Interplay between Anderson and Stark Localization in 2D Lattices

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This Letter studies the dynamics of a quantum particle in 2D lattices with on-site disorder in the presence of a static field. It is shown that the particle is localized along the field direction, while in the orthogonal direction to the field it shows diffusive dynamics for algebraically large times. For weak disorder an analytical expression for the diffusion coefficient is obtained by mapping the problem to a band random matrix. This expression is confirmed by numerical simulations of the particle's dynamics, which also indicate the existence of a universal equation for the diffusion coefficient, valid for an arbitrary disorder strength.

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1.—Nowadays one observes a resurgence of interest in the phenomenon of Anderson localization in disordered media, boosted by recent experiments with ultracold atoms in optical traps (see Refs. [1,2] and references therein). Because of the high degree of control over the parameters and a possibility for direct measurement of atomic density profiles, these systems have provided a new playground for studying Anderson localization and related phenomena. In the present Letter I discuss the localization properties of a quantum particle in disordered lattices in the presence of a static field. Besides the problem's intrinsic interest, its study is also relevant to that of conductivity with cold atoms [3–5]. Indeed, in solid-state physics the theory of Anderson localization is used to predict the conductivity of a "dirty" crystal in the limit of weak electric fields, where the Stark localization length, defined later in the text, is larger than the system size. While this is always the case for electrons in a crystal, the Stark localization length for cold atoms subject to typical laboratory fields (for example, the gravitational field) is smaller than the system size and, in principle, can be as small as one lattice site. Thus, when addressing atomic conductivity in disordered lattices, a static field should be included into the analysis from the very beginning.

For 1D lattices the question of the external fields effect on Anderson localization was addressed earlier in Ref. [6], where a special emphasis was given to the ac field. In this Letter I report findings on the interplay between Anderson and Stark localization in 2D disordered lattices, where the effect of the dc field leads to new phenomena. In particular, I show that the static field induces diffusive dynamics of a quantum particle (an atom) in the direction orthogonal to the field vector.

2.—I begin with the one-dimensional case of Stark and Anderson localization. In the tight-binding approximation the single-particle Hamiltonian of an atom in a 1D optical lattice reads

$$\hat{H} = -\frac{J}{2}\sum_n (|n+1\rangle\langle n| + \text{H.c.}) + g\sum_n \xi_n |n\rangle\langle n| + Fd\sum_n n|n\rangle\langle n|, \tag{1}$$

where J is the hopping matrix element, F the magnitude of the static field, d the lattice period, and $g\xi_n$ with $|\xi_n| \leq 1/2$ random on-site energies. It is well known that eigenfunctions of the Hamiltonian (1) are localized for $g \neq 0$ and $F = 0$ (Anderson localization), as well as for $F \neq 0$ and $g = 0$ (Stark localization).

First I discuss Stark localization. For $g = 0$ the Hamiltonian (1) can be easily diagonalized, resulting in the equidistant spectrum, $E_{m+1} - E_m = Fd$, and the wave functions, $|\psi_m\rangle = \sum_n \mathcal{J}_{n-m}(J/Fd)|n\rangle$ [here $\mathcal{J}_m(z)$ are Bessel functions of the first kind], which are localized within the Stark localization length,

$$l_{\rm St} \approx \begin{cases} 2J/Fd, & Fd \lesssim J \\ 1, & Fd \gtrsim J \end{cases}. \tag{2}$$

The equidistant spectrum of the system has a direct consequence on its dynamics. Namely, if one considers a localized wave packet, its time evolution will be the celebrated Bloch oscillations with the period $T_B = 2\pi\hbar/Fd$ and amplitude given by the Stark localization length (2). In this Letter I shall restrict myself to moderate static fields, where $l_{\rm St}$ varies from ten to hundred lattice sites [7].

Next I address Anderson localization. In the case of weak disorder and finite lattice size one typically meets localized states, associated with the bottom and top of the Bloch band (below and above the so-called mobility edges), and extended states, associated with central part of the band. With an increase in the lattice size the extended states also become localized, however, with essentially larger localization lengths. In what follows I concern myself mainly with the case of weak disorder, where the mean Anderson localization length $l_{\rm An}$ is larger than the Stark localization length $l_{\rm St}$.

Finally I analyze the situation where $g \neq 0$ and $F \neq 0$. Since I am interested in the case $l_{\rm An} > l_{\rm St}$, it is reasonable to consider the second term in the Hamiltonian (1) as a perturbation. It is easy to show numerically that this perturbation destroys Bloch oscillations and after a characteristic time, $\sim\hbar/g$, the initial wave packet becomes more or less uniformly distributed over the finite region $0 \lesssim n \lesssim l_{\rm St}$. (Initially the wave packet is assumed to be centered at $n = 0$.) This result can be understood analytically by considering the Hamiltonian (1) in the basis of the Wannier-Stark states $|\psi_n\rangle$. One has

$$\widehat{H}_{\rm eff} = g\sum_{m\geq 0}\sum_n I_n^{(m)}(|\psi_{n+m}\rangle\langle\psi_n| + {\rm H.c.}) + Fd\sum_n n|\psi_n\rangle\langle\psi_n|, \qquad (3)$$

where $I_n^{(m)} = \sum_l \xi_l \mathcal{J}_{l-(n+m)}(J/Fd)\mathcal{J}_{l-n}(J/Fd)$ are the new hopping matrix elements. Since these elements are essentially given by the weighted sum of random numbers ξ_n, they are distributed according to the normal law with zero average and a variance $\sim 1/\sqrt{l_{\rm St}} \sim \sqrt{F}$. Note that due to the exponentially small overlap between mutually remote Wannier-Stark states, the sum over m in (3) can be restricted to $m \lesssim l_{\rm St}$. Thus the Hamiltonian (3) is a band random matrix from the mathematical point of view [8]. Comparing the characteristic value of the off-diagonal elements, given by $g\sqrt{Fd/J}$, with the characteristic distance between unperturbed levels, given by Fd, one concludes that the considered case of weak disorder ($g/J < 1$) corresponds to a perturbative regime, where a typical eigenfunction of the Hamiltonian (3) is a random superposition of a few neighboring Wannier-Stark states. Since $l_{\rm St} \gg 1$, this only slightly corrects the final localization length. Therefore, the maximal distance a wave packet may travel is still defined by $l_{\rm St}$.

3.—I now turn to 2D lattices. A straightforward generalization of the 1D model (1) yields,

$$\widehat{H} = -\frac{J}{2}\sum_{\mathbf{n},\pm}(|\mathbf{n}\rangle\langle\mathbf{n}'| + {\rm H.c.}) + g\sum_{\mathbf{n}}\xi_{\mathbf{n}}|\mathbf{n}\rangle\langle\mathbf{n}| + d\sum_{\mathbf{n}}(\mathbf{F}\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|, \qquad (4)$$

where $\mathbf{n}$ amounts to the pair of indexes, $\mathbf{n} = (n, m)$, and $\mathbf{n}'$ is either $\mathbf{n}' = (n \pm 1, m)$, hopping along the x axis, or $\mathbf{n}' = (n, m \pm 1)$, hopping along the y axis. I shall discuss localization phenomena in 2D lattices in the same order as for 1D lattices.

Stark localization.—Without disorder the dynamics of a quantum particle in a 2D lattice induced by a static field is a superposition of Bloch oscillations along x and y directions, with the frequencies and amplitudes defined by projections of the static field vector on the lattice crystallographic axes. Thus the particle is bounded to a finite region in the configuration space, restricted by two Stark localization lengths $l_x \approx 2J/dF_x$ and $l_y \approx 2J/dF_y$ [9]. For future purposes it is convenient to introduce the total Stark localization length $l_{\rm St} = (l_x^2 + l_y^2)^{1/2}$.

Anderson localization.—Unlike the one-dimensional case, eigenfunctions of a quantum particle in a weakly disordered 2D lattice are known to be only marginally localized, which leads to an almost linear growth of the mean squared displacement, $M(t) = \sum_{n,m}(n^2 + m^2)P(n, m; t)$, for very long time (see lower panel in Fig. 1). To some extent this mimics diffusion of a classical particle in disordered media. However, an analysis of the wave-packet profile, $P(n, m; t) = |c_{\mathbf{n}}(t)|^2$, indicates that its spread is not diffusive. The upper panels in Fig. 1 show the integrated distributions $P_x(n) = \sum_m P(n, m)$ and $P_y(n) = \sum_m P(m, n)$ at the end of numerical simulation of the system dynamics for $g/J = 0.8$. A conelike profile, which is a characteristic signature of Anderson localization, is clearly seen in the figure. Let me also mention that for $F = 0$ the wave-packet spread is statistically isotropic, that is to say, that after averaging over different samples the integrated distributions for any direction are the same.

General case.—The results of numerical simulations for nonzero field and disorder are shown in Fig. 2. It is seen in Fig. 2 that, similar to 1D lattices, the wave-packet spread along the field direction is restricted by the Stark localization length $l_{\rm St}$. However, in the orthogonal direction the packet spreads diffusively, as is confirmed by its Gaussian profile together with linear growth of $M(t)$. The rest of the Letter is devoted to an explanation of this effect.

It is useful to rewrite the Hamiltonian (4) in the basis of two-dimensional Wannier-Stark states $|\psi_{\mathbf{n}}\rangle = |\psi_n^{(x)}\rangle|\psi_m^{(y)}\rangle$. One has

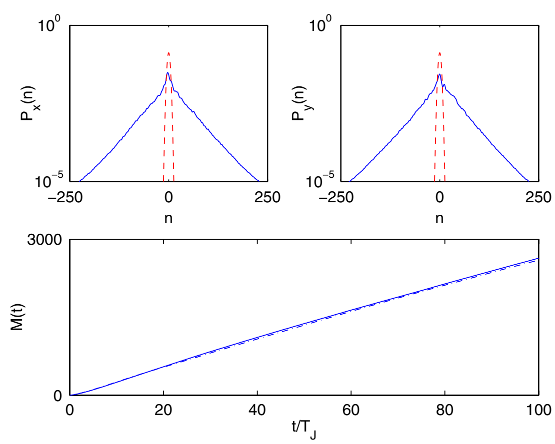

FIG. 1 (color online). Wave-packet spread in a 2D disordered lattice with $g = 0.8J$. Bottom: Mean squared displacement of the initially Gaussian packet along x (solid line) and y (dash-dotted line) directions. Time is measured in units of the tunneling period $T_J = 2\pi\hbar/J$. Upper row: Integrated distributions $P_x(n) = \sum_m P(n, m)$ and $P_y(n) = \sum_m P(m, n)$ at the end of numerical simulation, where the dashed lines indicate initial distributions. Averaged over 10 different samples.

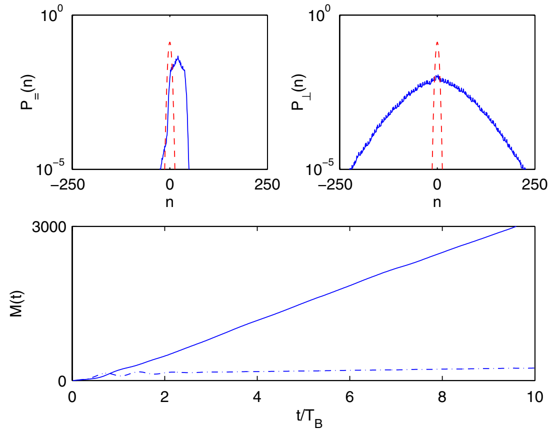

FIG. 2 (color online). Wave-packet spread in a 2D disordered lattice with $g = 0.4J$ in the presence of a static field $Fd = 0.1J$, $F_x/F_y = \sqrt{2/3}$. Bottom: Mean squared displacement of the initially Gaussian packet along (dash-dotted line) and across (solid line) the field. Time is measured in units of the Bloch period $T_B = 2\pi\hbar/Fd$. Upper row: Integrated distributions at the end of numerical simulation. Averaged over 10 different samples.

$$\hat{H} = d\sum_{\mathbf{n}}(\mathbf{F}\mathbf{n})|\psi_{\mathbf{n}}\rangle\langle\psi_{\mathbf{n}}| + g\sum_{\mathbf{n},\mathbf{n}'}V_{\mathbf{n},\mathbf{n}'}|\psi_{\mathbf{n}}\rangle\langle\psi_{\mathbf{n}'}|, \qquad (5)$$

where $V_{\mathbf{n},\mathbf{n}'} = \sum_{\mathbf{m}}\xi_{\mathbf{m}}\langle\psi_{\mathbf{n}}|\mathbf{m}\rangle\langle\mathbf{m}|\psi_{\mathbf{n}'}\rangle$. Let us assume for simplicity $F_x = F_y$. Then we notice that there is no energy mismatch between Wannier-Stark states in the direction which is orthogonal to the field vector. Thus an arbitrary weak random potential will couple these Wannier-Stark states, possibly forming an extended state. In the limit of small g an effective 1D Hamiltonian describing this coupling has the structure of the Hamiltonian (3), where index n now labels the 2D Wannier-Stark states with indexes $\mathbf{n} = (n_0 + n, n_0 - n)$ and where one should set $F = 0$ by construction. Repeating the arguments of the previous paragraph one has the hopping matrix elements $I_n^{(m)}$ to be distributed according to the normal law,

$$G(I^{(m)}) \equiv G_m(I) \sim \exp(-I^2/2\sigma_m^2), \qquad m \lesssim l_{\mathrm{St}},$$

with the variance $\sigma_m \sim \sqrt{F_xF_y} \sim F$.

One can make certain predictions about dynamics of the effective system (3), with the vanishing last term, by appealing to the theory of band random matrices (BRM). The properties of the BRM have attracted much attention during the past two decades with respect to the problems of electron propagation along a thin wire and dynamical chaos (see Refs. [10–12] and references therein). With relevance to the problem the results of BRM theory are summarized below. (i) The density of states $\rho(E)$ for an infinite BRM obeys the semicircle law $\rho(E) \sim \sqrt{1-(E/R)^2}$, where $R = \sqrt{8bv^2}$ with b being the bandwidth and v^2 the standard deviation of the distribution of off-diagonal elements, $v^2 = \langle H_{n,n'}^2\rangle$. Note that the parameter v^2 affects neither the statistical properties of the spectra nor the structure of the eigenstates, since it can be scaled out. (ii) In infinite BRM, all eigenstates are (asymptotically) exponentially localized with the localization length $l_\infty = \rho^2(E)b^2$. (iii) An initially localized wave packet shows a diffusive spread with the diffusion coefficient $D \sim l_\infty R$, for algebraically large time $t_D \sim l_\infty/R$. (iv) For $t \gg t_D$ diffusion saturates at $M(t=\infty) \sim b^4$. Adopting the above results to the current problem ($b \equiv l_{\mathrm{St}}$, $v^2 \equiv \langle\sigma_m^2\rangle$), one has

$$M(t) \approx Dt, \qquad t < t_D \sim \frac{\hbar}{g}\left(\frac{2J}{Fd}\right)^{5/2}, \qquad (6)$$

where

$$D \sim \frac{g}{\hbar}\left(\frac{2J}{Fd}\right)^{3/2}. \qquad (7)$$

It is worth noting with respect to Eq. (6), that the presented results in Fig. 2 correspond to $t \ll t_D$. To see a sign of localization, which manifests itself as a deviation from the linear dependence of $M(t)$, one has to simulate the system dynamics at least until $t = 40T_B = 400T_J$.

To check the predicted functional dependence (7) for the diffusion coefficient across the field, I simulate the wave-packet dynamics in 2D disordered lattices for different g, F, and field direction. As an example, Fig. 3 shows $M(t)$ at $t = 100T_J$ for $g/J = 0.4$, $F_x/F_y = 1$, and $Fd/J = 0.2$ (dashed line), 0.1 (solid line), and 0.05 (dash-dotted line). The figure inset shows the same curves yet scaled by factors $2^{3/2}$, 1, and $2^{-3/2}$, respectively. A nice coincidence of the scaled curves at small g confirms Eq. (7) for the diffusion coefficient [13]. It is also seen in the figure that a linear dependence of D on g holds only for g smaller than some critical g_{cr} which, in turn, is a function of F. The existence of g_{cr} is due to the "interline coupling," which the BRM model is unable to capture. Indeed, to reduce the

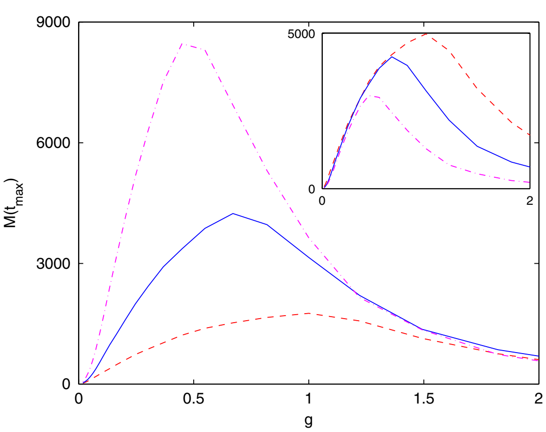

FIG. 3 (color online). Mean squared displacement at $t_{\max} = 100T_J$ for $g/J = 0.4$, $F_x/F_y = 1$, and $Fd/J = 0.2$ (dashed line), 0.1 (solid line), and 0.05 (dash-dotted line). The inset shows the same curves yet scaled by factors $2^{3/2}$, 1, and $2^{-3/2}$, respectively.

original 2D problem to an effective 1D problem I have neglected the coupling between Wannier-Stark states belonging to different lines across the field [i.e., different n_0 in the substitution $\mathbf{n} = (n_0 + n, n_0 - n)$, $F_x/F_y = 1$]. This approximation is justified for a small g, and this is the parameter region that Eq. (7) refers to. As g is increased, I eventually violate the above condition and, hence, should observe a deviation from (7). One might naively expect the interline coupling to enhance diffusion. However, this thought contradicts with the ''common sense'' conclusion that there can be no diffusion in the limit $g \to \infty$. Indeed, I found that the diffusion coefficient always goes to zero for a large g, faster than $D \sim 1/g^2$. It is an open problem to find an analytical expression for $D = D(g, F)$ valid in the whole parameter space.

4.—In summary, I studied dynamics of a quantum particle in 2D disordered lattices, subject to a static field. The static field is shown to localize the particle *along* the field within the Stark localization length. At the same time it induces the diffusivelike dynamics of the particle *across* the field. Numerical simulations of the system dynamics indicate that there is a universal dependence for the diffusion coefficient on the field magnitude and disorder strength. Moreover, for weak disorder this dependence was found analytically by mapping the problem to a band random matrix. This mapping also shows that the discussed diffusion is a temporal phenomenon which, however, persists for algebraically large times [14].

I left aside a number of important questions, like the dependence of the diffusion coefficient on the field direction and the effect of residual atomic interactions [15]. These results will be published elsewhere. Nevertheless, concluding the discussion I would like to stop briefly on the effect of a noise, which is inevitably present in any laboratory experiment and, as a very rough approximation, substitutes for atomic interactions. It is well known that noise destroys all localization phenomena and, hence, the particle will diffuse both across and along the field. I have found, however, that the resulting diffusion coefficients across and along the field may differ by an order of magnitude. This asymmetric diffusion would be a clear indication of the discussed effect in a laboratory experiment with cold atoms in optical lattices.

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[13] A deviation from the linear dependence of the diffusion coefficient D on g at very small g is due to the fact that linear growth of $M(t)$ in BRM is preceded by the short-time ballistic regime, where $M(t) \sim t^2$ [12].

[14] It is interesting to estimate the diffusion time t_D in a laboratory experiment with cold atoms. As an example, I consider rubidium atoms in a 3D optical lattice superimposed with speckle pattern, where the lattice z axis is misaligned by some angle α to gravitational field. Taking the lattice period $d = \pi/k_L = 426 \times 10^{-9}$ m and the lattice depth $V_x = V_y = 2E_R$, $E_R = \hbar^2 k_L^2/2M$, one has $J = 0.3E_R$. Then $l_{\mathrm{St}} = 20$ and $g = 0.4J$, used in my numerical simulations, correspond to declination angle $\sin\alpha \approx 0.1$ and lattice imperfection $g = 0.12E_R$, respectively. Substituting these numbers into Eq. (6) I obtain $t_D \approx 0.9s$, which should be compared with the Bloch period $T_B = 2\pi\hbar/dF \sin\alpha = 0.011s$ and the tunneling period $T_J = 2\pi\hbar/J = 0.001s$.

[15] There is a bunch of papers devoted to the effect of particle-particle interactions on Anderson localization. [To cite a few of them, the recent papers G. Kopidakis, S. Komineas, S. Flach, and S. Aubry, Phys. Rev. Lett. **100**, 084103 (2008); A. S. Pikovsky and D. L. Shepelyansky, Phys. Rev. Lett. **100**, 094101 (2008) discuss the case of Bose atoms, and the papers P. Schmitteckert, T. Schulze, C. Schuster, P. Schwab, and U. Eckern, Phys. Rev. Lett. **80**, 560 (1998); V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007) are devoted to fermions.] However, a clear understanding of the interplay between interaction and disorder has not yet been obtained, even in the simplest case of unbiased 1D lattices.