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Gaussian Random Waves in Elastic Media[¶]

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Similar to the Berry conjecture of quantum chaos, an elastic analogue which incorporates longitudinal and transverse elastic displacements with corresponding wave vectors is considered. The correlation functions are derived for the amplitudes and intensities of elastic displacements. A comparison to the numerics in a quarter-Bunimovich stadium demonstrates excellent agreement.

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1. INTRODUCTION

Attracting interest in the field of wave chaos [1], elastomechanical systems are being studied analytically, numerically, and experimentally. Weaver first measured the few hundred lower eigenfrequencies of an aluminum block and worked out the spectral statistics [2]. Spectral statistics coinciding with random matrix theory were observed in experiments for monocrystalline quartz blocks shaped as three-dimensional Sinai billiards [3], as well as in experimental and numerical studies of flexural modes [4, 5] and in-plane modes [6, 7] for stadium-shaped plates. The statistical properties of the eigenfunctions describing standing waves in elastic billiards were first reported by Schaadt et al. [8]. The authors measured the displacement field of several eigenmodes of a thin plate shaped as a Sinai stadium. Due to the good preservation of the up–down symmetry in the case of thin plates, they dealt with two types of modes. The flexural modes with displacement perpendicular to the plane of the plate are well described by the scalar biharmonic Kirchoff–Love equation [9, 10]. In this case, good agreement with the theoretical prediction for both the intensity statistical distribution and the intensity correlation function was found. However, in the case of the in-plane displacements described by the vector Navier–Cauchy equation [9, 10], agreement between the intensity correlator experimental data and the theory was not achieved [8].

The aim of present Letter is to present an analogue of the Berry conjecture for elastic vibrating solids and derive the amplitude and intensity correlation functions with corresponding comparison to numerics. Quite recently, Acolzin and Weaver suggested a method to calculate the intensity correlator of vibrating elastic solids [11]. Based on the Green's function averaging technique, they succeeded in deriving the intensity correlator of flexural modes generalized due to the finite thickness of a plate. Although the method might be used for the in-plane modes in elastic chaotic billiards, we propose here a simpler and physically transparent approach based on the random superposition of traveling plane waves (Gaussian random wave (GRW) or the Berry function [12]). We show that the approach allows us to derive all kinds of correlation functions of RGW, not only in infinite elastic media, but also to take into account the double-ray splitting at the boundary of a plate that plays a significant role in the elastomechanical chaotic motion [13, 14]. We restrict ourselves to the two-dimensional case, because of the current experiments available. Note, however, that the method can be easily generalized for the three-dimensional case.

2. ANALOGUE OF THE BERRY CONJECTURE IN ELASTIC MEDIA

Shapiro and Goelman [15] first presented the statistics of the eigenfunctions in a chaotic quantum billiard although their numerical histogram was not compared with the Gaussian distribution. This was done by McDonnell and Kaufmann [16], who concluded that the majority (≈90%) of the eigenfunctions of the Bunimovich billiard are a Gaussian random field. Later, this was confirmed by numerous numerical and experimental studies. The simple way to construct RGF is random superposition of particular solutions of Eq. (4) [12, 17, 18] with a sufficient number N. Thus, we come to the Berry conjecture in the form [19, 1]

$$\psi_B(\mathbf{x}) = \sqrt{\frac{1}{N}}\sum_{n=1}^{N}\exp[(i(\theta_n + \mathbf{k}_n\mathbf{x})], \qquad (1)$$

where the phases θ_n are randomly distributed uniformly in the range $[0, 2\pi)$ and all of the amplitudes are taken to be equal (one could assume random independent amplitudes without any change in the results). The

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wave vectors $\mathbf{k}_n$ are uniformly distributed on a d-dimensional sphere of radius k. It follows now from the central limit theorem that both $\mathrm{Re}\psi_B$ and $\mathrm{Im}\psi_B$ are independent Gaussian variables. In a closed billiard, the Berry function is viewed as a sum of many standing waves, which is simply the real or imaginary part of function (1).

In our case, one has to construct a RGW function describing the acoustic in-plane modes. These modes are described by a two-dimensional Navier–Cauchy equation [10, 20]

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\mathbf{u}) + \rho\Omega^2\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}(x, y)$ is the displacement field in the plate, λ, μ are the material dependent Lamé coefficients, and ρ is the density. Introducing elastic potentials ψ and $\mathbf{A}$ with the help of the Helmholtz decomposition [20], the displacement field $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \mathbf{u}_l = \nabla\psi, \quad \mathbf{u}_t = \nabla\times\mathbf{A}. \tag{3}$$

Equation (2) reduces to two Helmholtz equations for the elastic potentials

$$-\nabla^2\psi = k_l^2\psi, \quad -\nabla^2\mathbf{A} = k_t^2\mathbf{A}. \tag{4}$$

Here, $k_l = \omega/c_l$, $k_t = \omega/c_t$ are the wavenumbers for the longitudinal and transverse waves, respectively, and $\omega^2 = \rho\Omega^2/E$, where E is Young's modulus. In the two-dimensional case, potential $\mathbf{A}$ has only one none-zero component A_z and the dimensionless longitudinal and transverse sound velocities $c_{l,\,t}$ are given by

$$c_l^2 = \frac{1}{1-\sigma^2}, \quad c_t^2 = \frac{1}{2(1+\sigma)}, \tag{5}$$

where σ is Poisson's ratio [10, 20]. E and σ are functions of the Lamé coefficients [10, 20]. Our conjecture is that both elastic potentials be statically independent Berry-like functions (1). We write the potentials in the following form:

$$\begin{aligned}\psi(\mathbf{x}) &= \frac{a_l}{ik_l}\sqrt{\frac{1}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{ln}\mathbf{x}+\theta_{ln})],\\ A_z(\mathbf{x}) &= \frac{a_t}{ik_t}\sqrt{\frac{1}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{tn}\mathbf{x}+\theta_{tn})],\end{aligned} \tag{6}$$

where θ_{ln}, θ_{tn} are statistically independent random phases. The wave vectors $\mathbf{k}_{ln}$ and $\mathbf{k}_{tn}$ are uniformly distributed on circles of radii k_l and k_t, respectively. According to (3), the components u, v of the vector displacement field $\mathbf{u}$ can now be written as

$$\begin{aligned}u(\mathbf{x}) &= \sqrt{\frac{1-\gamma}{N}}\sum_{n=1}^{N}\cos\phi_{ln}\exp[i(\mathbf{k}_{ln}\mathbf{x}+\theta_{ln})]\\ &\quad+\sqrt{\frac{\gamma}{N}}\sum_{n=1}^{N}\sin\phi_{tn}\exp[i(\mathbf{k}_{tn}\mathbf{x}+\theta_{tn})],\\ v(\mathbf{x}) &= \sqrt{\frac{1-\gamma}{N}}\sum_{n=1}^{N}\sin\phi_{ln}\exp[i(\mathbf{k}_{ln}\mathbf{x}+\theta_{ln})]\\ &\quad-\sqrt{\frac{\gamma}{N}}\sum_{n=1}^{N}\cos\phi_{tn}\exp[i(\mathbf{k}_{tn}\mathbf{x}+\theta_{tn})],\end{aligned} \tag{7}$$

where ϕ_{ln} and ϕ_{tn} are the angles between $\mathbf{k}_{ln}$ and $\mathbf{k}_{tn}$ and the x axis, respectively. The prefactors $a_l = \sqrt{\gamma}$ and $a_t = \sqrt{1-\gamma}$ are chosen from the normalization condition $\langle\mathbf{u}^\dagger\mathbf{u}\rangle = 1$, and $\langle\ldots\rangle$ mean average over the random-phase ensembles. Parameter γ ranges from 0 (pure transverse waves) to 1 (pure longitudinal waves). Similarly, one can construct the elastomechanical GRW for a closed system. One can see that the Berry analogue of chaotic displacements (7) is not a sum of two independent GRWs (or two independent Berry functions) $a_l\psi_l + a_t\psi_t$, as was argued by Schaadt et al. [8], with the arbitrary coefficients a_l and a_t. In fact, each component u and v in Eq. (7) is related to Berry functions (6) via space derivatives in accordance with relations (3).

3. CORRELATION FUNCTIONS

First, we calculate the amplitude correlation functions in a chaotic elastic plate for in-plane GRW (7). For quantum mechanical GRW (1), the two-dimensional correlation function

$$\langle\psi_B(\mathbf{x}+\mathbf{s})\psi_B^*(\mathbf{x})\rangle = J_0(s)$$

was found firstly by Berry [12]. A straightforward procedure of averaging over the ensembles of random phases θ_{ln} and θ_{tn} and, then, over the angles of the k vectors gives

$$\begin{aligned}\langle u(\mathbf{x}+\mathbf{s})u(\mathbf{x})\rangle &= \frac{\gamma}{2}(\cos^2\alpha f(k_ls) + \sin^2\alpha g(k_ls))\\ &\quad+\frac{1-\gamma}{2}(\sin^2\alpha f(k_ts) + \cos^2\alpha g(k_ts)),\\ \langle v(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle &= \frac{\gamma}{2}(\sin^2\alpha f(k_ls) + \cos^2\alpha g(k_ls))\\ &\quad+\frac{1-\gamma}{2}(\cos^2\alpha f(k_ts) + \sin^2\alpha g(k_ts)),\\ \langle u(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle &= \sin2\alpha\left(\frac{1-\gamma}{2}J_2(k_ts) - \frac{\gamma}{2}J_2(k_ls)\right),\end{aligned} \tag{8}$$

where

$$f(s) = J_0(s) - J_2(s), \quad g(s) = J_0(s) + J_2(s). \tag{9}$$

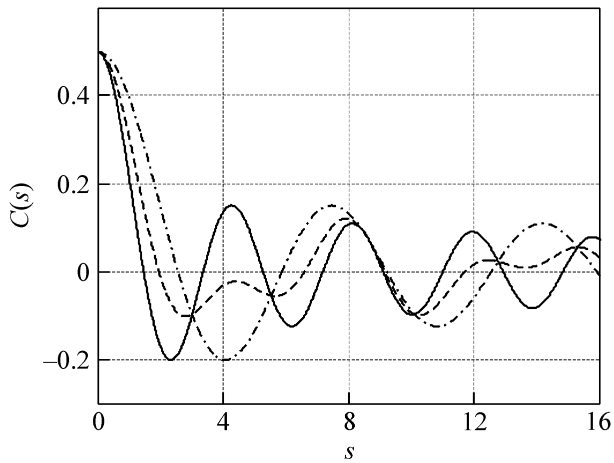

Fig. 1. Correlation function (10) for γ = (solid line) 0, (dashed line) 0.5, and (dash–dotted line) 1 at $\sigma = 0.345$ (aluminum).

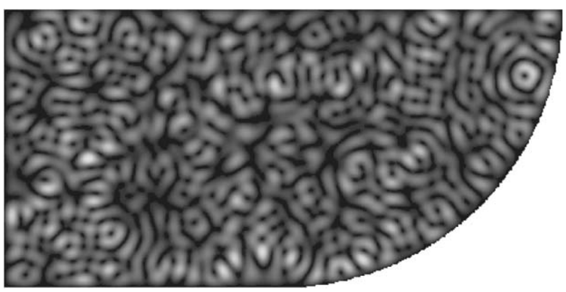

Fig. 2. Intensity $I = |\mathbf{u}|^2$ of the eigenstate at frequency ω = 28.4 in a quarter of the Bunimovich billiard with a fixed boundary at $\sigma = 0.345$.

It is important to note that correlation functions (8) were obtained for a given direction of the vector **s**, where α is the angle included between vector **s** and the x axis. However, averaged over all directions of **s**, the first two correlation functions simplify to

$$C(s) = \overline{\langle u(\mathbf{x}+\mathbf{s})u(\mathbf{x})\rangle} = \overline{\langle v(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle} = \frac{1-\gamma}{2}J_0(k_l s) + \frac{\gamma}{2}J_0(k_t s), \tag{10}$$

while the third vanishes $\overline{\langle u(\mathbf{x}+\mathbf{s})v(\mathbf{x})\rangle} = 0$. One can see that, in the averaged case, the amplitude correlation function is defined by two scales because of two different sound velocities c_l and c_t; this is obvious. The correlation function $C(s)$ is shown in Fig. 1.

Next, we calculate the intensity correlation functions $P(s) = \langle I(\mathbf{x}+\mathbf{s})I(\mathbf{x})\rangle$, where the intensity $I = |\mathbf{u}|^2$ is proportional to the elastic energy of the in-plane oscillations. In quantum mechanics, this value is analogous to the probability density, the correlation function of which was calculated by Prigodin et al. [21]. For the in-plane chaotic GRW of the form $a_l\psi_l + a_t\psi_t$, Schaadt et al. [8] derived the intensity correlation function as

$$P(s) = 1 + 2[a_l^2 J_0(k_l s) + a_t^2 J_0(k_t s)]^2. \tag{11}$$

Our calculations similar to those for amplitude correlation functions (8) give a different result:

$$P(s) = 1 + \frac{1}{2\eta}[(\gamma J_0(k_l s) + (1-\gamma)J_0(k_t s)]^2 + \frac{1}{2\eta}[(\gamma J_2(k_l s) - (1-\gamma)J_2(k_t s)]^2, \tag{12}$$

where $\eta = 1$ for a real GRW and $\eta = 2$ for a complex one. Although the first term in (12) corresponds to (11), there is a different term consisting of the Bessel functions J_2. The mathematical origin of the deviation is that formula (7) contains the contributions of the components of the wave vectors $\mathbf{k}_l$ and $\mathbf{k}_t$ via space derivatives.

4. WAVE CONVERSION AT THE BOUNDARY

Waves propagate freely inside the billiard, that is, the longitudinal and transverse components are decoupled. Wave conversion occurs at the boundary according to Snell's law

$$c_l \sin(\theta_t) = c_t \sin(\theta_l). \tag{13}$$

The reflection amplitudes for each event of the reflection can be easily found following the procedure described in [10]. At first, we consider the simpler case of the Dirichlet boundary condition (where the boundary is fixed). Approximating the boundary as the straight lines for the wavelengths that are much less than the radius of the curvature, we have for the reflection amplitudes

$$\begin{aligned} t_{ll} &= \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \\ t_{lt} &= \frac{2\sin(\theta_l)\cos(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \\ t_{tl} &= \frac{2\sin(\theta_t)\cos(\theta_t)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \\ t_{tt} &= \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}. \end{aligned} \tag{14}$$

Next, we assume that all wave directions are statistically equivalent. Then, we have for the energy density of reflected wave

$$\rho_{\rm out} = \gamma(\overline{T}_{ll} + \overline{T}_{lt}) + (1-\gamma)(\overline{T}_{tt} + \overline{T}_{tl}), \tag{15}$$

where

$$\overline{T}_{ij} = \frac{1}{\pi}\int_0^\pi t_{ij}^2 d\theta_i, \quad i = l, t.$$

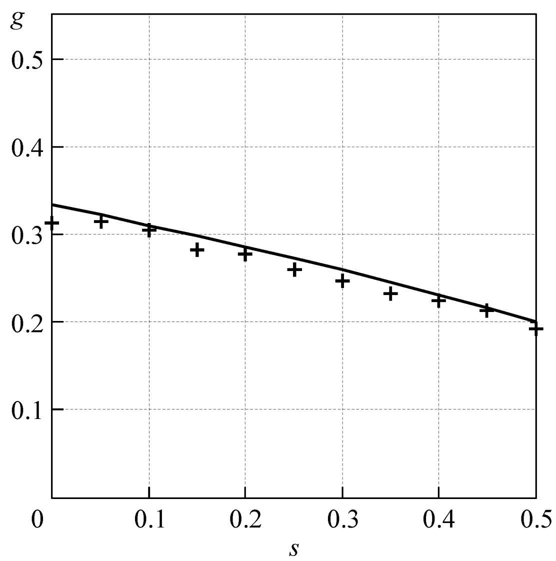

Fig. 3. Cross marks show the numerical results for ratio γ averaged over 200 eigenfunctions of the quarter-Bunimovich plate. The solid line is plotted using formula (18).

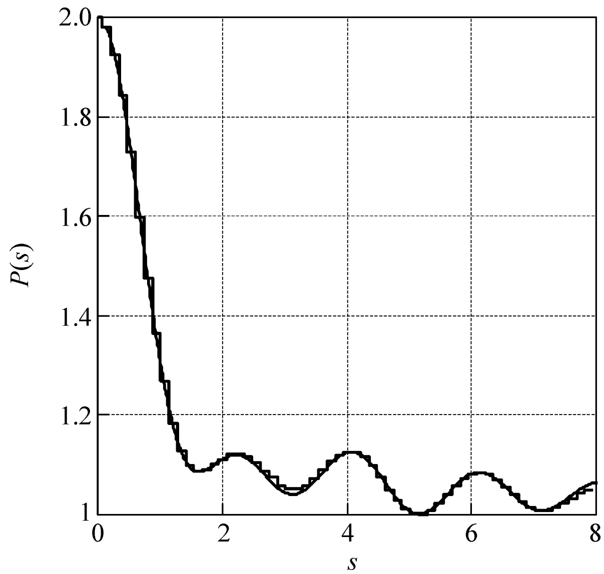

Fig. 4. Intensity correlation function (12) compared to the numerics for the same situation as in Fig. 2.

Substituting into here (14), one can obtain, after elementary calculations,

$$\bar{T}_{ll} = 1 - \frac{c_t}{c_l} I_1, \quad \bar{T}_{lt} = I_2,$$

$$\bar{T}_{tt} = 1 - \frac{2}{\pi}\arcsin\frac{c_t}{c_l} + \left(\frac{c_t}{c_l}\right)^3 I_1, \tag{16}$$

$$\bar{T}_{tl} = \frac{2}{\pi}\arcsin\frac{c_t}{c_l} - \left(\frac{c_t}{c_l}\right)^2 I_2.$$

We do not present here integrals I_1 and I_2, since, after the substitution of (16) into (15), they cancel each other. The equality $\rho_{\text{in}} = 1 = \rho_{\text{out}}$ gives a very simple evaluation:

$$\gamma = \frac{c_t^2}{c_t^2 + c_l^2}. \tag{17}$$

The next remarkable result is that, although the reflection amplitudes for the free boundary condition [10] have a form different from (14) (see, for example, the formulas in [10]), the evaluation of γ using the same procedure gives the same form as for the fixed boundary condition. Therefore, we can conclude that the result does not depend on whether the free boundary condition or the fixed boundary condition is applied. Using (5), formula (17) could be written in a simpler form:

$$\gamma = \frac{1-\sigma}{3-\sigma}. \tag{18}$$

5. NUMERICAL RESULTS AND CONCLUSIONS

For numerical tests, we took a quarter of the Bunimovich billiard and calculated the eigenstates of Navier–Cauchy equation (2) with a fixed boundary condition: $u = 0$ and $v = 0$ at the boundary of the billiard using the finite-difference method. An example of the eigenstate in the form of intensity $I = |\mathbf{u}|^2$ is presented in Fig. 2. First of all, we verified formula (18). For each value of Poisson's ratio σ in the range [0, 0.5] with the step 0.05, 200 eigenfunctions of the billiard were found to calculate an averaged γ. The resulting dependence of γ on σ is shown in Fig. 3, which demonstrates a good agreement with formula (5). Therefore, we can evaluate γ for specific $\sigma = 0.345$, which corresponds to the aluminum plate, and plot the correlation functions. Intensity correlation function (12) is shown in Fig. 4 compared to the numerics calculated for the eigenfunction presented in Fig. 2. One can see a good coincidence between the theory and the numerical results, which demonstrates the correctness of GRW approach (7) to chaos in elastic billiards.

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