THE CRITICAL EXPONENT OF THE TREE LATTICE GENERATING FUNCTION IN THE EDEN MODEL

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We consider the increase in the number of trees as their size increases in the Eden growth model on simple and face-centered hypercubic lattices in different space dimensions. We propose a first-order partial differential equation for the tree generating function, which allows relating the exponent at the critical point of this function to the perimeter of the most probable tree. We estimate tree perimeters for the lattices considered. The theoretical values of the exponents agree well with the values previously obtained by computer modeling. We thus explain the closeness of the dimension dependences of the exponents of the simple and face-centered lattices and their difference from the results in the Bethe lattice approximation.

1. Introduction

The cluster growth in the Eden model results from randomly adding particles (sites, bonds) on the cluster boundary with equal probability [1]–[6]. This process is especially simple on the Bethe lattice [2], [3], where it is assumed that each new bond is added in a new dimension, and the lattice therefore has an infinite dimension. The obtained cluster is a tree because the bonds do not intersect. We have

$$T_1 = Z, \qquad T_2 = 2Z(Z-1), \qquad T_n = \prod_{m=1}^{n} [m(Z-2)+2] = (Z-2)^n \frac{\Gamma(n+p+1)}{\Gamma(p+1)}, \qquad (1.1)$$

for the number of trees depending not only on the shape but also on the order of adding bonds and

$$E_b(x) = \sum_{n=0}^{\infty} \frac{T_n x^n}{n!} = \left(1 - \frac{x}{x_b}\right)^{-1-p} \qquad (1.2)$$

for the generating function, where the critical point coordinate $x_b = 1/Z_b$ is the reciprocal of the growth parameter $Z_b = Z - 2$, $p = p_b = 2/(Z-2)$ is the exponent of the critical point of the generating function, Z is the coordination number, and $\Gamma(x)$ is the gamma function. In the limit as $n \to \infty$, formula (1.1) transforms into the simple asymptotic expression

$$T_n \sim A n! \, (Z_c)^n n^p, \qquad (1.3)$$

where $p = p_b$, $Z_c = Z_b$, and $A = A_b = 1/\Gamma(p_b + 1)$ in the Bethe approximation.

We encounter difficulties in constructing a tree on a hypercubic lattice of dimension d following the same rules if we prohibit intersections of branches (repeated site occupations). The branch interaction caused

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by this restriction, called the excluded volume interaction [7], does not allow finding a rigorous analytic solution of the problem, and approximate methods [1], [3] or computer modeling [4], [6] are therefore used. The excluded volume interaction affects tree properties and makes them dependent on variants of the tree construction. The mean squared radius and the fractal dimension in the growth models are the commonly sought properties of a large cluster [4], [6]. The averaging is customarily taken over a relatively small number of clusters. It is assumed that a dense cluster grows in the Eden model [5], [6].

It is interesting to investigate properties of the whole ensemble of trees, which are encoded in their generating function introduced above in the example of Bethe trees. Such an analysis is necessary, for example, when studying analytic properties of time spin correlation functions at high temperatures [8], [9]. Tree properties obtained by averaging over the whole ensemble differ from those obtained by averaging over a small sample because all possibilities are realized when constructing the whole ensemble of trees and all the perimeter bonds are taken into account one by one. Trees with a greater number of descendants, i.e., trees with larger perimeters therefore contribute more to the ensemble. In contrast, when constructing a sample comprising a small number of trees, trees with smaller perimeters enter the sum with larger weight factors because the perimeter area appears in the denominator when we determine a lattice site occupation probability [5].

The abovementioned properties of the tree ensemble must be reflected in the analytic properties of the generating function. Above, we advanced the hypothesis that this function has a critical point at a finite value x_c of the variable x. Because we cannot prove this statement by an exact calculation, we must use other tools: on one hand, we perform a numerical experiment; on the other hand, we calculate the critical point characteristics by approximate methods. The coincidence of the results of the two independent calculations would support our hypothesis. In pursuing this program, we used Monte Carlo simulations to estimate how the total number of trees in the ensemble increases as the number n of bonds in the tree increases for simple (SC) [10] and face-centered (FCC) [11] hypercubic lattices of various dimensions d ($d = 2, 3, 4, 6, 8, 10$). Using the $1/d$-expansion, we then derived an asymptotic formula for the tree growth parameter, which is determined as the quantity reciprocal to the coordinate of the generating function critical point [8], [11]. We observed a good agreement between the theoretical predictions and the results of the numerical experiment. We also extracted the generating function critical exponent from the numerical data. The critical exponents for the SC and FCC lattices turn out to be close to each other and differ drastically from those for Bethe lattice (1.2).

In this paper, we obtain a theoretical expression for the critical exponent and explain its dependence on the space dimension. We derive a differential equation for the tree ensemble generating function in the next section. We describe the tree ensemble growth using the most probable tree, which is characterized by two parameters: the growth (external) perimeter and the inner (dead) perimeter. In Sec. 3, we calculate these parameters for the SC and FCC lattices in various dimensions. In Sec. 4, we compare the obtained critical exponents with those from the simulations. We analyze the tree characteristics of the Bethe lattice in the appendix.

2. The equation for the generating function

When a tree is constructed on the Bethe lattice of an infinite space dimension, each new vertex can be developed into a branch of arbitrary length. When trees are constructed on a finite-dimensional hypercubic lattice under the nonintersecting-branch condition, some of the branches cannot be continued (dead ends), and only the other branches constituting the skeleton can grow further. Here we use the terminology proposed for describing the structure of an infinite cluster in percolation theory [12]. Two variants are realized in passing from the ensemble N_n of trees with n bonds to an ensemble of N_{n+1} trees with $n+1$ bonds:

1. The bond is added at a dead end, and the perimeter does not increase.

2. The bond is added to a skeleton and the perimeter increases (the added bond is then actually inside the branch, not at its end, because adding a bond at a branch end results in the shift of the surface when constructing a large tree, with the result that the inner tree part increases).

We therefore consider the perimeter S of a tree on a hypercubic lattice to be composed of three parts:

$$\nu + m\nu_1 + m\nu_2, \tag{2.1}$$

where ν is the initial perimeter ($\nu = Z$), ν_1 is the part of the perimeter that results in the skeleton increasing (the external or growth perimeter), and ν_2 is the part of the perimeter from which dead ends grow (the internal or dead perimeter). For the Bethe lattice, $\nu_2 = 0$ and $\nu_1 = Z - 2$.

Partition (2.1) of the perimeter S into the three parts is a rough simplification that we use to describe the pattern qualitatively. Each separate tree can have its own perimeter. When calculating the total number of trees in the ensemble exactly, we must sum over all the perimeter values,

$$N_{n+1} = \sum_S S N_n(S) \cong \sum_m (\nu + m\nu_1 + m\nu_2) N_n(\nu + m\nu_1 + m\nu_2), \tag{2.2}$$

where $N_n(S)$ is the number of trees with the perimeter S that are composed of n bounds. When an ensemble of large trees is analyzed, these values tend to some means. If the distribution of trees with respect to the perimeter is narrow, then the growth comes from trees with the most probable mean parameters,

$$N_{n+1} \approx (\bar{m}\bar{\nu}_1 + \bar{m}\bar{\nu}_2) \sum_m N_n(\nu + m\nu_1 + m\nu_2) = \bar{m}(\bar{\nu}_1 + \bar{\nu}_2) N_n. \tag{2.3}$$

On the other hand,

$$N_{n+1} = Z_c n N_n.$$

Hence,

$$Z_c = \frac{\bar{m}(\bar{\nu}_1 + \bar{\nu}_2)}{n} \tag{2.4}$$

(we omit the bar over the quantities in what follows). We note that such a transition was used, for example, in theory of disordered systems when determining the electron spectrum level density by the optimal fluctuation method (seeking the most probable configuration of impurities) [12].

To better understand the proposed description of the tree ensemble growth, we turn to the clear picture of a continuous model. We consider two occupied sites to be connected if their mutual distance is less than r. Let a cluster composed of n occupied connected sites be characterized by the free volume of $\nu + m\nu_1 + m\nu_2$ accessible sites. When a new site is added, it can fall either into the internal volume or on the surface, and the number of trees in the ensemble increases $\nu + m\nu_1 + m\nu_2$ times. The number of fallings on the surface is $\nu + m\nu_1$. In this case, both the internal volume and the surface of the tree increase, and the latter becomes $\nu + (m+1)(\nu_1 + \nu_2)$. The number of fallings into the internal volume is $m\nu_2$; both the internal volume and the tree surface then remain unchanged.

We have recurrence relations for the number of trees with a given perimeter value $N_n(m) \equiv N_n(\nu + m\nu_1 + m\nu_2)$ growing in accordance with the above rules:

$$N_0(0) = 1, \qquad N_1(1) = \nu N_0(0) = \nu,$$

$$N_2(1) = \nu_2 N_1(1) = \nu_2 \nu,$$

$$N_2(2) = (\nu + \nu_1) N_1(1) = (\nu + \nu_1)\nu,$$

or, in general form,

$$N_n(m) = [\nu + (m-1)\nu_1]N_{n-1}(m-1) + m\nu_2 N_{n-1}(m).$$ (2.5)

We introduce the exponential tree generating function,

$$F(x,y) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{x^n y^m N_n(m)}{n!}.$$ (2.6)

Multiplying both sides of Eq. (2.5) by $x^n y^m / n!$ and summing, we obtain the equation for the generating function:

$$F(x,y) = 1 + \int_0^x \left[\nu y F(x_1, y) + \nu_1 y^2 \frac{\partial}{\partial y} F(x_1, y) + \nu_2 y \frac{\partial}{\partial y} F(x_1, y) \right] dx_1.$$ (2.7)

Differentiating both parts of Eq. (2.7) with respect to x, we obtain the inhomogeneous linear first-order partial differential equation

$$\frac{\partial F}{\partial x} = (\nu_1 y^2 + \nu_2 y) \frac{\partial F}{\partial y} + \nu y F.$$ (2.8)

We solve Eq. (2.8) using the method of characteristics [13],

$$F(x,y) = \left[1 - y \frac{\nu_1}{\nu_2} (e^{\nu_2 x} - 1) \right]^{-\nu/\nu_1}.$$ (2.9)

According to formula (2.6), the parameter x determines the number of trees, and the parameter y determines their distribution with respect to the perimeter. Generating function (2.9) treated as a function of x develops a singular point at $x = x_c$:

$$x_c = \frac{1}{\nu_2} \log\left(1 + \frac{\nu_2}{y\nu_1}\right).$$ (2.10)

The critical exponent is

$$1 + p = \frac{\nu}{\nu_1}.$$ (2.11)

Based on formulas (2.6) and (2.9), we can approximately set

$$N_n = \sum_{m=1}^{n} y^m N_n(m) \approx x_c^{-n}$$ (2.12)

for large trees. It hence follows that $Z_c = 1/x_c$. We obtain the mean perimeter from (2.12),

$$\bar{m} = \lim_{y=1} \sum_{m=1}^{n} m y^m \frac{N_n(m)}{N_n} = \lim_{y=1} \frac{d}{dy} \log N_n \approx -n \lim_{y=1} \frac{d}{dy} \log x_c = \frac{n}{(1 + \nu_1/\nu_2) \log(1 + \nu_2/\nu_1)},$$

and we can verify that formula (2.4) is satisfied.

For the Bethe lattice, $\nu_2 = 0$, and it follows that $\bar{m} \approx n$. We can easily see that the known results are reproduced from (2.10) and (2.11) by setting $\nu_1 = Z_b = Z - 2$ and $\nu = Z$.
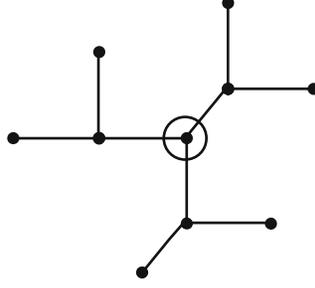
Fig. 1. A part of the most probable tree (the selected vertex is encircled).

3. The perimeter of the most probable tree

We set the origin at a lattice site. We write the coordinates of other sites measured in lattice parameters in the form

$$(a_1, a_2, \ldots, a_d). \tag{3.1}$$

The nearest neighbors in the SC lattice are situated along the coordinate axes and have only one nonzero coordinate $a_i = \pm 1$ in representation (3.1), where $i = 1, \ldots, d$. Two coordinates of the nearest neighbors differ from zero in formula (3.1) for the FCC lattice:

$$a_i = \pm\frac{1}{2}, \qquad a_j = \pm\frac{1}{2}, \quad i, j = 1, \ldots, d.$$

For the coordination number Z, we hence have $Z = 2d$ for the SC lattice and $Z = 2d(d-1)$ for the FCC lattice.

In the appendix, we find a useful characteristic for large trees on the Bethe lattice: their mean number of branches per internal vertex is

$$L = \frac{Z-2}{Z-1}. \tag{3.2}$$

For large Z, the most probable tree therefore contains mostly vertices at which three occupied bonds meet (see Fig. 1). We call occupied bonds edges. Vertices with other numbers of incident edges can also appear in a tree, but they must be rare. When constructing trees on hypercubic lattices, we assume that the growth of the number of trees in the ensemble is governed by three-valent vertices, and we determine their contribution ν_1 to the perimeter.

The SC lattice. We consider a most probable tree on the SC lattice of dimension d. We choose an internal vertex with three incident edges, which we call close edges. Such a vertex has $Z-3$ free bonds. Two edges, which we call far edges, come from each vertex at the end of a close edge. In the most probable configuration, all these six far edges have different directions. Then one of the free bonds of the given vertex is parallel to each such edge. The site at the end of this bond is the site closest to the end of the parallel edge (see Fig. 2a). Such a site can be equiprobably joint to either the starting or the terminal vertex of this edge. Therefore, these six free bonds all enter the mean perimeter of the selected vertex ν_1 with the factor $1/2$:

$$\nu_1 = Z - 3 - \frac{6}{2} = Z - 6. \tag{3.3}$$

We now find corrections of the order $1/Z$ to the obtained solution. First, the direction of a far edge can coincide with the direction of a close edge. In this case, one of the six directions of the selected vertex bonds becomes free, i.e., the perimeter increases by $1/2$. The probability that the direction of one of the six
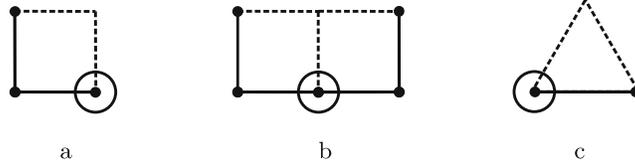
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Fig. 2. Reducing the contribution of a free bond in the perimeter because the site is close to several tree vertices in the SC lattice with (a) $\nu_{11} = 1/2$ and with (b) $\nu_{11} = 1/3$ and in the FCC lattice with (c) $\nu_{11} = 1/2$.

far edges coincides with the direction of one of the three close edges is $6 \cdot 3/Z$. The corresponding correction to perimeter (3.3) is $9/Z$.

Second, the directions of two of the six far edges may coincide. The probability is $12/Z$. In this case, one of the directions becomes free, and the contribution to ν_1 of the free bond whose direction coincides with the directions of these two far edges (see Fig. 2b) changes from $1/2$ to $1/3$. The corresponding correction to perimeter (3.3) is then $4/Z$.

Taking the corrections into account, we obtain

$$\nu_1 = Z - 6 + \frac{13}{Z}. \tag{3.4}$$

For three-dimensional lattices with $Z = 6$, the correction is not small, and we must improve result (3.4) by taking the finite number of positions into account. We begin with the placement of the three close edges around the selected vertex. These edges can be situated either in one plane or along three different coordinate axes. The respective probabilities of these placements are

$$p_{11} = \frac{1}{d-1}, \qquad p_{12} = \frac{d-2}{d-1}. \tag{3.5}$$

We now segregate one of $Z - 3 = 2d - 3$ free bonds of the selected vertex. This bond is in a new dimension with the probability

$$p_{21} = \frac{2(d-2)}{Z-3}, \qquad p_{22} = \frac{2(d-3)}{Z-3} \tag{3.6}$$

depending on the type of close-edge placement (3.5). But this bond can also be on the same axis as a close edge with the respective probabilities

$$p_{31} = 1 - p_{21} = \frac{1}{Z-3}, \qquad p_{32} = 1 - p_{22} = \frac{3}{Z-3}. \tag{3.7}$$

We now consider the placement of far edges. Neglecting the probability that two ends of far edges meet at a site, which is of the second order of smallness, the total number of variants of their placements is

$$D^3 = \left[\frac{(Z-1)(Z-2)}{2} \right]^3. \tag{3.8}$$

We now segregate these variants with respect to the site at the end of the chosen free bond:

1. No edge is in close proximity; the contribution of the free bond to the perimeter is $\nu_{11} = 1$.

2. One edge is in close proximity; the contribution of the free bond to the perimeter is $\nu_{11} = 1/2$.

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3. Two edges are in close proximity; the contribution of the free bond to the perimeter is $\nu_{11} = 1/3$.

4. Three edges are in close proximity; the contribution of the free bond to the perimeter is $\nu_{11} = 1/4$.

It turns out that the number of variants in these classes is independent of the above segregated patterns of mutual close-edge placement (3.5) but does depend on the position of the chosen free bond. If this bond is directed along the same axis as one of the close edges but oppositely to it, then edges coming from the end of this edge cannot be close to the end of the chosen bond. For the number of variants in this case, we have

1. No edge is in close proximity; $W'_0 = D[(Z-2)(Z-3)/2]^2$.

2. One edge is in close proximity; $W'_1 = 2D(Z-2)[(Z-2)(Z-3)/2]$.

3. Two edges are in close proximity; $W'_2 = D(Z-2)^2$.

If the chosen free bond has a new direction, then we have the following variants:

1. No edge is in close proximity; $W_0 = [(Z-2)(Z-3)/2]^3$.

2. One edge is in close proximity; $W_1 = 3(Z-2)[(Z-2)(Z-3)/2]^2$.

3. Two edges are in close proximity; $W_2 = 3(Z-2)^2[(Z-2)(Z-3)/2]$.

4. Three edges are in close proximity; $W_3 = (Z-2)^3$.

From formulas (3.5)–(3.7), we find that the sum of probabilities for the two positions of the chosen free bond is

$$P_c = p_{11}p_{31} + p_{12}p_{32} = \frac{3d-5}{(d-1)(Z-3)}$$

in the case where the free bond direction coincides with the direction of a close edge and

$$P_n = p_{11}p_{21} + p_{12}p_{22} = \frac{2(d-2)^2}{(d-1)(Z-3)}$$

in the opposite case.

We find the averaged contribution of the free bond by the formula

$$\bar{\nu}_{11} = P_c \sum_{m=0}^{2} \frac{W'_m}{(m+1)D^3} + P_n \sum_{m=0}^{3} \frac{W_m}{(m+1)D^3}. \tag{3.9}$$

Multiplying this result by the total number of free bonds of the selected vertex, we find its mean perimeter $\nu_1 = \bar{\nu}_{11}(Z-3)$.

The square lattice. Result (3.9) is also formally applicable to the case $d = 2$. But according to (3.2), $L = 2/3$ for $Z = 4$, and to increase the accuracy, we must take vertices with two edges and the prohibition of intersections of far edges into account. This results in the following interpretation: if we arbitrarily choose three internal vertices of a large tree, then on the average, two of them are incident to three edges, and one is incident to two edges. The possible types of environment of such vertices in the square lattice are shown in Fig. 3 (we do not take vertices with four incident edges into account, because, first, they are rare and, second, they do not have free bonds and therefore contribute nothing to ν_1). For each diagram in the figure, we indicate its contribution to the perimeter ν_{1i} and its relative weight K_i equal to the number of possible placements of the far edges on the lattice for a given placement of the close edges and a given topology

$$\nu_1 = \frac{1}{2} + \frac{1}{2} = 1, \ K = 6 \qquad \nu_1 = 1 + \frac{1}{2} = \frac{3}{2} = 1, \ K = 4 \qquad \nu_1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \ K = 4$$

$$\nu_1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \ K = 1 \qquad \nu_1 = \frac{1}{2} + \frac{1}{2} = 1, \ K = 2 \qquad \nu_1 = 1 + \frac{1}{3} = \frac{4}{3}, \ K = 2$$

a

$$\nu_1 = \frac{1}{2}, \ K = 2 \qquad \nu_1 = \frac{1}{2}, \ K = 2 \qquad \nu_1 = \frac{1}{2}, \ K = 8 \qquad \nu_1 = \frac{1}{2}, \ K = 6$$

$$\nu_1 = 1, \ K = 1 \qquad \nu_1 = 1, \ K = 2 \qquad \nu_1 = \frac{1}{3}, \ K = 8 \qquad \nu_1 = \frac{1}{3}, \ K = 8$$
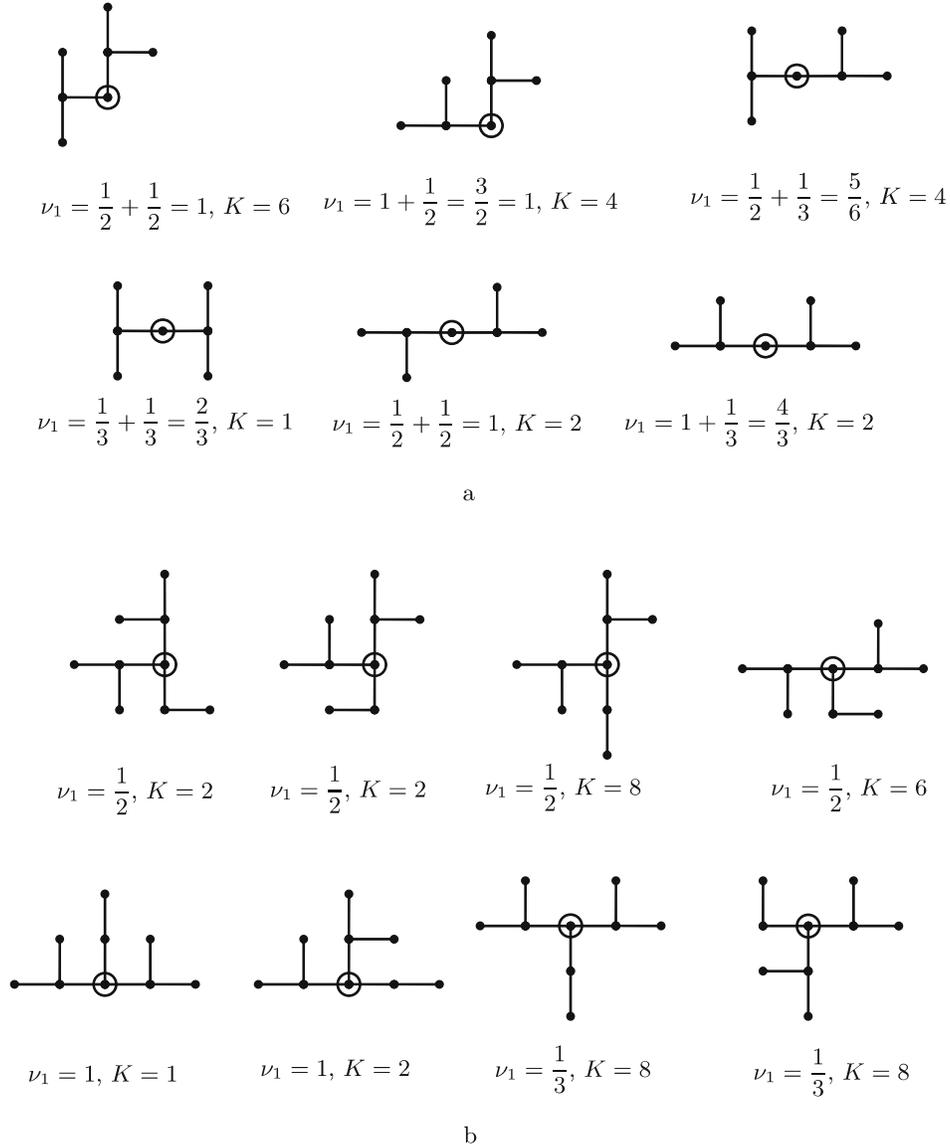
b

Fig. 3. The variants of the spatial placement of (a) two branches and (b) three branches around the selected vertex in the square lattice.

of the diagram. We obtain the mean contribution of a vertex by separately summing the contributions of vertices with different numbers of branches by the formula

$$\nu_1 = \frac{1}{3} \frac{\sum_i K_i \nu_{1i}}{\sum_i K_i} + \frac{2}{3} \frac{\sum_j K_j \nu_{1j}}{\sum_j K_j}.$$

We therefore have

$$\nu_1 = \frac{1}{3} \frac{62}{57} + \frac{2}{3} \frac{104}{222} \cong 0.67. \tag{3.10}$$

The FCC lattice. We take an internal vertex incident to three edges of a most probable tree situated on the FCC lattice. We choose one of the three edges starting at this vertex and terminating at the site $(0, \ldots, a_i, 0, \ldots, a_j, \ldots, 0) \equiv (a_i, a_j)$. For convenience here and hereafter, we omit zero coordinates in

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formula (3.1). The bonds close to the given edge are those that begin at the selected vertex and terminate at the sites (a_j, a_f) and (a_i, a_f), where we choose the coordinate a_f among $d-2$ free directions and it may take the values $\pm 1/2$. We therefore have $\omega = 4(d-2)$ close bonds. The site at the end of each such bond is also closest to the second end of the chosen edge (see Fig. 2c), i.e., we have ω triangles of bonds around each edge. Such a site can be equiprobably joint to either the starting or the terminating vertex of the given edge. Therefore, 3ω bonds in these triangles constructed on the base of three edges enter the mean perimeter ν_1 of the selected vertex with the coefficient $1/2$:

$$\nu_1 = Z - 3 - \frac{3\omega}{2} = Z - 3 - 6(d-2).$$

We next find corrections of the order $1/Z$ to the obtained solution. The first correction comes from the overlapping of free bonds of two triangles constructed at different close edges. The second correction comes from the far edges.

To estimate corrections of the first type, we consider a site in the center incident to Z vacant bonds. We choose an arbitrary vacant bond that links this site to an empty site with the coordinates (a_i, a_j). There remain $\Omega = Z - 1$ vacant bonds into which we can distribute three indistinguishable edges in M ways, where

$$M = \frac{\Omega(\Omega - 1)(\Omega - 2)}{6}. \tag{3.11}$$

This set of states can be segregated into four classes with respect to the chosen bond:

1. No edge is close to the given bond; $M_0 = (\Omega - \omega)(\Omega - \omega - 1)(\Omega - \omega - 2)/6$.

2. One edge is close to the given bond; $M_1 = \omega(\Omega - \omega)(\Omega - \omega - 1)/2$.

3. Two edges are close to the given bond; $M_2 = (\Omega - \omega)\omega(\omega - 1)/2$.

4. Three edges are close to the given bond; $M_3 = \omega(\omega - 1)(\omega - 2)/6$.

For the mean contribution of a given bond to the perimeter, we obtain

$$\bar{\nu}_{11} = \sum_{m=0}^{3} \frac{M_m}{(m+1)M}. \tag{3.12}$$

We hence obtain the perimeter as the total contribution of $Z - 3$ vacant bonds,

$$\nu_1 = \bar{\nu}_{11}(Z - 3) \approx Z\left(1 - \frac{3}{d} + \frac{11}{2d^2}\right). \tag{3.13}$$

We now consider the contributions of far edges. In the general form, these are edges linking the sites (a_i, a_j) and (a_i, a_j, a_k, a_f). Taking all possible edge placements into account, we find the formula for the number of sites accessible to the end of a far edge,

$$V = \frac{Z(d-2)(d-3)}{3} + 2Z(d-2) + 2Z + 2d. \tag{3.14}$$

Let three close edges and six far edges be distributed randomly and independently. We choose a vacant bond beginning at the center and calculate its contribution to the perimeter diminished by the possibility of edges to be in a close proximity to this bond.

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We first distribute three close edges with the above probabilities M_m/M of m edges falling into the domain ω. We then distribute six far edges. The total number of distributions of six indistinguishable far edges into the domain $G = V - 4$ that remains after distributing three close edges is

$$Q = \frac{G(G-1)(G-2)(G-3)(G-4)(G-5)}{6!}. \tag{3.15}$$

We now segregate these states into classes depending on the number of far-edge terminal points falling into the closest neighborhood of the given vacant bond terminal point consisting of $\Omega = Z - 1$ sites. For the mean contribution of one bond to the perimeter, we obtain

$$\bar{\nu}_{11} = \sum_{n=0}^{6} \sum_{m=0}^{3} \frac{P_{mn}}{(m+n+1)}, \tag{3.16}$$

where

$$P_{mn} = \frac{M_m(G-\Omega+m)!\,(\Omega-m)!}{MQ(G-\Omega+n+m-6)!\,(\Omega-m-n)!\,(6-n)!\,n!}. \tag{3.17}$$

We hence find the selected vertex perimeter, which is the sum of the contributions of $Z-3$ vacant bonds,

$$\nu_1 = \bar{\nu}_{11}(Z-3) \approx Z\left(1 - \frac{3}{d} - \frac{7}{2d^2}\right). \tag{3.18}$$

We recall that we assume that the far-edge terminal points are distributed homogeneously in the domain V given by (3.14). If we consider an isolated system of two joined edges, then the probability of coming to a site depends on the distance of this site from the center. It is well known that the position of the end of a long chain without excluded volume interactions is described by the Gaussian distribution. Volume interactions diminish the probability of coming to the center [7]. Our homogeneous distribution, on one hand, simplifies estimates and, on the other hand, somehow takes the effect of the excluded volume interaction on the structure of a most probable tree into account.

4. Calculating the exponent and discussing the results

The generating function critical exponent is determined by formula (2.11) in terms of the mean perimeter. The dependence of the exponent p on d calculated by formulas (3.12) and (3.16) is depicted in Fig. 4 together with results based on data that we obtained previously using Monte Carlo simulations [10], [11]. For $d = 2$, we take the value $\nu_1 \cong 0.67$ obtained above (see (3.10)), which results in $p \approx 5$. We observe that theoretical estimates agree well with the results of numerical experiment. At the same time, we observe that these values of the exponent differ substantially from those obtained by Bethe lattice approximation (1.2).

Another important parameter determining the growth of trees in the ensemble is the growth parameter Z_c. We studied its dependence on d for the SC and FCC lattices in [8]–[11]. The results of the simulations and the $1/d$-expansion agree well. In the above model, the growth parameter is determined by formula (2.10). We assume that $Z = \nu = \nu_1 + \nu_2/c + \delta$ and transform formula (2.10):

$$\frac{Z}{Z_c} = (1+p)\frac{\log(1+a)}{a}, \tag{4.1}$$

where $a = \nu_2/\nu_1 = cp(1 - \delta/Z) - c\delta/Z$. In the Bethe lattice case, $\nu = Z$, $\nu_2 = 0$, $\nu_1 = Z_c = Z - 2$, and $\delta = 2$. The results of numerical experiments for hypercubic lattices demonstrate that $\delta > 2$ and $2 > c > 1$. The physical meaning of the parameter c is that when all varieties of trees are constructed, bonds entering
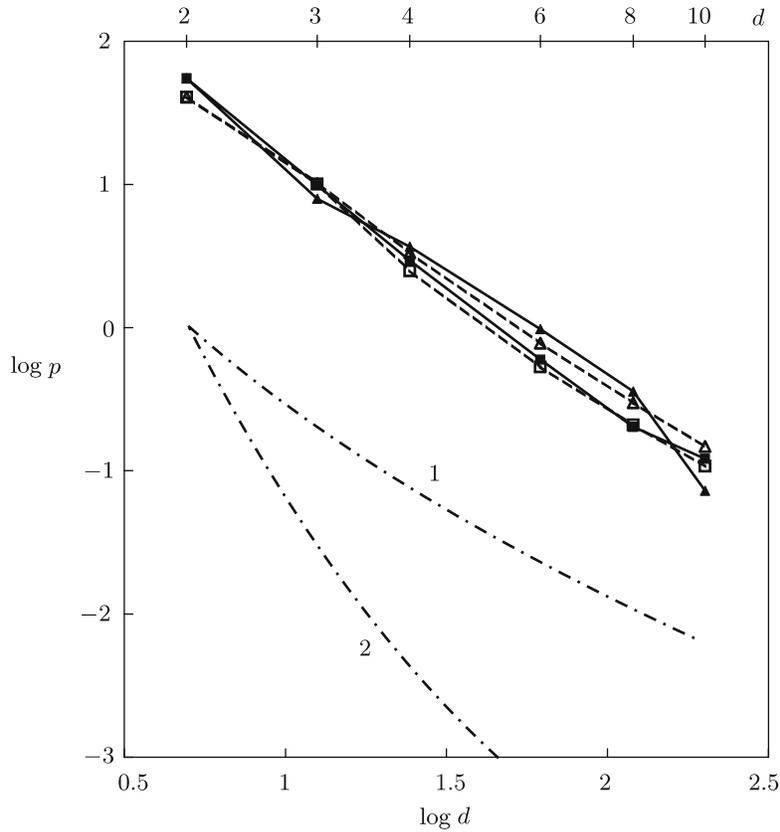
Fig. 4. The dependences of exponents on the space dimension for the SC lattices (squares) and for the FCC lattices (triangles): ■, ▲, and solid lines are the results of numerical experiments in [10], [11]; □, △, and dashed lines are the results of calculations using the obtained formulas. Dash–dot lines correspond to the Bethe approximations of (1) the SC lattices and (2) the FCC lattices.

ν_2 can be built in trees in different ways, for example, a branch can pass them in two ways: either in one direction or in the opposite one.

Substituting the asymptotic expressions for the perimeter ν_1, we obtain an approximate expression for the exponent at $d \gg 1$. For the SC lattice, we obtain

$$p = \frac{3 - 13/4d}{d - 3 + 13/4d} \qquad (4.2)$$

from formula (3.4), and for the FCC lattice, we obtain

$$p = \frac{3 + 7/2d}{d - 3 - 7/2d} \qquad (4.3)$$

from formula (3.18).

Exponents (4.2) and (4.3) coincide up to corrections linear in $1/d$. The second-order corrections differ. An insufficient accuracy of the numerical experiment (20%), which is due to restrictions on the size of trees accessible for simulation does not allow proving or disproving these small differences. On the other hand, theoretical results (4.2) and (4.3) may differ because of inaccuracy in determining the most probable tree used when calculating the mean perimeter ν_1.

Substituting the values of Z, we find

$$p_b = \frac{1}{d-1}, \qquad p_b = \frac{1}{d(d-1)-1} \tag{4.4}$$

for the respective SC and FCC lattices in the Bethe approximation.

The principal difference between the two approaches is how we take branches infinitely growing from a vertex into account. In the above Bethe lattice approximation, these are all $Z-1$ branches. The independence of the growing branches is reflected in the structure of Eqs. (A.2) and (A.3) (see the appendix) for the generating functions, while only a small fraction of branches can grow infinitely on the hypercubic lattices because branch intersections are prohibited. To take the branch excluded volume interaction into account, we attacked the problem of deriving an equation for the generating function by analyzing the growth of the whole perimeter, not the growth of separate branches. A new form of generating function equation (2.8) allowed reducing the problem of taking the excluded volume interaction into account to the problem of determining local properties of the most probable tree vertex environment, which ultimately allowed explaining the simulation results.

Appendix: Tree characteristics on the Bethe lattice

We consider the Bethe lattice with the coordination number Z. In addition to tree generating function (1.2), we use the generating function of the so-called trees with pending root such that only one bond is attached to the root,

$$E_1(x) = \left(1 - \frac{x}{x_b}\right)^{-1/(Z-2)}. \tag{A.1}$$

Different tree branches are constructed independently in the Bethe approximation, and generating function (A.1) therefore satisfies the self-consistency equation

$$E_1(x) = 1 + \int_0^x \left(E_1(x)\right)^q dx_1, \tag{A.2}$$

where $q = Z - 1$. We introduce the function $E_q(x) = \left(E_1(x_1)\right)^q$ by the equation

$$E_q(x) = 1 + \sum_{k=1}^{q} \binom{q}{k} \left[\int_0^x E_q(x_1)\, dx_1\right]^k, \tag{A.3}$$

which is obtained from (A.2) and is represented as a sum over the number k of branches growing from the vertex. In the summation, we insert the weight factor b^{k-1} of the number of branchings,

$$E_q(x) = \sum_{k=0}^{q} b^{k-1} \binom{q}{k} \left[\int_0^x E_q(x_1)\, dx_1\right]^k. \tag{A.4}$$

We introduce the function

$$Y = 1 + b \int_0^x E_q(x_1)\, dx_1, \tag{A.5}$$

in which we set $a = 1 - b$ and obtain the equation

$$\frac{dY}{dx} = Y^q - a$$

from (A.4). Its solution can be written in the form

$$x = \int_1^Y \frac{dy}{y^q - a}.$$

We hence obtain the equation for the critical point coordinate:

$$x_c = \int_1^\infty \frac{dy}{y^q - a}, \qquad \frac{d}{db} x_c = -\int_1^\infty \frac{dy}{(y^q - a)^2}.$$

In particular, at $b = 1$ ($a = 0$), we have

$$x_c = \frac{1}{Z - 2}, \qquad \frac{d}{db} x_c = -\frac{1}{2Z - 3}.$$

We introduce two mean characteristics of trees composed of n bonds: the mean number B of branchings and the mean number L of branchings per internal site, which we define by the formulas

$$B = \frac{d}{db} \log T_n(b) \approx -n \frac{d}{db} \log x_c, \qquad L = \frac{B}{n - B - 1}. \tag{A.6}$$

In particular, for $b = 1$ ($a = 0$), we have

$$B = n \frac{Z - 2}{2Z - 3}, \qquad L = \frac{Z - 2}{Z - 1}.$$

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