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Phase correlation function of complex random Gaussian fields

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Abstract – The phase correlation function $\langle\exp[i\theta(\mathbf{x}+\mathbf{s})-i\theta(\mathbf{x})]\rangle$ for the complex random Gaussian field $\psi(\mathbf{x})=|\psi(\mathbf{x})|\exp[i\theta(\mathbf{x})]$ is derived. It is compared to the numerical scattering wave function in the open Sinai billiard.

Introduction. – Understanding of the statistical properties of the eigenfunctions of a quantum system whose classical counterpart is chaotic and their relation to the underlying classical dynamics is one of the key problems studied in the field of quantum chaos. Among various applications, wave function correlations are important for the statistics of electron transport through quantum chaotic billiards, see ref. [1] and references therein, and for the statistics of nodal points [2,3]. Almost thirty years ago, Berry [4] conjectured that an eigenfunction of a classically chaotic system (quantum billiard) can be represented as a random superposition of plane waves with fixed absolute value k of the wave vector (determined by the energy $E=\hbar^2k^2/2m$). This implies Gaussian statistics of the eigenfunction amplitude $\psi(\mathbf{r})$. Berry [4] derived the space-averaged spatial correlation function (in what follows correlator) of the wave function:

$$C(r)=\langle\psi(\mathbf{x}+\mathbf{r})\psi^*(\mathbf{x})\rangle=\Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d}{2}-1}(kr)}{(kr)^{\frac{d}{2}-1}}, \qquad (1)$$

where d is dimension of the space, Γ is the Gamma-function, $J_n(x)$ is the Bessel function of the n-th order. Later, similar correlations were rigorously derived for quantum dots in ref. [5]. Moreover Prigodin [6] derived the correlator of the square of the wave function $\langle|\psi(\mathbf{x}+\mathbf{s})\psi(\mathbf{x})|^2\rangle$ and of the higher degrees of the wave functions $\langle|\psi(\mathbf{x}+\mathbf{s})|^{2m}|\psi(\mathbf{x})|^{2n}\rangle$ where m, n are integers.

The underlying random phase fields, however, have received much less attention, and many important properties of these fields are still unknown from either theory or experiment. The concept of phase is usually introduced as a property of coherent wave fields. Such wave fields are described by a complex function $\psi(\mathbf{r})=|\psi(\mathbf{r})|\exp[i\theta(\mathbf{r})]$. The surfaces of constant $\theta(\mathbf{r})$ can be identified with the wavefronts. In free space, $\mathbf{v}=\frac{1}{m}\nabla\theta(\mathbf{r})$ describes the local velocity [7]. The phase of an optical wave field can be determined from correlation measurements [8,9]. Also it can be determined indirectly by independent measuring of the real and imaginary parts of the complex wave field as it was done in microwave experiments [10]. The statistical properties of the correlator (1) are defined by fluctuations of the wave field modulus $|\psi|$ as well as by the phase fluctuations θ. If the random field $\psi(\mathbf{r})$ had constant absolute value $|\psi(\mathbf{r})|=\text{const}$, the correlator (1) would be given purely by the phase correlator

$$Z(s)=\langle\exp[i\theta(\mathbf{x}+\mathbf{s})-i\theta(\mathbf{x})]\rangle. \qquad (2)$$

In the present letter we find analytically the phase correlation function (2) and compare it to the wave function one (1) for the case of the random Gaussian field. Surprisingly, the space behavior of the wave function correlator (1) is mainly given by the phase one (2) as it will be shown below.

The correlator (2) is free of 2π phase discontinuities contrary to the spatial phase correlator $\Xi(\mathbf{s})=\langle\theta(\mathbf{x}+\mathbf{s})\theta(\mathbf{x})\rangle$. Such a kind of the phase correlator was firstly calculated by Middleton [11]. However, as it was shown by Freund and Kessler [12] (see also [13]) the result by Middleton disagrees with numerics because it apparently fails to correctly handle the complex topology of two-dimensional random phase fields. This complex topology arises from the presence of vortices (phase singularities) (see, for example patterns of the phase fields

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$$Z(s)=\frac{1-C(s)^2}{2\pi}\int_0^\pi \mathrm{d}\varphi\mathrm{d}\varphi'\frac{C(s)\cos(\varphi-\varphi')(\sin\varphi\sin\varphi'+\epsilon^2\cos\varphi\cos\varphi')}{\sqrt{1+(\epsilon^2-1)\cos\varphi^2}\sqrt{1+(\epsilon^2-1)\cos\varphi'^2}\left(1-C(s)^2\cos(\varphi-\varphi')^2\right)^{\frac{3}{2}}}. \quad (8)$$

in [13–15]). The ways to cope with this problem were considered in [12,16].

Phase correlator. – Representing the complex Gaussian random field (RGF) ψ as sum of two independent RGFs $\psi=u+iv$, one can rewrite (2) as

$$Z(s)=\left\langle\frac{uv_s+u_su+i(uv_s-vu_s)}{\sqrt{u_s^2+v_s^2}\sqrt{u^2+v^2}}\right\rangle. \quad (3)$$

For brevity the dependence on s is shown by the index.

We consider u and v as two independent RGFs $\langle uv\rangle=0$. If $\langle uv\rangle\neq 0$, the phase transformation given in [17] could make u and v statistically independent. Open chaotic billiards are characterized by the phase rigidity of the scattering wave function ψ_S inside the billiard $\rho=\langle\psi_S^2\rangle/\langle|\psi_S|^2\rangle$ [18] which is related to the ratio $\epsilon^2=\langle v^2\rangle/\langle u^2\rangle$ by the equation $\rho=(1-\epsilon^2)/(1+\epsilon^2)$. The factor ϵ was introduced by Pnini and Shapiro [19] to present the scattering wave function in the Berry-like form

$$\psi=\sum_j\cos(\mathbf{k}_j\mathbf{r}+\theta_j)+i\epsilon\sum_j\sin(\mathbf{k}_j\mathbf{r}+\phi_j). \quad (4)$$

In order to calculate the phase correlator (3), we need a couple of the joint distributions, respectively [2],

$$\begin{aligned}w(u,u_s) &= \frac{1+\epsilon^2}{2\pi\sqrt{1-C(s)^2}}\\ &\quad\times\exp\left\{-\frac{(u^2+u_s^2-2C(s)uu_s)(1+\epsilon^2)}{2(1-C(s)^2)}\right\},\\ w(v,v_s) &= \frac{1+\epsilon^2}{2\pi\epsilon^2\sqrt{1-C(s)^2}}\\ &\quad\times\exp\left\{-\frac{(v^2+v_s^2-2C(s)vv_s)(1+\epsilon^2)}{2\epsilon^2(1-C(s)^2)}\right\},\end{aligned} \quad (5)$$

which are written under condition that $\langle|\psi|^2\rangle=1$. Then the phase correlator (3) can be calculated by integration:

$$Z(s)=\int_{-\infty}^{+\infty}\mathrm{d}u\mathrm{d}u_s\mathrm{d}v\mathrm{d}v_s\frac{uu_s+vv_s}{\sqrt{u^2+v^2}\sqrt{u_s^2+v_s^2}}\times w(u,u_s)w(v,v_s). \quad (6)$$

Because of the symmetry of the joint distributions (5), the imaginary part in (3) does not contribute to the phase correlator, *i.e.*

$$Z(s)=\langle\cos[\theta(\mathbf{x}+\mathbf{s})-\theta(\mathbf{x})]\rangle. \quad (7)$$

Substituting $u=r\sin\varphi$, $v=r\epsilon^2\cos\varphi$, $u_s=r'\sin\varphi'$, $v_s=r'\epsilon^2\cos\varphi'$ into (6), one can obtain

see eq. (8) *above*

Analytical expressions for the phase correlators can be obtained for the cases $\epsilon=0,1$. For $\epsilon=1$ the expression (8) reduces to the following form:

$$Z_1(s)=\frac{1}{C(s)}[E(C^2(s))-(1-C(s)^2)K(C^2(s))], \quad (9)$$

where $K(x),E(x)$ are the elliptic integrals of the first and the second order, respectively. The second case $\epsilon=0$ gives

$$Z_0(s)=\frac{2}{\pi}\arctan\left(\frac{C(s)}{\sqrt{1-C(s)^2}}\right)=\frac{2}{\pi}\arcsin C(s), \quad (10)$$

where the amplitude correlator $C(s)$ is given by (1). As seen from (4) RGF $\psi(\mathbf{x})$ is real for the case $\epsilon=0$. Then the phase of RGF takes only values 0 or π, and correspondingly the value $\exp[i\theta(\mathbf{x})]$ is a random binary process. Therefore for $\epsilon=0$ the correlation function (2) is that for clipped noise which was calculated by Van Hove and Middleton [20] just in the form of (10). Measurements of such correlations called the one-bit correlations are reported, for example, in [21].

Plots of the phase correlators (9) and (10) for the two-dimensional case are presented in fig. 1. The figure shows that the case $\epsilon=0$ is close to the case $\epsilon=1$. The numerical computations of (8) for arbitrary ϵ shows that the phase correlator varies between (9) and (10), being extremely insensitive to ϵ. Moreover, the phase correlation functions are compared to the amplitude one (1) in fig. 1. The comparison shows that the phase correlator $\langle\exp[i\theta(\mathbf{x}+\mathbf{s})-i\theta(\mathbf{x})]\rangle$ mainly contributes to the amplitude one $\langle\psi(x+r)\psi^*(\mathbf{x})\rangle$. Especially, for $\epsilon=1$ we can write the approximate equality

$$\langle\psi(x+r)\psi^*(\mathbf{x})\rangle\approx\langle\exp[i\theta(\mathbf{x}+\mathbf{s})-i\theta(\mathbf{x})]\rangle, \quad (11)$$

to stress that for the closed chaotic billiards fluctuations of the absolute value of the RGF $|\psi|$ give a negligible contribution into the wave field correlator.

In fig. 2 we compare the phase correlators (9) and (10) with the numerically computed function obtained for the Sinai billiard opened by attachment of two leads. For each energy of the incident quantum particle we computed the scattering wave function inside the billiard ψ_S and consequently the phase rigidity $\rho=\langle\psi_S^2\rangle/\langle|\psi_S|^2\rangle$ and parameter $\epsilon^2=\frac{1-\rho}{1+\rho}$. Here $\langle\dots\rangle$ means integration over the

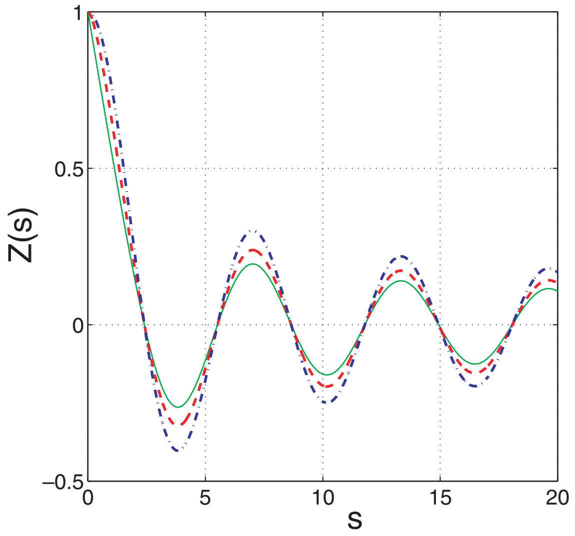

Fig. 1: (Color online) Plots of the phase correlators given by (9) for $\epsilon = 1$ (dashed red line) and by (10) for $\epsilon = 0$ (green solid line). They are compared to the amplitude correlator $C(s) = J_0(s)$ shown by the blue dash-dotted line.

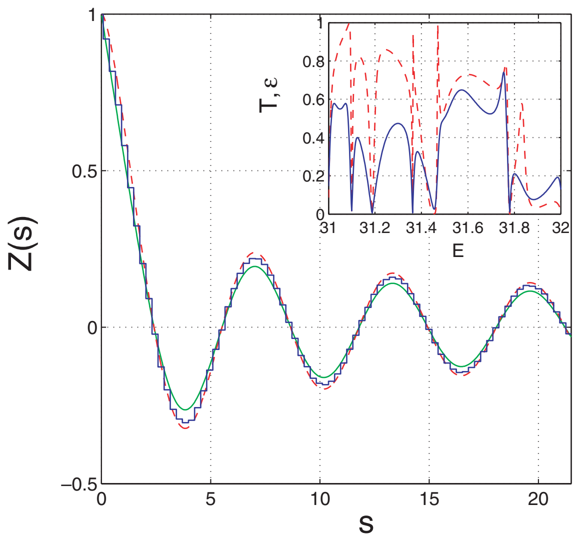

Fig. 2: (Color online) The numerically computed phase correlation function for the open Sinai billiard with length 16, width 8 and radius 4 in terms of the unit width of the leads. Inset shows fluctuations of the parameter ϵ in the energy window which corresponds to the first channel transmission with $\langle\epsilon\rangle = 0.35$ complemented by the energy dependence of the conductance T. The numerical histogram is compared to the analytical phase correlators (9) (red dashed line) and (10) (green solid line).

area of the billiard [18]. As the inset in fig. 2 shows, the parameter ϵ strongly fluctuates with the energy with mean value $\langle\epsilon\rangle = 0.35$. Hence we averaged the phase correlator over the energy window shown in the inset of fig. 2. As a result the numerically computed phase correlator ranges between the analytical results (9) and (10). The next interesting result is that the phase correlator Z is much less sensitive to the fluctuations of the phase rigidity compared to the intensity correlator [22].

* * *

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