Partial restoration of the wave spectrum of a superlattice due to cross correlations between one- and three-dimensional inhomogeneities

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Effects of cross correlations between one-dimensional (1D) and three-dimensional (3D) random inhomogeneities on the wave spectrum in sinusoidal superlattices are studied theoretically. The situation when the gap in the spectrum (the forbidden zone) at the first Brillouin zone boundary of the superlattice is closed under the action of the 1D inhomogeneities is considered. The phenomenon of the partial opening of this gap is found when the 3D inhomogeneities cross correlated with the 1D inhomogeneities add to the superlattice. The appearance of the logarithmic resonance in the center of the forbidden zone under the action of the cross correlations is shown. The physical nature of the effects in the wave spectrum of the superlattice that are caused by the cross correlations and the relation of these effects with the asymptotic properties of the correlation function of inhomogeneities are discussed.

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I. INTRODUCTION

Wave spectra in partly randomized superlattices have been intensely studied in recent years in view of wide application of such materials in various high-technology devices. In addition, these materials are convenient models for developing new methods in theoretical physics for studying media without translational symmetry. Several theoretical approaches have been developed to investigate such superlattices: the introduction of the one-dimensional (1D) random phase,[1,2] modeling of violation of ordering in the sequence of layers of two different materials,[3–10] numerical simulation of random deviations of the interfaces between the layers from their initial periodic arrangement,[11–13] postulation of the form of the correlation function of the superlattice with inhomogeneities,[14,15] application of geometrical optics approximations,[16] and development of a dynamic theory of composite elastic media.[17]

One more method for investigating effects of inhomogeneities in a superlattice on the wave spectrum was proposed in our earlier publication (Ref. 18), the method of random spatial modulation of the superlattice period. It can be briefly described as follows. The only parameter describing the random medium and appearing in the expression for the averaged Green's function is the correlation function $K(\mathbf{r})$ that depends on the distance $\mathbf{r}$ between two points of the medium ($\mathbf{r}=\mathbf{x}-\mathbf{x}'$). Consequently, the first stage of solving the problem is reduced to determining function $K(\mathbf{r})$ for the superlattice that has some inhomogeneities of its structure; the second stage involves the extraction of the spectral parameters from the expression for the Green's function containing this correlation function using the standard approximate methods. To determine the superlattice correlation function $K(\mathbf{r})$, we developed an approach which is a generalization of the well-known method for determining the time correlation function for a random frequency (phase) modulation of a radio signal[19,20] to the case of a spatial (three dimensions in the general case) random modulation of the superlattice period. The advantage of this method is that the form of the superlattice correlation function is not postulated in it but is derived from the most general assumptions concerning the nature of random spatial modulation of the superlattice period. It was shown that this function, in the general has a quite complicated form, which is determined to a considerable extent by the dimensionality of inhomogeneities, the structure of the interface between the layers, and so on. Knowledge of the correlation functions corresponding to various types and dimensionalities of inhomogeneities allowed us to apply the methods to studying the averaged Green's functions for determining the eigenfrequencies, damping factor, and other wave parameters in superlattices.[18,21–27]

Effects of inhomogeneities in the superlattices with the finite thicknesses of interfaces,[24] inhomogeneities with anisotropic correlation properties,[25] as well as the mixture of the 1D and three-dimensional (3D) inhomogeneities[26,27] on the wave spectrum of the superlattices have been studied in these papers. Analyzing the mixture of inhomogeneities in Ref. 26, we assumed that cross correlations between inhomogeneities with different dimensionalities are absent. In Ref. 27, effects of the mixture of cross-correlated 1D and 3D inhomogeneities on the wave spectrum in the initially sinusoidal superlattice were studied. It was found out that the cross correlation led to the partial suppression of the influence of inhomogeneities on the wave spectrum: the width of the gap in the spectrum at the first Brillouin zone boundary increased and the damping of waves decreased in comparison with the values of these parameters caused by the mixture of 1D and 3D inhomogeneities in the absence of the cross correlations. It was found out also that the cross correlations led to the emergence of unexpected resonance effects: a narrow dip or narrow peak at the center of the band gap (depending on the sign of the correlation coefficient). In Ref. 27, phenomena of the partial suppression of the influence of inhomogeneities on the wave spectrum and the emergence of the resonances at the center of the forbidden band were considered in the situation when rms fluctuations of the 1D inhomogeneities were not so large that the gap in the spectrum could be closed only under the action of this inhomogeneities. In the present work, we consider the opposite case of enough large rms variations of the 1D inhomogeneities which lead to the closing of the gap in the spectrum. We show that adding of the 3D inhomogeneities cross correlated with these 1D inhomogeneities leads to a qualitatively another phenomenon: the partial opening of the gap in the spectrum with the increase of rms fluctuations of the 3D inhomogeneities. The analysis of the physical reasons of the effects studied in Refs. 26 and 27, as well as in the present paper, is carried out.

II. MODEL AND METHOD

We recall shortly the main features of the model and method used in Refs. 26 and 27. A superlattice is characterized by the dependence of a material parameter A on spatial coordinates $\mathbf{x}=\{x,y,z\}$. Parameter $A(\mathbf{x})$ can be of any physical origin. It can be the density of the material or a force constant for the elastic system of the medium, the magnetic anisotropy, magnetization, or exchange constant for the magnetic system, and so on. We represent $A(\mathbf{x})$ in the form

$$A(\mathbf{x}) = A + \Delta A \rho(\mathbf{x}), \tag{1}$$

where A is the mean value of the parameter, ΔA is its rms deviation, and $\rho(\mathbf{x})$ is a centered ($\langle\rho(\mathbf{x})\rangle=0$) and normalized ($\langle\rho^2(\mathbf{x})\rangle=1$) function. The function $\rho(\mathbf{x})$ describes both the periodic dependence of the parameter along the z axis of the superlattice and random spatial modulation of this parameter, which can generally be a function of all three coordinates x, y, and z.

We consider a superlattice with a sinusoidal dependence of the material parameter on the z coordinate in the initial state (in the absence of inhomogeneities). Following Refs. 26 and 27, we present the function $\rho(\mathbf{x})$ in the form

$$\rho(\mathbf{x}) = \sqrt{2}\cos\{q[z - u_1(z) - u_3(\mathbf{x})] + \psi\}, \tag{2}$$

where $q=2\pi/l$ is the wave number of the superlattice and l is the superlattice period. The sinusoidal superlattice can be considered as a particular case of the multilayer structure with very smooth boundaries between the layers. In this the positive and negative regions of the function $\rho(\mathbf{x})$ along the z axis correspond to the alternate layers of this multilayer structure and zero points of the function $\rho(\mathbf{x})$ correspond to the boundaries (interfaces) between these layers. In the framework of this interpretation, the function $u_1(z)$ models random displacements of these interfaces from their initial periodic arrangement (1D inhomogeneities), and the function $u_3(\mathbf{x})$ models random deformations of surfaces of these interfaces (3D inhomogeneities). The coordinate-independent phase ψ is characterized by a uniform distribution on the interval $(-\pi,\pi)$.

The wave equation for the time Fourier transformation in the superlattice can be represented in the form

$$\nabla^2 m + \left[\nu - \frac{\Lambda}{\sqrt{2}}\rho(\mathbf{x})\right] m = 0, \tag{3}$$

where the function $m=m(\mathbf{x},\omega)$ and parameters ν and Λ are different for the waves, of different nature. For spin waves, Eq. (3) corresponds to the ferromagnetic superlattice with the inhomogeneous parameter of the anisotropy $\beta(\mathbf{x})$ [$A=\beta$, $\Delta A=\Delta\beta$ in Eq. (1)] in the situation when the direction of the external magnetic field $\mathbf{H}$, the constant component of the magnetization $\mathbf{M}_0$, and the magnetic anisotropy axis coincide with the direction of the superlattice z axis. In this case, $m=M_x+iM_y$, $\nu=(\omega-\omega_0)/\alpha g M_0$, $\Lambda=\sqrt{2}\Delta\beta/\alpha$, where ω is the frequency, $\omega_0=g[H+(\beta-4\pi)M_0]$ is the frequency of the ferromagnetic resonance, g is the gyromagnetic ratio, and α is the exchange parameter. For elastic waves in the scalar approximation in the superlattice with the inhomogeneous density of the material $p(\mathbf{x})$ ($A=p$, $\Delta A=\Delta p$), we have $\nu=(\omega/s)^2$ and $\Lambda=\sqrt{2}\omega^2(\Delta p)/ps^2$, where s is the elastic wave velocity. For electromagnetic waves in the same approximation in the superlattice with the inhomogeneous dielectric permeability $\varepsilon(\mathbf{x})$ ($A=\varepsilon$, $\Delta A=\Delta\varepsilon$), we have $\nu=\varepsilon(\omega/c)^2$ and $\Lambda=\sqrt{2}\omega^2(\Delta\varepsilon)/\varepsilon c^2$, where c is the speed of light.

The averaged Green's function for Eq. (3) has the form

$$G(\nu,\mathbf{k}) = \frac{1}{\nu - k^2 - M(\nu,\mathbf{k})}, \tag{4}$$

where $M(\nu,\mathbf{k})$ is the classical analog of the mass operator, which can be represented in the Bourret approximation[28] in the form obtained in Ref. 23,

$$M(\nu,\mathbf{k}) = -\frac{\Lambda^2}{8\pi}\int \frac{K(\mathbf{r})}{|\mathbf{r}|}\exp[-i(\mathbf{kr} + \sqrt{\nu}|\mathbf{r}|)]d\mathbf{r}. \tag{5}$$

The correlation function $K(\mathbf{r})$ for a sinusoidal superlattice is defined by the expression

$$K(\mathbf{r}) = \langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle_{\psi\chi_1\chi_3}. \tag{6}$$

The averaging in this formula must be carried out over the random phase ψ, as well as over the random functions χ_1 and χ_3, where

$$\chi_1 = q[u_1(z+r_z) - u_1(z)],$$

$$\chi_3 = q[u_3(\mathbf{x}+\mathbf{r}) - u_3(\mathbf{x})]. \tag{7}$$

In Ref. 26, we assumed that the random functions χ_1 and χ_3 are mutually uncorrelated and each of these functions obeys a Gaussian distribution. In Ref. 27, the definition of the cross correlations between the inhomogeneities of different dimensionalities has been introduced. It was assumed that the positive cross correlations mean that the enhancement of rms fluctuations of inhomogeneities of some dimensionality in some region of the space leads to the enhancement of rms fluctuations of inhomogeneities of the other dimensionality in the same region and vice versa, irrespective of the sign of these fluctuations. For a superlattice, this means that the increase in the displacement of the interfaces irrespective of its sign (1D inhomogeneities) must lead for the positive cross correlations to the increase in the deformation of surfaces of these interfaces (3D inhomogeneities).

The reason for the emergence of such correlations in the most general form may lie in the reasonable assumption that any random instability in the setup for obtaining superlattices, which causes an increase in the deviation of the thickness of a layer from the preset value, might also increase the probability that the deformations of the surface of this layer increase. To simulate such cross correlations, we introduced in Ref. 27 a distribution function that described correlation between absolute values $|\chi_1|$ and $|\chi_3|$ of random functions χ_1 and χ_3, leaving the functions themselves uncorrelated. The averaging in Eq. (6) carried out with that function had led to the following form of the correlation function:

$$K(\mathbf{r}) = \cos qr_z\{K_1(r_z)K_3(r) + K_{13}(\mathbf{r})\}, \quad (8)$$

where the expression in the braces is the decreasing part of the correlation function, which consists of the sum of the products of the decreasing parts of the correlation function for the components of a mixture of 1D and 3D inhomogeneities, $K_1(r_z)$ and $K_3(r)$, and the cross correlation function $K_{13}(\mathbf{r})$. Analytic expressions of these function were given in Refs. 27 and 26. Asymptotic formulas for these functions will be given in Sec. IV, where we discuss the nature of phenomena in the wave spectrum of superlattices caused by the inhomogeneities. The correlation function, Eq. (8) was substituted to Eq. (5). After that, the integration in Eq. (5) was performed using approximate expression for $K_1(r_z)$, $K_3(r)$, and $K_{13}(\mathbf{r})$. At the boundary of the first Brillouin zone of a superlattice $k=k_r\equiv q/2$, we obtained[27] the following expression for the Green's function in the two-wave approximation for Λ, $k_\parallel^2$, and $k_0^2 \ll k_r^2$:

$$G(\nu) = \frac{1}{\Lambda}\left\{X - \frac{1}{4}\left[\frac{1-L}{X - i\eta_1\gamma_1^2 - i\eta_3\gamma_3^2} + \frac{L}{X - i\eta_1\gamma_1^2}\right] - \kappa P(X)\right\}^{-1}. \quad (9)$$

Here,

$$P(X) = \frac{iN\eta_1}{12\pi\eta_3^2\gamma_1^2}[e^{v}E_1(v) + e^{u_-}E_1(u_-) + e^{u_+}E_1(u_+)], \quad (10)$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt$$

is the integral exponential function, $X=(\nu-k_r^2)/\Lambda$ is the dimensionless frequency detuning from the value of $\nu=k_r^2$, $\eta_1=k_\parallel q/\Lambda$ and $\eta_3=k_0 q/\Lambda$ are the dimensionless correlation wave numbers for 1D and 3D inhomogeneities, respectively, γ_1 and γ_3 are the relative rms fluctuations of the random functions $u_1(z)$ and $u_3(\mathbf{x})$,

$$v = ibX, \quad u_\pm = -\frac{1}{2}ibX(1 \pm i\sqrt{3}),$$

$$b = \left[\frac{N}{2\gamma_1^4\eta_3^3\gamma_3^2(1-L)}\right]^{1/3}, \quad N = L + 2\sqrt{3}\gamma_3 D(\sqrt{3}\gamma_3) - 1. \quad (11)$$

Here,

$$D(z) = e^{-t^2}\int_0^z e^{t^2}dt$$

is the Dawson's integral and $L=\exp(-3\gamma_3^2)$ is the asymptotic form for $K_3(r)$ for $r\to\infty$. The numerical cross correlation factor κ in Eq. (9) lies, in the general case, in the interval $-1<\kappa<1$, but Eq. (9) was obtained with the proviso that $|\kappa|\ll 1$.

For $\kappa=0$, Eq. (9) can be reduced to the expression for a mixture of uncorrelated 1D and 3D inhomogeneities,[26] while for $\gamma_3=0$ or $\gamma_1=0$, Eq. (9) can be reduced to the expressions for 1D or 3D inhomogeneities, respectively.

III. SUSCEPTIBILITY AND WAVE SPECTRUM

The Green's function given by Eq. (9) describes the complex dynamic susceptibility of the superlattice at $k=k_r$, and the equation for the complex frequency $\nu(k_r)=\nu'(k_r) - i\nu''(k_r)$ follows from the equality to zero of the function $G(\nu)$ denominator.

It is well known that the spectrum $\nu(k)$ of waves in a superlattice has a band structure. The first gap (forbidden band) is formed in the spectrum for $k=k_r$. In the absence of inhomogeneities and the intrinsic damping of waves, the gap width in the spectrum for $k=k_r$ [which corresponds to the spacing between levels $\nu_+(k_r)$ and $\nu_-(k_r)$ of the split spectrum] is equal to Λ. In this case, two δ-shaped peaks spaced by Λ will be observed on the imaginary part of the Green's function $G''(\nu)$. With increasing rms fluctuations γ_1 of 1D inhomogeneities, the gap $\Delta\nu=\nu'_+-\nu'_-$ between the spectral levels decreases and ultimately vanishes for a certain critical value of γ_1. The increase in γ_1 is accompanied by the increase of the damping $\nu''(k)$. The peaks on the $G''(\nu)$ dependence decrease and become closer with increasing γ_1, while their half-widths Γ increase until these peaks merge into one at a certain value of $\gamma_1=\gamma_{1c}$. The mode of variation of the spacing $\Delta\nu_m$ between the tops of the peaks corresponds to the change in the difference $\nu'_+-\nu'_-$ between the eigenfrequencies; however, there is no exact quantitative correspondence between these quantities for $\gamma_1\neq 0$ [the critical value of $\gamma_1=\gamma_{1c}$ corresponding to the $\Delta\nu_m$ vanishing is less of the critical value of γ_1 corresponding to the $\Delta\nu$ vanishing by 10–15% (Ref. 21)].

Forms of the dependencies $G''(\nu)$ which follow from Eq. (9) at $\kappa=0$ and $\gamma_3=0$ are shown in Fig. 1 by the thin curves 1 and 2, corresponding to the values of $\gamma_1^2=0.053$ and $\gamma_1^2=0.1$, respectively (thick solid and dotted-dashed curves in this figure will be discussed below). The spacing between the peaks $\Delta\nu_m$ and half-width of these peaks Γ as functions of γ_1^2 are shown by thin solid curves 1D in Figs. 2(a) and 2(b), respectively. Since the shape of the peaks on the $G''(\nu)$ dependence becomes asymmetric with increasing γ_1, this graph depicts the values of Γ corresponding to the right half-width of the right peak (and, accordingly, the left half-width of the left peak).

Dependencies of $\Delta\nu_m$ and Γ on γ_3^2 for 3D inhomogeneities [thin solid curves 3D in Figs. 2(a) and 2(b)] strongly differ from the corresponding dependencies for the 1D inho-

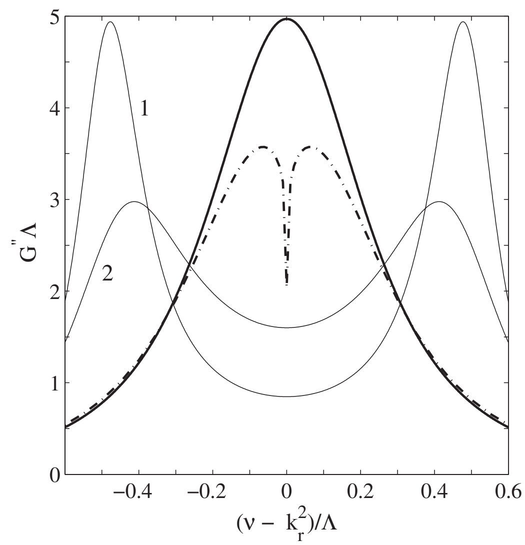

FIG. 1. The imaginary part of the Green's function G'' vs the frequency ν at the boundary of the Brillouin zone $k=k_r$ for the case of 1D inhomogeneities with $\gamma_1^2=0.053$ (curve 1) and $\gamma_1^2=0.1$ (curve 2) and for the case of the mixture of 1D and 3D inhomogeneities with $\gamma_1^2=0.2$ and $\gamma_3^2=0.3$ at the absence of crossing correlations (thick solid curve, $\kappa=0$) and in the presence of the crossing correlations (dotted-dashed curve, $\kappa=0.6$).

mogeneities: the spacing $\Delta\nu_m$ decreases more slowly than for the 1D inhomogeneities and the gap in the spectrum, decreasing with the increase in γ_3^2, remains open at any values of γ_3^2.

We are now coming to consideration of the phenomena at the simultaneous presence of 1D and 3D inhomogeneities or, in other words, to consideration of the mixture of 1D and 3D inhomogeneities. In Ref. 27, these phenomena were studied at values of rms fluctuations of the 1D inhomogeneities $\gamma_1 < \gamma_{1c}$ corresponding to the nonzero spacing $\Delta\nu_m$ between the peaks in the spectrum in the absence of the 3D inhomogeneities. Recall shortly the results obtained for this case. In Fig. 2(a), the dependence of $\Delta\nu_m$ on the sum of γ_1^2 and γ_3^2 is shown, and this permits us to analyze the spacing $\Delta\nu_m$ at different relationships between rms fluctuations of 1D and 3D inhomogeneities. We begin our consideration with the situation when 3D inhomogeneities are absent $(\gamma_3=0)$ and rms fluctuations of the 1D inhomogeneities increase from zero to $\gamma_1^2=0.14$. The normalized spacing $\Delta\nu_m/\Lambda$ in this case decreases according to the initial part of the curve 1D in Fig. 2(a) from 1 to 0.6. Then, we add 3D inhomogeneities increasing γ_3 and keeping $\gamma_1=0.14$. If cross correlations between the 1D and 3D inhomogeneities are absent, the dependence of $\Delta\nu_m/\Lambda$ on $\gamma_1^2+\gamma_3^2$ is described by the thin dashed curve in Fig. 2(a) and the gap is closed ($\Delta\nu_m$ vanishes) at $\gamma_1^2+\gamma_3^2\approx 0.35$ $(\gamma_1^2=0.14,\ \gamma_3^2\approx 0.21)$. At the presence of the positive $(\kappa>0)$ cross correlations between 1D and 3D inhomogeneities, the dependence of $\Delta\nu_m/\Lambda$ on $\gamma_1^2+\gamma_3^2$ is sharply changed and is described by the thin dotted-dashed curve in Fig. 2(a). The decrease of $\Delta\nu_m$ on γ_3^2 becomes smoother and the curve $\Delta\nu_m(\gamma_3^2)$ lies between those describing the dependence of $\Delta\nu_m$ on γ_3^2 for an uncorrelated $(\kappa=0)$ mixture and for a medium with 3D inhomogeneities. Negative correlations lead to a steeper descent of the function $\Delta\nu_m(\gamma_3^2)$ as compared to that for $\kappa=0$ and, hence, to the closure of the gap for smaller values of γ_3^2 [see Fig. 3(a) in Ref. 27].

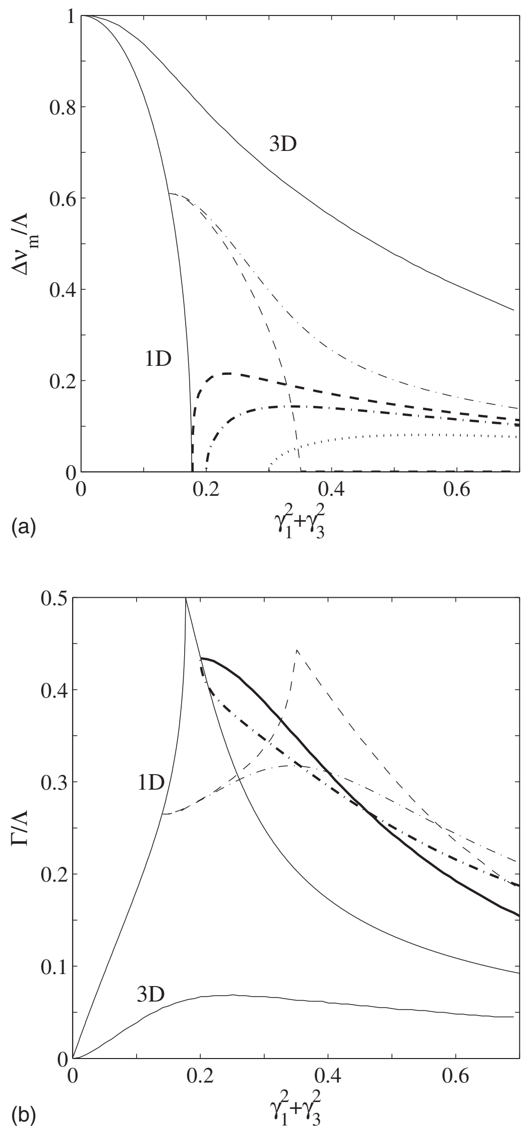

FIG. 2. (a) Spacing $\Delta\nu_m$ between the peaks on the function $G''(\nu)$ and (b) half-width of these peaks Γ as functions of the sum $\gamma_1^2+\gamma_3^2$ for $\gamma_1^2\neq 0$, $\gamma_3^2=0$ (curve 1D), $\gamma_1^2=0$, $\gamma_3^2\neq 0$ (curve 3D), for the case of the mixture of 1D and 3D inhomogeneities without crossing correlations for $\gamma_1^2=0.14$, $\gamma_3^2\neq 0$ (thin dashed curves), and for the case of the mixture 1D and 3D inhomogeneities in the presence of the positive crossing correlations ($\kappa=0.6$, $\gamma_3^2\neq 0$) for $\gamma_1^2=0.14$ (thin dotted-dashed curve), $\gamma_1^2=0.177$ (thick dashed curve), $\gamma_1^2=0.2$ (thick dotted-dashed curve), and $\gamma_1^2=0.3$ (dotted curve). Thick solid curve (b) corresponds to the closed gap at $\kappa=0$, $\gamma_1^2=0.2$, and $\gamma_3^2\neq 0$.

These results match the behavior of the $\Gamma(\gamma_3^2)$ curves for a mixture of inhomogeneities [see Fig. 2(b)]: the resonance line half-width decreases for $\kappa>0$ (thin dotted-dashed curve) as compared to the $\Gamma(\gamma_3^2)$ curve for an uncorrelated mixture of 1D and 3D inhomogeneities (thin dashed curve). Thus, the positive cross correlations for which the stochastic spatial synchronization of intensity fluctuations of 1D and 3D inho-

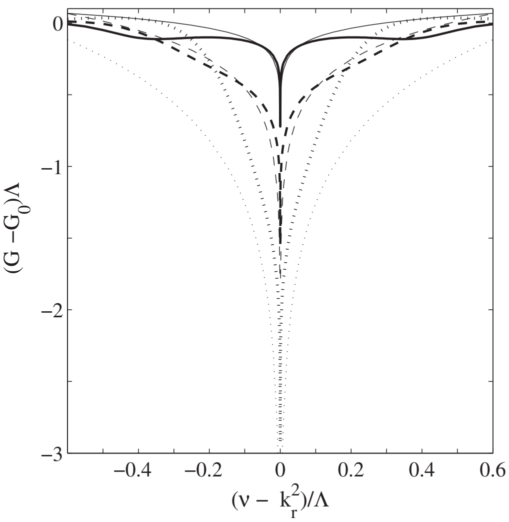

FIG. 3. The difference between the imaginary parts of the Green's function for the mixture of the cross-correlated and uncorrelated 1D and 3D inhomogeneities (thick curves) and the form of the logarithmic resonance described by the approximate formula [Eq. (14)] for $\gamma_1^2=0.1$, $\gamma_3^2=0.1$ (solid curves), $\gamma_1^2=0.14$, $\gamma_3^2=0.2$ (dashed curves), and $\gamma_1^2=0.2$, $\gamma_3^2=0.3$ (dotted curves); $\kappa=0.6$.

mogeneities takes place partly suppress the effect of a mixture of 1D and 3D inhomogeneities on the wave spectrum: the gap width at the boundary of the Brillouin zone increases and wave damping decreases as compared to the effect observed in a mixture of 1D and 3D inhomogeneities for $\kappa=0$. The negative cross correlations for which the intensity fluctuations of 1D and 3D inhomogeneities have a tendency to lie in different spatial regions lead to the opposite effect: the gap decreases and the damping increases as compared to those for $\kappa=0$ [see Fig. 3(b) in Ref. 27].

Let us consider now the phenomenon of suppressing the effects of 1D inhomogeneities for the unexplored earlier case of $\gamma_1 \geqslant \gamma_{1c}$ corresponding to the closed gap in the spectrum at the absence of 3D inhomogeneities. Let the rms fluctuations of 1D inhomogeneities increase from zero to $\gamma_{1c}^2=0.177$ at the absence of 3D inhomogeneities [curve 1D in Fig. 2(a)]. The spacing $\Delta\nu_m$ vanishes at this point, and only one maximum remains in the curve $G''(\nu)$. Then, the rms fluctuations γ_3 of 3D inhomogeneities increase at $\gamma_1=\gamma_{1c}$. If cross correlations between 1D and 3D inhomogeneities are absent, the spacing $\Delta\nu_m$ remains equal to zero at any values of γ_3^2 [the shape of the maximum in the curve $G''(\nu)$ will, certainly, change]. At the presence of cross correlations, an unexpected phenomenon happens with the increase of γ_3: the gap in the spectrum again partly opens and, correspondingly, the maximum in the curve $G''(\nu)$ again splits up into two peaks. The spacing between these peaks $\Delta\nu_m/\Lambda$ vs $\gamma_1^2+\gamma_3^2$ is shown in Fig. 2(a) by the thick dashed curve. One can see that this spacing may be as much as 20% of the initial width of the gap. This phenomenon can be observed also for $\gamma_1 > \gamma_{1c}$. Let us increase the rms fluctuations of 1D inhomogeneities to $\gamma_1^2=0.2$ at $\gamma_3=0$. Then, we grow the rms fluctuations γ_3 of 3D inhomogeneities saving $\gamma_1^2=0.2$. At the presence of the positive cross correlations, the gap in the spectrum is opening as it is shown in Fig. 2(a) by thick dotted-dashed curve. The function $G''(\nu)$ is shown in Fig. 1, namely, for this situation at $\gamma_1^2+\gamma_3^2=0.5$ ($\gamma_1^2=0.2$, $\gamma_3^2=0.3$): the thick solid curve corresponds to the absence of cross correlations ($\kappa=0$), and the thick dotted-dashed curve corresponds to $\kappa=0.6$. The dotted curve in Fig. 2(a) describes the process of opening the gap with increasing γ_3 at $\gamma_1^2=0.3$. By comparing the thick dashed, dotted-dashed, and dotted curves in Fig. 2(a), one can see how the effect of opening the gap decreases with the increase of the rms fluctuations γ_1 of the 1D inhomogeneities. In Fig. 2(b), the change of the resonance line half-width with the increase of γ_3 at $\gamma_1^2=0.2$ at the absence of the cross correlations (thick solid curve) and at $\kappa=0.6$ (thick dotted-dashed curve) is shown. It is seen that the cross correlations at $\gamma_1^2+\gamma_3^2<0.4$ lead to the some decrease of the damping in the system. However, this effect is not so clearly expressed as compared with the case when the increase of γ_3^2 occurs at values of $\gamma_1 < \gamma_{1c}$ [thin dashed ($\kappa=0$) and dotted-dashed ($\kappa=0.6$) curves in Fig. 2(b)].

IV. RELATION OF EFFECTS IN THE WAVE SPECTRUM WITH ASYMPTOTIC PROPERTIES OF THE SUPERLATTICE CORRELATION FUNCTION

It was shown earlier[25–27] that effects of inhomogeneities on the wave spectrum strongly depend on the asymptotic behavior of the superlattice correlation function. We consider in this section in more detail the relation between the phenomena in the superlattice wave spectrum, which is due to inhomogeneities and the asymptotic properties of correlation functions of these inhomogeneities. Let the only 1D inhomogeneities $u_1(z)$ be present in the superlattice, which are characterized by the rms fluctuation γ_1 and the correlation wave number $k_\parallel$. The decreasing part $K_1(r_z)$ of the superlattice correlation function $K(\mathbf{r})$ in this case has the finite correlation radius $(k_\parallel\gamma_1^2)^{-1}$ and vanishes exponentially at $r_z\to\infty$ (see Refs. 26 and 27). In this case ($\kappa=0$, $\gamma_3=0$, and $L=1$), only one resonance term $1/4(X-i\eta_1\gamma_1^2)$ is left in the braces in Eq. (9) for the Green's function $G(\nu)$. The interaction of this resonance with the first term in the braces X describes all peculiarities of the wave spectrum in the superlattice. The damping of the waves in the resonance term is proportional to the inverse correlation radius $k_\parallel\gamma_1^2$, and its increase leads to the vanishing of the spacing $\Delta\nu_m$ at $\gamma_1=\gamma_{1c}$. Now, let the only 3D inhomogeneities $u_3(\mathbf{r})$ be present in the superlattice, which are characterized by the rms fluctuation γ_3 and the correlation wave number k_0. It was shown earlier[26] that the decreasing part of the correlation function $K_3(r)$ was divided in that case on two parts by the asymptote $K_3(\infty)=L=\exp(-3\gamma_3^2)$. The part of $K_3(r)$ which lies above this asymptote is proportional to $1-L$, characterized by the finite correlation radius $(k_0\gamma_3^2)^{-1}$, and exponentially tends to zero for $r\to\infty$. The part of $K_3(r)$ which lies under this asymptote is proportional to L and has infinite correlation radius. Two resonance terms correspond to these parts of $K_3(r)$, which remain in the case of 3D inhomogeneities ($\kappa=0$, $\gamma_1=0$) in the braces in Eq. (9): the term $(1-L)/4(X-i\eta_3\gamma_3^2)$, where the damping is proportional to the inverse correlation radius

$k_0\gamma_3^2$, and the term $L/4X$, which describes waves propagating without the damping. Owing to this latter term, the damping of waves in the superlattice with 3D inhomogeneities is essentially smaller than in the superlattice with 1D inhomogeneities with the same rms fluctuation [Fig. 2(b)] and the spacing $\Delta\nu_m$ is essentially larger [Fig. 2(a)]. If the mixture of both 1D and 3D inhomogeneities without crossing correlations between them there is in the superlattice, the decreasing part of the correlation function will be a product of the decreasing parts of the correlation functions of 1D and 3D inhomogeneities $K_1(r_z)K_3(r)$. This correlation function has a complex structure, and its correlation radius is determined by two parameters: $(k_\parallel\gamma_1^2)^{-1}$ and $(k_0\gamma_3^2)^{-1}$; the product $K_1(r_z)K_3(r)$ exponentially tends to zero for $r\to\infty$. Two resonance terms are in the braces in Eq. (9) in this case ($\kappa=0$, $\gamma_1\neq 0$, and $\gamma_3\neq 0$), as in the case of 3D inhomogeneities. However, each of these terms describes damped waves: the term proportional to $1-L$ makes a contribution to the damping determined by the sum of $k_\parallel\gamma_1^2$ and $k_0\gamma_3^2$ and the term proportional to L makes a contribution to the damping determined only by $k_\parallel\gamma_1^2$. Because of that, with the increase of rms fluctuations of both 1D and 3D inhomogeneities, the decrease of $\Delta\nu_m$ to zero and, correspondingly, the closing of the gap in the spectrum occur [thin dashed curve in Fig. 2(a)].

When cross correlations between 1D and 3D inhomogeneities appear, the function of the cross correlation $K_{13}(\mathbf{r})$ is added to the product $K_1(r_z)K_3(r)$. The function $K_{13}(\mathbf{r})$ has long weakly decaying ($\propto r^{-1}$) correlations. It is known that for a correlation function with such asymptotic properties, a finite correlation radius cannot be introduced. The term $P(X)$ determined by Eq. (10) corresponds to the function of the cross correlation $K_{13}(\mathbf{r})$ in Eq. (9). For discussing the effects which are due to the cross correlations, we use the expansion of the function $E_1(z)$ at $|z|\ll 1$ (see, for example, Ref. 29) and the expansion of the term $P(X)$ in the vicinity of $X=0$. As a result, the approximate expression for $P(X)$ takes the form

$$P(X)\approx -\frac{iN\eta_1}{4\pi\eta_3^2\gamma_1^2}\left[\ln(\gamma b|X|)+i\,\mathrm{sgn}(X)\frac{\pi}{6}\right], \qquad (12)$$

where $\gamma\approx 1.78$ is the Euler's constant. It is seen that the cross correlations lead to the appearance, in the braces of Eq. (9), of the additional resonance term of the logarithmic form. This form of the resonance is completely determined by the asymptotic form r^{-1} of the cross correlation function $K_{13}(\mathbf{r})$. Let us consider the manifestation of this logarithmic resonance in the form of the imaginary part of the Green's function $G''(\nu)$. We study for this purpose the function

$$\Delta G''(\nu)=G''(\nu)-G_0''(\nu), \qquad (13)$$

where both $G''(\nu)$ and $G_0''(\nu)$ are determined by Eq. (9), but $G''(\nu)$ corresponds to $\kappa>0$ while $G_0''(\nu)$ corresponds to $\kappa=0$. The function $\Delta G''(\nu)$ is shown in Fig. 3 by thick curves for the three pairs of the values of γ_1^2 and γ_3^2: $\gamma_1^2=0.1$, $\gamma_3^2=0.1$ (thick solid curve); $\gamma_1^2=0.14$, $\gamma_3^2=0.2$ (thick dashed curve); and $\gamma_1^2=0.2$, $\gamma_3^2=0.3$ (thick dotted curve). These curves clearly depict the logarithmic resonance at the center of the gap $\nu=k_r^2$. For comparison, we consider an approximate expression for $\Delta G''(\nu)$ that corresponds to the expansion of the exact Eq. (13) in powers of both κ and X,

$$\Delta G''|_{\substack{X\to 0\\ \kappa\to 0}}=-\kappa\left[\frac{4\eta_1\gamma_1^2(\eta_1\gamma_1^2+\eta_3\gamma_3^2)}{\eta_1\gamma_1^2+L\eta_3\gamma_3^2}\right]^2 P''(X), \qquad (14)$$

where $P''(X)=\mathrm{Im}\,P(X)$, and $P(X)$ is described by Eq. (12).

This approximate function $\Delta G''(\nu)$ is shown in Fig. 3 by thin curves for the same three pairs of the values of γ_1^2 and γ_3^2 for which the exact function $\Delta G''(\nu)$ is shown by thick curves. It is seen that differences between the thin and the corresponding thick curves are not so large. The pictures looks like that the switching on of cross correlations leads to subtraction (at $\kappa>0$) of the logarithmic resonance approximately described by Eq. (12) from the Green's function $G_0''(\nu)$, corresponding to the absence of the cross correlations. Thick solid curve in Fig. 3 corresponds to the function $G''(\nu)$ that is shown in Fig. 4 of Ref. 27 for $\gamma_1^2=0.1$, $\gamma_3^2=0.1$. In this case, the width of the logarithmic resonance line is much less than the spacing $\Delta\nu_m$ between the maxima of the function $G''(\nu)$ for $\kappa=0$ (solid curve in Fig. 4 of Ref. 27). That is why the switching on of crossing correlations between 1D and 3D inhomogeneities gives rise to the resonance effects in the center of the forbidden band: a narrow dip (at $\kappa>0$) or narrow peak (at $\kappa<0$). Let us consider now the case of $\gamma_1^2=0.14$ and $\gamma_3^2=0.2$ when the gap is closed under the action of uncorrelated 1D and 3D inhomogeneities [Fig. 2(a), the right end of the thin dashed curve] and only one resonance peak remains on the curve $G''(\nu)$. The logarithmic resonance at these values of γ_1^2 and γ_3^2 becomes larger (dashed curves at Fig. 3), and the switching on of the positive cross correlation leads to the splitting of one resonance peak onto two peaks, that is, to the partial opening of the gap [the point on the thin dotted-dashed curve at $\gamma_1^2+\gamma_3^2=3.4$ in Fig. 2(a)]. At last, we consider the case $\gamma_1^2=0.2$ when the gap is closed due to only 1D inhomogeneities. The adding of the correlated with the 3D inhomogeneities leads to the further enhancement of the logarithmic resonance (the dotted curves in Fig. 3) that will cause the partial opening of the gap in the spectrum with the increase of the rms fluctuation γ_3^2.

Thus, the effects of crossing correlations between 1D and 3D inhomogeneities considered earlier[27] (the appearance of the narrow resonance in the center of the forbidden band in the superlattice wave spectrum and the slowing down of the closing of the gap with the increase of γ_3), as well as the effect of opening the gap with the increase of γ_3 that was found in the present work, are due to the same reason: the appearance of the logarithmic resonance in the center of the gap. This resonance, in its turn, is conditioned by the slowly decaying ($\propto r^{-1}$) term $K_{13}(\mathbf{r})$ in the correlation function of the superlattice $K(\mathbf{r})$. From the physical point of view, it means that the stochastic spatial synchronization between fluctuations of 1D and 3D inhomogeneities leads to the appearance of regions in the space, in which waves propagate with very small damping. The experimental observation of the logarithmic resonance could be a test of the existence of the cross correlations between 1D and 3D inhomogeneities in the material.

It should be emphasized that the slow decay ($\propto r^{-1}$) of the correlation function of the cross-correlated 1D and 3D inhomogeneities is the fundamental property of this function that has no relations with any model. This law of the decay of correlations corresponds to the intuitive supposition that the correlation function of the cross-correlated 1D and 3D inhomogeneities must have the asymptotic behavior at $r \to \infty$ that is the intermediate one between the asymptotic behaviors of the 1D inhomogeneities $[K_1 \propto \exp(-\gamma_1^2 k_\parallel r)]$ and 3D inhomogeneities ($K_3 \propto L = \text{const}$).

Effects of cross correlations between 1D and 3D inhomogeneities were investigated in the present paper as well as in Ref. 27 for the superlattice with the sinusoidal initial modulation. The question whether the phenomena found in these papers are characteristics for another modulation profile or only for the sinusoidal one arises. As it was shown earlier,[22,24] the spectral characteristics at the first Brillouin zone boundary and their modifications under the action of inhomogeneities are determined, mainly, by the first Fourier harmonic of the function describing the modulation profile. This harmonic depends only weakly on the form of this function when the latter changes from the sinusoidal to rectangular form. That is why one would expect that the results obtained here for the sinusoidal superlattice will be qualitatively the same and for another forms of the modulation profile.

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