{29}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{30}{\left\{\frac{5}{4}\pi\right\}_X}\right)^{5-6}$$

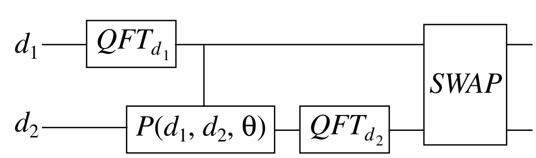
$$\to\left(\underset{31}{\left\{\frac{1}{2}\pi\right\}_Y} \to \underset{32}{\left\{\frac{7}{4}\pi\right\}_X}\right)^{7-8}$$

$$\to\left(\underset{33}{\{2\pi\}_Z} \to \underset{34}{\{\pi\}_Y}\right)^{2-5}$$

$$\to \underset{35}{\{2\pi\}_Z^{4-5}} \to \underset{36}{\{3\pi\}_Y^{4-7}}.$$

We note that the resulting matrices differ from Eq. (1) by the phase factors $\exp\left(i\frac{11\pi}{8}\right)$ and $\exp\left(i\frac{29\pi}{16}\right)$ for $d =$ 4 and 8, respectively.

Let us consider the case $N \neq 2^n$ and the simplest six-level system. In order to determine QFT_6, we represent this system as consisting of a virtual qutrit ($d_1 = 3$) and qubit ($d_2 = 2$) with the states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, $|20\rangle$, and $|21\rangle$. We generalize the above quantum Fourier transform schemes for the case of two different qudits:



(8)

where $k = 0, 1, 2\ldots(d_1 - 1)$, $m = 0, 1, 2\ldots(d_2 - 1)$, and $\theta = 2\pi/d_1d_2$ in Eq. (7). Using scheme (8), we obtain the following sequence of 33 selective pulses for QFT_6:

$$\left(\underset{1}{\left\{\frac{2}{3}\pi\right\}_Z} \longrightarrow \underset{2}{\left\{\frac{3}{2}\pi\right\}_Y} \longrightarrow \underset{3}{\left\{\frac{4}{3}\pi\right\}_Z}\right)^{3-5}$$

$$\longrightarrow \left(\underset{4}{\left\{\frac{13}{6}\pi\right\}_Z} \longrightarrow \underset{5}{\left\{2\arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right\}_X} \longrightarrow \underset{6}{\left\{\frac{11}{6}\pi\right\}_Z}\right)^{1-3}$$

$$\longrightarrow \left(\underset{7}{\left\{\frac{4}{3}\pi\right\}_Z} \longrightarrow \underset{8}{\left\{\frac{3}{2}\pi\right\}_X} \longrightarrow \underset{9}{\left\{\frac{2}{3}\pi\right\}_Z}\right)^{1-5}$$

$$\longrightarrow \underset{10}{\{\pi\}_Z^{1-3}} \longrightarrow \underset{11}{\left\{\frac{2}{3}\pi\right\}_Z^{3-5}}$$

$$\longrightarrow \left(\underset{12}{\left\{\frac{2}{3}\pi\right\}_Z} \longrightarrow \underset{13}{\left\{\frac{3}{2}\pi\right\}_Y} \longrightarrow \underset{14}{\left\{\frac{4}{3}\pi\right\}_Z}\right)^{4-6}$$

$$\longrightarrow \left(\underset{15}{\left\{\frac{13}{6}\pi\right\}_Z} \longrightarrow \underset{16}{\left\{2\arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right\}_X} \longrightarrow \underset{17}{\left\{\frac{11}{6}\pi\right\}_Z}\right)^{2-4}$$

$$\longrightarrow \left(\underset{18}{\left\{\frac{4}{3}\pi\right\}_Z} \longrightarrow \underset{19}{\left\{\frac{3}{2}\pi\right\}_X} \longrightarrow \underset{20}{\left\{\frac{2}{3}\pi\right\}_Z}\right)^{2-6}$$

$$\longrightarrow \underset{21}{\{\pi\}_Z^{2-4}} \longrightarrow \underset{22}{\left\{\frac{2}{3}\pi\right\}_Z^{4-6}} \longrightarrow \underset{23}{\{\pi\}_Z^{1-2}}$$

$$\longrightarrow \underset{24}{\{\pi\}_Z^{3-5}} \longrightarrow \underset{25}{\{2\pi\}_Z^{1-3}} \longrightarrow \underset{26}{\left\{\frac{7}{3}\pi\right\}_Z^{4-6}} \longrightarrow \underset{27}{\left\{\frac{7}{2}\pi\right\}_Y^{1-2}}$$

$$\longrightarrow \left(\underset{28}{\{3\pi\}_X} \longrightarrow \underset{29}{\left\{\frac{1}{2}\pi\right\}_Y}\right)^{3-4} \longrightarrow \underset{30}{\left\{\frac{3}{2}\pi\right\}_Y^{5-6}}$$

$$\longrightarrow \underset{31}{\{\pi\}_X^{2-5}} \longrightarrow \underset{32}{\{\pi\}_X^{2-3}} \longrightarrow \underset{33}{\{\pi\}_X^{4-5}}.$$

Pulses with rotation about the z axis are realized in the form of a composed z pulse [10] $\{\theta\}_Z^{r-s} \equiv \{-\pi/2\}_X^{r-s} \longrightarrow \{\theta\}_Y^{r-s} \longrightarrow \{\pi/2\}_X^{r-s}$. Other methods also exist, in particular, by means of the phase shift of following pulses [2] or by means of two phase-shifted π pulses $\{\pi\}_Y^{r-s} \longrightarrow \{\pi\}_{Y+\pi+\theta/2}^{r-s}$ [13]. Selective pulses on forbidden transitions with $\Delta m > 1$ can also be realized using several sequential pulses as discussed, e.g., in [9]. It is worth noting that the presented sequences of pulses are not the only possible sequences.

The above expressions show that the number of operations increases when we pass from pulses acting on real spins to pulses acting on individual transitions in the multilevel system. This completely agrees with the general theory of quantum computations [14], according to which the fundamental advantages of quantum algorithms are manifested on systems consisting of many quantum objects. The situation can be partially improved by using the possibility of parallel action on uncoupled levels [15].

In conclusion, we note that the results obtained above are also important for other quantum algorithms, because the QFT_d operators enter into the basic two-qudit gate, the SUM gate [12], which is a generalization of the basic two-qubit gate CNOT:

$$\begin{gathered}(E_1 \otimes (QFT_{d2})^{-1})P(d_1, d_2, 2\pi/d_2) \\ \times (E_1 \otimes QFT_{d2}),\end{gathered} \tag{9}$$

where E_1 is the identity matrix in the state space of the first, control, qudit. We emphasize that, in contrast to Eq. (8), both QFT_d operators in Eq. (9) act on the second, working, qudit and the angle acquires another value. Finally, the sequences obtained in this work after the replacement of RF pulses by laser pulses [1, 5, 6] are suitable for realizing the quantum Fourier transform on ions or atoms in traps and other multilevel quantum systems.

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