Low Temperature Physics

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Cite as: Low Temp. Phys. **36**, 741 (2010); https://doi.org/10.1063/1.3490860
Published Online: 23 November 2010

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Effect of cross correlations between inhomogeneities on the spectrum and damping of spin and elastic waves

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(Submitted December 25, 2009)
Fiz. Nizk. Temp. **36**, 933–940 (August–September 2010)

This is a study of the dispersion relations and damping of spin waves in ferromagnets with inhomogeneities in their exchange and magnetic anisotropy parameters and of elastic waves in isotropic media with inhomogeneities in the density of matter and in the elastic force constants, with cross correlations between these inhomogeneities taken into account. A general behavior is found which is independent of the physical nature of the waves: the kind of effect cross correlations between inhomogeneities in any two parameters of a material will have on the wave spectrum depends on whether both parameters belong to the same (i.e., both belong to the kinetic part or both belong to the potential part) or to different parts of the hamiltonian. In the former case, positive cross correlations lead to a substantial modification of the dispersion relation and to increased wave damping, and in the latter case, to a smaller modification of these characteristics. On the other hand, negative cross correlations lead to the opposite effects in both cases. This behavior is explained qualitatively.
[doi:10.1063/1.3490860]

I. INTRODUCTION. CORRELATIONS AND CROSS CORRELATIONS

Amorphous and nanocrystalline materials are used extensively in various modern electronic devices based on the propagation and conversion of electromagnetic, elastic, and spin waves. From a theoretical standpoint these materials are characterized by two basic properties: (1) inhomogeneities in all the parameters of the hamiltonian (density of matter, elastic force constants, exchange parameters, magnetic anisotropy, etc.), and (2) extended correlations of these inhomogeneities with correlation radii that are determined by both the topological and compositional disorder and may vary over wide limits (tens to hundreds of interatomic distances). Large correlation radii make it impossible to use the well developed theoretical techniques which account for uncorrelated (δ-correlated) inhomogeneities in calculating a whole range of effects in these materials.

The idea that spin and elastic waves in ferromagnets should be treated in terms of a unified magnetoelastic continuum was advanced in the pioneering work of Turov and Irkhinin,[1] Akhiezer, Bar'yakhtar, and Peletminskii,[2] and Kittel,[3] who developed a theory of coupled magnetoelastic waves and of the magnetoelastic resonance phenomenon which arises in the region where the dispersion curves of the unperturbed spin and elastic waves intersect. The problems involved in accounting for correlations of the inhomogeneities are, however, quite complicated and here we examine them separately for spin and elastic waves, while neglecting the magnetoelastic interaction. This treatment is possible either for materials with small magnetoelastic parameters or at frequencies far from the magnetoelastic resonance frequencies.

The effect of inhomogeneities with arbitrary correlation radii on the spectrum and damping of spin waves has been accounted for in a continuum model[4–7] in the first nonvanishing perturbation theory approximation. The effect of inhomogeneities with arbitrary correlation radii on the spectrum and damping of elastic waves in isotropic media has also been examined.[4,8,9] The effect of each fluctuating parameter of the medium, $A_i(\mathbf{x})$, where $\mathbf{x}=\{x,y,z\}$, has been examined separately.[4–7,9]

The random function $A_i(\mathbf{x})$ for each parameter i of the medium can be written in the form

$$A_i(\mathbf{x}) = A_i[1 + \gamma_i\rho_i(\mathbf{x})], \tag{1}$$

where A_i and γ_i are, respectively, the average value and relative mean square deviation of $A_i(\mathbf{x})$, and $\rho_i(\mathbf{x})$ is a centered ($\langle\rho_i(\mathbf{x})\rangle=0$) and normalized ($\langle\rho_i^2(\mathbf{x})\rangle=1$) homogeneous random function of the coordinates. The stochastic characteristics of the random function $\rho_i(\mathbf{x})$ are described by the correlation function $K_{ii}(\mathbf{r})$ or by the associated Fourier spectral density $S_{ii}(\mathbf{k})$ of the inhomogeneities, with

$$K_{ii}(\mathbf{r}) = \langle\rho_i(\mathbf{x})\rho_i(\mathbf{x}+\mathbf{r})\rangle, \tag{2}$$

$$S_{ii}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int K_{ii}(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r},$$

where $|\mathbf{r}|$ is the distance between two points in space and the angle brackets denote averaging over the ensemble of possible values of the random function $\rho_i(\mathbf{x})$.

The main result of the theory developed in Ref. 4 is that in the neighborhood of the wave number $k=k_{ii}/2$, a change should be observed in the dispersion relation $\omega'(k)$ and damping constant $\omega''(k)$ and that this should be different for inhomogeneities in different physical parameters. These effects arise because the waves scatter differently on the correlated ($k>k_{ii}/2$) and uncorrelated ($k\ll k_{ii}/2$) parts of the inhomogeneities. The theory, in which the inhomogeneity of each parameter is examined separately, is approximately valid in a number of cases.

In general, however, averaging stochastic wave equations containing several inhomogeneous parameters $A_i(\mathbf{x})$ means that $\omega'(k)$ and $\omega''(k)$, along with all the nonrandom characteristics of the stochastic system, become functionals of the mutual correlation functions (cross correlations) $K_{ij}(\mathbf{r})$ between the parameters (or their spectral densities $S_{ij}(\mathbf{r})$), as well as of the correlation (autocorrelation) functions $K_{ii}(\mathbf{r})$ for each parameter A_i, with

$$K_{ij}(\mathbf{r}) = \langle \rho_i(\mathbf{x})\rho_j(\mathbf{x}+\mathbf{r})\rangle, \quad S_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int K_{ij}(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}, \tag{3}$$

where $i \neq j$ and the average is taken over the ensemble of possible realizations of the two random functions $\rho_i(\mathbf{x})$ and $\rho_i(\mathbf{x})$.

As opposed to the autocorrelation function $K_{ii}(\mathbf{r})$, which equals 1 for $\mathbf{r}=0$, the cross correlation function $K_{ij}(\mathbf{r})$ is equal to a dimensionless coefficient κ_{ij} at $\mathbf{r}=0$. This coefficient characterizes the magnitude and sign of the cross correlations between the parameters A_i and A_j and can take arbitrary values between −1 and 1. The particular magnitude and sign of κ_{ij} should be determined by experiment or calculated using a microscopic model that accounts for the actual, sometimes very complicated, physical relationships between the parameters A_i and A_j. This kind of formal description makes it possible to study the effect of cross correlations on the spectrum of a system in a general form, without discussing the physical mechanisms responsible for these cross correlations in detail.

In the limiting cases of $\kappa_{ij} = \pm 1$, the stochastic cross correlations transform into deterministic couplings between the inhomogeneities in the different parameters. When $\kappa_{ij} = 1$, the random functions $\rho_i(\mathbf{x})$ and $\rho_j(\mathbf{x})$ coincide, while when $\kappa_{ij} = -1$, they are mirror images of one another. In general, cross correlations, while not changing the mean square deviations γ_i and γ_j of the random functions $A_i(\mathbf{x})$ and $A_j(\mathbf{x})$, do cause a partial stochastic spatial synchronization of these functions, the degree of which is controlled by the magnitude of the cross correlation coefficient κ_{ij}.

This paper is devoted to calculating the effect of cross correlations of arbitrary magnitude and sign between different parameters of a medium on the spectrum and damping of spin and elastic waves.

II. SPIN WAVES

We have previously examined[10] a model for ferromagnets in which the exchange parameter $\alpha(\mathbf{x})$ and the uniaxial magnetic anisotropy $\beta(\mathbf{x})$ are inhomogeneous. The direction $\mathbf{n}$ of the anisotropy is assumed constant and coincident with the direction of the direction of the external magnetic field $\mathbf{H}$. The energy density W in this model is given by

$$W = \frac{1}{2}\alpha(\mathbf{x})\left(\frac{\partial \mathbf{M}}{\partial \mathbf{x}}\right)^2 - \mathbf{MH} - \frac{1}{2}\beta(\mathbf{x})(\mathbf{Mn})^2. \tag{4}$$

We write the exchange $\alpha(\mathbf{x})$ and anisotropy $\beta(\mathbf{x})$ parameters in the form (1), where $A_i = \alpha$, β and $\gamma_i = \gamma_\alpha$, γ_β. The transverse projections of the magnetization, $m_x(\mathbf{x},t)$ and $m_y(\mathbf{x},t) \propto \exp(i\omega t)$, are described by a system of linear equations with coefficients depending on $\mathbf{x}$:

$$-\frac{i\omega}{g}m_x = M_0\beta m_y + M_0\beta\gamma_\beta\rho_\beta m_y + Hm_y - M_0\alpha(\nabla^2 m_y + \gamma_\alpha\rho_\alpha\nabla^2 m_y + \gamma_\alpha \nabla\rho_\alpha \nabla m_y),$$

$$\frac{i\omega}{g}m_y = M_0\beta m_x + M_0\beta\gamma_\beta\rho_\beta m_x + Hm_x - M_0\alpha(\nabla^2 m_x + \gamma_\alpha\rho_\alpha\nabla^2 m_x + \gamma_\alpha \nabla\rho_\alpha \nabla m_x), \tag{5}$$

where g is the gyromagnetic ratio.

Taking the Fourier transform and introducing the circular projection, we obtain the following integral equation for the resonance projection $m(\mathbf{k})$:

$$(\nu - k^2)m(\mathbf{k}) = \gamma_\alpha \int (\mathbf{kk}_1)\rho_\alpha(\mathbf{k}-\mathbf{k}_1)m(\mathbf{k}_1)d\mathbf{k}_1 + \frac{\beta\gamma_\beta}{\alpha}\int \rho_\beta(\mathbf{k}-\mathbf{k}_1)m(\mathbf{k}_1)d\mathbf{k}_1, \tag{6}$$

where we have introduced the notation $\nu = (\omega - \omega_0)/gM_0\alpha$. Here ω_0 is the ferromagnetic resonance frequency.

We average Eq. (6) over the random realizations of the functions $\rho_\alpha(\mathbf{k})$ and $\rho_\beta(\mathbf{k})$. We uncouple the averages of the derivatives of the random functions ρ and m under the integral signs in the first nonvanishing approximation of perturbation theory (the Bourret approximation[11]). This yields the following general form of the dispersion relation for spin waves:

$$\nu - k^2 = \int \frac{d\mathbf{k}_1}{\nu - k_1^2}[\gamma^2(\mathbf{kk}_1)^2 S_{\alpha\alpha}(\mathbf{k}-\mathbf{k}_1) + 2\gamma\eta_\beta(\mathbf{kk}_1)S_{\alpha\beta}(\mathbf{k}-\mathbf{k}_1) + \eta_\beta^2 S_{\beta\beta}(\mathbf{k}-\mathbf{k}_1)], \tag{7}$$

where we have introduced the notation $\gamma = \gamma_\alpha$ and $\eta_\beta = \beta\gamma_\beta/\alpha$. In this equation, the terms proportional to γ^2 and η_β^2 account for the effect of the exchange and anisotropy inhomogeneities, respectively. The term proportional to $\gamma\eta_\beta$ accounts for the effect of cross correlations between the exchange and anisotropy inhomogeneities.

A. 1D inhomogeneities

We assume that the correlations, both the exchange $K_{\alpha\alpha}(r)$ and anisotropy $K_{\beta\beta}(r)$ autocorrelation functions and the cross correlation functions $K_{\alpha\beta}(r)$ between exchange and anisotropy fluctuations, fall off exponentially, i.e.,

$$K_{ii}(r) = \exp(-k_{ii}r), \quad K_{ij}(r) = \kappa \exp(-k_{ij}r), \tag{8}$$

where, in our case $K_{ii} = K_{\alpha\alpha}$ or $K_{\beta\beta}$; $K_{ij} = K_{\alpha\beta}$; $k_{ii} = r_{ii}^{-1}$ and $k_{ij} = r_{ij}^{-1}$ are the correlation wave numbers, while r_{ii} and r_{ij} are the correlation radii.

In general, the correlation radii r_{ii} for the inhomogeneities of each parameter i can be different. The cross correlation radii r_{ij} between the inhomogeneities of different parameters i and j can also be different. For simplicity, in the following we limit our selves to the case where all the cross correlation radii are roughly equal, i.e., $r_{ii} \approx r_{ij} \approx r_c$.

According to Eqs. (2) and (3), these correlation functions correspond to spectral densities of the form

$$S_{\alpha\alpha}(k)=S_{\beta\beta}(k)=\frac{1}{\pi}\frac{k_c}{k_c^2+k^2},\quad S_{\alpha\beta}(k)=\frac{\kappa}{\pi}\frac{k_c}{k_c^2+k^2}. \tag{9}$$

Calculating the integrals in Eq. (7) by the methods of the theory of residues, we obtain a complex dispersion relation on setting $\sqrt{\nu}\approx k$ on the right hand side of Eq. (7). Writing ν in the form $\nu=\nu'+i\nu''$, we obtain a dispersion relation for spin waves including mutual correlations between inhomogeneities in the exchange and magnetic anisotropy parameters of the form

$$\nu'=k^2\left(1-\gamma^2\frac{1+3u^2}{1+4u^2}-\frac{2\kappa\gamma\eta}{1+4u^2}\right)+\frac{\eta^2k_c^2}{1+4u^2} \tag{10}$$

and a damping constant for spin waves of the form

$$\nu''=k^2\left(\gamma^2\frac{u(1+2u^2)}{1+4u^2}+\frac{4\kappa\gamma\eta u}{1+4u^2}\right)+\frac{\eta^2(1+2u^2)k_c^2}{u(1+4u^2)}, \tag{11}$$

where we have introduced the dimensionless notation $u=k/k_c$ and $\eta=\eta_\beta/k_c^2$.

B. 3D inhomogeneities

We assume that the decrease in the correlations is characterized by isotropic correlation functions which depend only on the modulus of the radius vector $r=|\mathbf{r}|$ in the form of Eq. (8). According to Eqs. (2) and (3), these correlation functions correspond to spectral densities of the form

$$S_{\alpha\alpha}(\mathbf{k})=S_{\beta\beta}(\mathbf{k})=\frac{1}{\pi^2}\frac{k_c}{(k_c^2+k^2)^2},$$

$$S_{\alpha\beta}(\mathbf{k})=\frac{\kappa}{\pi^2}\frac{k_c}{(k_c^2+k^2)^2}. \tag{12}$$

Substituting these expressions in Eq. (7) and taking the integrals in a spherical coordinate system, we obtain the following dispersion relation $\nu''(\mathrm{k})$ and damping for spin waves $\nu''(\mathrm{k})$ with mutual correlations between 3D inhomogeneities in the exchange and magnetic anisotropy taken into account:

$$\frac{\nu'}{k_c^2}=u^2\left\{1-\gamma^2\left[\frac{1+5u^2}{1+4u^2}+\frac{1}{u^2}-\frac{(1+2u^2)\tan^{-1}(2u)}{2u^3}\right]\right\}$$
$$+2\kappa\gamma\eta\left(\frac{\tan^{-1}(2u)}{2u}-\frac{1+3u^2}{1+4u^2}\right)-\frac{\eta^2}{1+4u^2}, \tag{13}$$

$$\frac{\nu''}{k_c^2}=\gamma^2u^3\left[\frac{1}{u^2}+\frac{2u^2}{1+4u^2}-\frac{(1+2u^2)\ln(1+4u^2)}{4u^4}\right]$$
$$+2\kappa\gamma\eta\left[\frac{u(1+2u^2)}{1+4u^2}-\frac{\ln(1+4u^2)}{4u}\right]+\frac{2\eta^2u}{1+4u^2}. \tag{14}$$

The changes in the dispersion relation given by Eq. (13) are shown in Fig. 1a. The dotted curve in this figure shows the dispersion relation $\nu'=\mathrm{k}^2$ corresponding to a uniform ferromagnet ($\gamma=\eta=0$). The smooth curve corresponds to the combined changes in the dispersion relation owing to the simultaneous presence of exchange and anisotropy inhomogeneities without mutual correlations between them ($\gamma\neq0$, $\eta\neq0$, $\kappa=0$). It can be seen that positive cross correlations

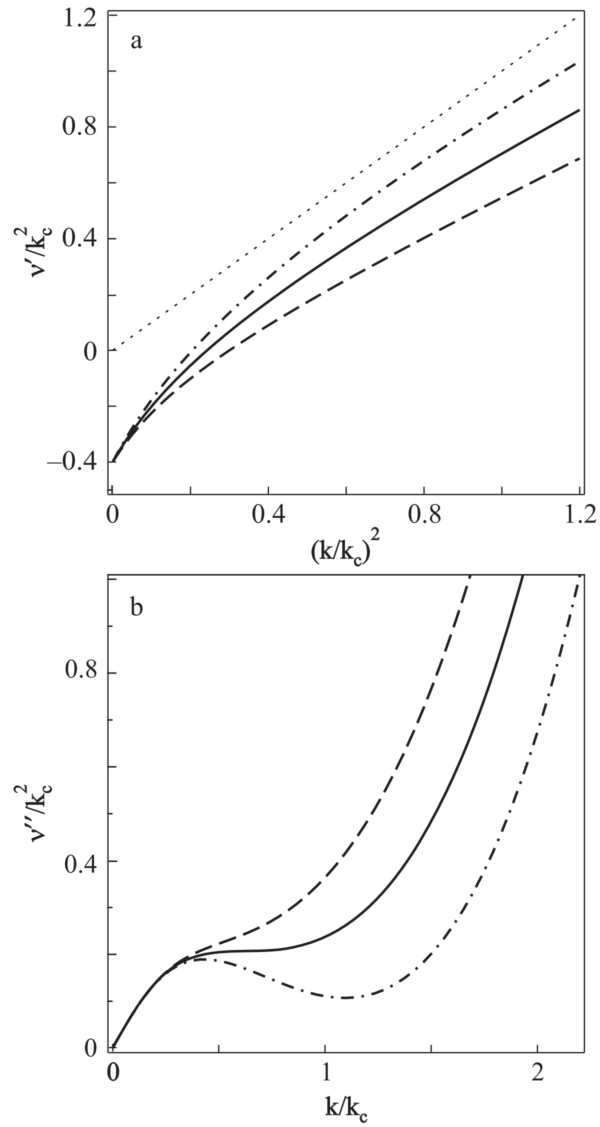

FIG. 1. Dispersion relation (a) and damping (b) for spin waves in a ferromagnet with 3D exchange and anisotropy inhomogeneities: $\kappa=0$ (smooth curves); $\kappa=0.8$ (dashed curves); $\kappa=-0.8$ (dash-dot curves). The dotted line is the dispersion relation for a homogeneous ferromagnet.

(dashed cure $\kappa>0$) lead to a greater shift in the dispersion relation, while negative correlations (dash-dot curve, $\kappa<0$), on the other hand, reduce the change in this dispersion relation relative to that caused by the combined effect of uncorrelated exchange and anisotropy inhomogeneities.

The damping $\nu''(\mathrm{k})$ caused by 3D exchange and anisotropy inhomogeneities and described by Eq. (14) is illustrated in Fig. 1b. The smooth curve in this figure corresponds to the combined effect owing to simultaneous exchange and anisotropy inhomogeneities when there are no mutual correlations between them ($\gamma\neq0$, $\eta\neq0$, $\kappa=0$). It is clear that positive cross correlations (dashed curve, $\kappa>0$) lead to an increase, and negative cross correlations (dash-dot curve, $\kappa<0$), to a decrease in the wave damping. The change in the magnitude of the damping owing to the cross correlations is a function of k or, accordingly, the frequency. For $\kappa<0$ the strongest drop in damping should be observed in the neighborhood of $\mathrm{k}\sim\mathrm{k_c}$.

III. ELASTIC WAVES

We have examined[12] a model of an isotropic elastic medium in which the force constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ and the density of the material $p(\mathbf{x})$ are inhomogeneous. The equation of motion for the displacement vector $\mathbf{u}(\mathbf{x},t)$ is

$$-p(\mathbf{x})\frac{\partial^2 u_s}{\partial t^2}+\frac{\partial}{\partial x_s}\left(\lambda(\mathbf{x})\frac{\partial u_f}{\partial x_f}\right)+\frac{\partial}{\partial x_f}\left(\mu(\mathbf{x})\frac{\partial u_s}{\partial x_f}\right)+\frac{\partial}{\partial x_f}\left(\mu(\mathbf{x})\frac{\partial u_f}{\partial x_s}\right)=0, \quad (15)$$

where the subscripts s and f run over x, y, z, and a repeated index f is taken to mean summation over all the coordinates. We write the position dependent parameters $p(\mathbf{x})$, $\lambda(\mathbf{x})$, and $\mu(\mathbf{x})$ in the form (1).

Assuming that $\mathbf{u}(\mathbf{x},t)\sim e^{-i\omega t}\mathbf{u}(\mathbf{x})$ and taking the Fourier transform, from Eq. (15) we obtain a vector equation for the transforms of the Fourier functions $\mathbf{u}(\mathbf{x})$:

$$(\omega^2-v_t^2k^2)\mathbf{u}(\mathbf{k})-(v_l^2-v_t^2)\mathbf{k}(\mathbf{k}\mathbf{u}(\mathbf{k}))=-\gamma_p\omega^2\int\rho_p(\mathbf{k}-\mathbf{k}_1)\mathbf{u}(\mathbf{k}_1)d\mathbf{k}_1+\gamma_\mu v_t^2\int(\mathbf{k}\mathbf{k}_1)\rho_\mu(\mathbf{k}-\mathbf{k}_1)\mathbf{u}(\mathbf{k}_1)d\mathbf{k}_1+\gamma_\lambda(v_l^2-2v_t^2)\mathbf{k}\int\rho_\lambda(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}_1\mathbf{u}(\mathbf{k}_1))d\mathbf{k}_1+\gamma_\mu v_t^2\int\mathbf{k}_1\rho_\mu(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}\mathbf{u}(\mathbf{k}_1))d\mathbf{k}_1. \quad (16)$$

We raise the subscript on $\mathbf{k}$ in Eq. (16) by unity and calculate the function $\mathbf{u}(\mathbf{k}_1)$ from the resulting equation:

$$\mathbf{u}(\mathbf{k}_1)=-\gamma_p\omega^2\int\frac{\rho_p(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2}{\omega^2-v_t^2k_1^2}-\gamma_p\omega^2(v_l^2-v_t^2)\int\frac{\mathbf{k}_1\rho_p(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2-v_t^2k_1^2)(\omega^2-v_l^2k_1^2)}+\gamma_\lambda(v_l^2-2v_t^2)\int\frac{\mathbf{k}_1\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{\omega^2-v_t^2k_1^2}+\gamma_\mu v_t^2\int\frac{(\mathbf{k}_1\mathbf{k}_2)\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2}{\omega^2-v_t^2k_1^2}+\gamma_\lambda(v_l^2-2v_t^2)(v_l^2-v_t^2)\int\frac{k_1^2\mathbf{k}_1\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2-v_t^2k_1^2)(\omega^2-v_l^2k_1^2)}+\gamma_\mu v_t^2\int\frac{\mathbf{k}_2\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{\omega^2-v_t^2k_1^2}+2\gamma_\mu v_t^2(v_l^2-v_t^2)\int\frac{\mathbf{k}_1(\mathbf{k}_1\mathbf{k}_2)\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2-v_t^2k_1^2)(\omega^2-v_l^2k_1^2)}. \quad (17)$$

Substituting this expression in the right hand side of Eq. (16), averaging the resulting equation over the random variations in the functions $\rho_p(\mathbf{k})$, $\rho_\mu(\mathbf{k})$, and $\rho\lambda(\mathbf{k})$, and splitting the resulting correlators in the Bourret approximation,[11] we obtain the following complex dispersion relations $\omega(k)$ for transverse waves:

$$\omega=v_tk\left\{1-\frac{v_t^2}{2}[\gamma_p^2u^2(2L_{tpp}^{20}-(v_l^2-v_t^2)(L_{ltpp}^{t40}-L_{ltpp}^{t42}))-4\gamma_p\gamma_\mu u(L_{tp\mu}^{t31}-(v_l^2-v_t^2)(L_{ltp\mu}^{t51}-L_{ltp\mu}^{t53}))+\gamma_\mu^2(L_{t\mu\mu}^{t40}+L_{t\mu\mu}^{t42}-4(v_l^2-v_t^2)(L_{lt\mu\mu}^{t62}-L_{lt\mu\mu}^{t64}))]\right\}, \quad (18)$$

and for longitudinal waves:

$$\omega=v_lk\{1-v_l^2[\gamma_p^2u^2(L_{tpp}^{l20}-(v_l^2-v_t^2)L_{ltpp}^{l42})-2\gamma_p\gamma_\lambda u(1-2\beta^2)(L_{tp\lambda}^{l31}-(v_l^2-v_t^2)L_{ltp\lambda}^{l51})-4\gamma_p\gamma_\mu\beta^2(L_{tp\mu}^{l31}-(v_l^2-v_t^2)L_{ltp\mu}^{l53})+\gamma_\lambda^2(1-2\beta^2)^2(L_{t\lambda\lambda}^{l40}-(v_l^2-v_t^2)L_{lt\lambda\lambda}^{l60})+4\gamma_\lambda\gamma_\mu\beta^2(1-2\beta^2)(L_{t\lambda\mu}^{l42}-(v_l^2-v_t^2)L_{lt\lambda\mu}^{l62})+4\gamma_\mu^2\beta^4(L_{t\mu\mu}^{l42}-(v_l^2-v_t^2)L_{lt\mu\mu}^{l64})]\}, \quad (19)$$

where $\beta=v_t/v_l$.

These equations contain 22 complex integral expressions, which can be written in general form as

$$L_{tij}^{lmn}=\frac{1}{v_t^2\pi}\int_{-\infty}^{+\infty}\int_0^1\frac{u_1^mx^nS_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_1}du_1dx,\quad L_{ltij}^{lmn}=\frac{1}{v_l^2v_t^2\pi}\int_{-\infty}^{+\infty}\int_0^1\frac{u_1^mx^nS_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_2}du_1dx,$$

$$L_{ltpp}^{tmn}=\frac{1}{v_l^2v_t^2\pi}\int_{-\infty}^{+\infty}\int_0^1\frac{u_1^mx^nS_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_4}du_1dx,\quad L_{tij}^{tmn}=\frac{1}{v_t^2\pi}\int_{-\infty}^{+\infty}\int_0^1\frac{u_1^mx^nS_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_3}du_1dx, \quad (20)$$

where $Z_1=u_1^2-u^2/\beta^2$, $Z_2=(u_1^2-u^2/\beta^2)(u_1^2-u^2)$, $Z_3=u_1^2-u^2$, $Z_4=(u_1^2-\beta^2u^2)(u_1^2-u^2)$.

Two groups of integrals can be distinguished in Eqs. (20). These describe processes of different physical types: the integrals L_{tij}^{tmn} which do not contain β determine the contribution to the change in the dispersion relation from scattering processes for waves of one and the same type, while all the other integrals, which contain β, describe the contribution to the change in the dispersion relation owing to scattering processes with a change in the type of waves.

The integrals in Eq. (20) with respect to u_1 were calculated using the theory of residues, then the integrals with respect to x were reduced to tabulated integrals.[13]

The form of $\omega'(k)$ and $\omega''(k^2)$ for transverse waves is shown in Fig. 2 for different values of the cross correlation coefficients k_{ij} between the inhomogeneities. Figure 2a shows that without cross correlation (smooth curve) the dispersion relation for transverse waves deviates from the unperturbed dispersion relation (dotted curve) toward the low frequency side, and there is a further bend in this curve in the same direction near $k/k_c=0.5$. The appearance of positive cross correlations between the inhomogeneities in p and μ brings the dispersion curve (dashed curve) closer to the unperturbed dispersion curve and causes a smaller bend in it. A negative cross correlation causes a larger change in the dispersion relation (dash-dot curve). The damping ω'' of the transverse waves as a function of k^2 is shown in Fig. 2b. In these coordinates, to the left of the crossover point $(k/k_c)^2$

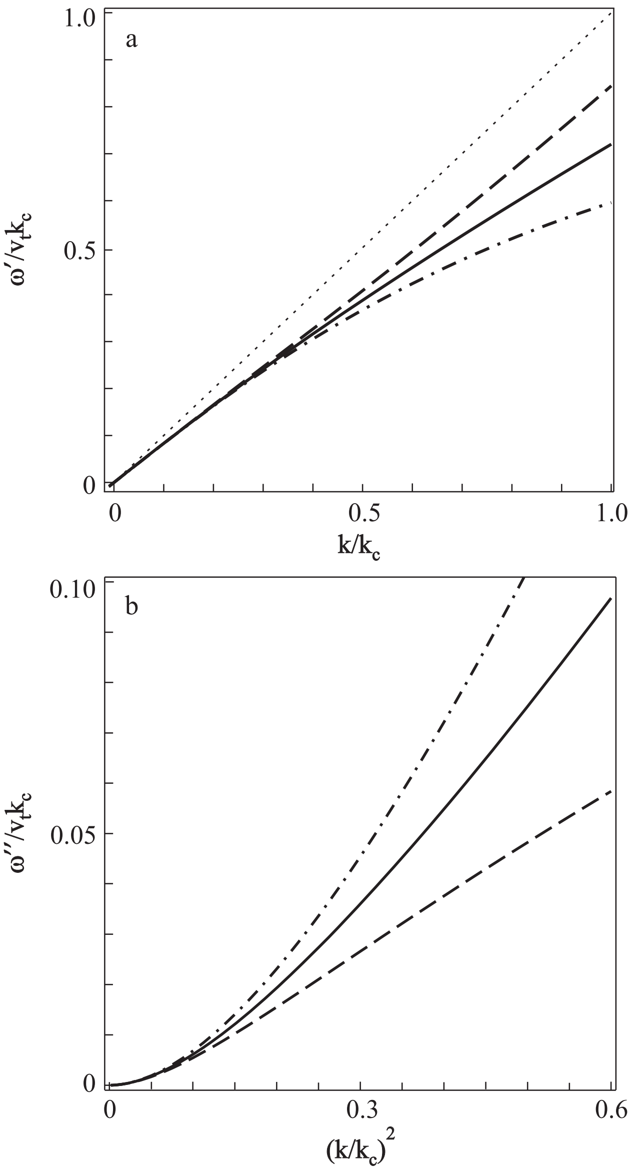

FIG. 2. Dispersion relation (a) and damping (b) for transverse elastic waves in a medium with different values of the cross correlation coefficient $\kappa_{p\mu}$ between 3D inhomogeneities in the density and force constants of the material: $\kappa_{p\mu}=0$ (smooth curves); $\kappa_{p\mu}=0.9$ (dashed curves); $\kappa_{p\mu}=-0.9$ (dash-dot curve). The dotted line indicates the dispersion relation for a homogeneous medium.

$=0.25$ the function $\omega''(k^2)$ has the form of a parabola, and to the right, it is a straight line. The appearance of positive cross correlations between the inhomogeneities in p and μ causes a reduction (dashed curve), and negative cross correlations, an increase (dash-dot curve) in the damping. In this case, therefore, cross correlations produce effects in the wave spectrum of elastic waves that are directly opposite to those produced by cross correlations between exchange and anisotropy inhomogeneities in the spectrum of spin waves. (See Fig. 1.)

For longitudinal waves, cross correlations between inhomogeneities in the density and elastic constants ($\kappa_{p\mu}$ and $\kappa_{p\lambda}$) produce effects similar to those shown in Fig. 2. However, cross correlations between the elastic constants μ and λ lead to directly opposite effects: when $\kappa_{\lambda\mu}$ is positive there are increased changes in the dispersion relation and in the damping curve, but when $\kappa_{\lambda\mu}$ is negative, the changes are smaller.

IV. DISCUSSION AND CONCLUSION

In this paper we have obtained a result, paradoxical at first glance, to the effect that positive cross correlations ($\kappa_{ij}>0$) between inhomogeneities in some material parameters cause a greater modification of the dispersion relation and increased wave damping, while cross correlations of the same sign between inhomogeneities in other parameters produce a smaller modification of the dispersion relation and reduced wave damping. Since negative ($\kappa_{ij}<0$) cross correlations always lead to effects opposite those produced by positive cross correlations, their effect is similarly ambiguous.

This ambiguity is eliminated if we take note of the fact that the parameter p goes with the kinetic part of the hamiltonian, while μ and λ, like the parameters α and β, go with the potential part. This suggests that the effect of cross correlations between parameters changes, depending on whether the two parameters both belong to one or the other part of the hamiltonian., or the two belong to different parts of the hamiltonian.

The physical mechanism for these differences in the effect of cross correlations can be understood from the following simplified model of an inhomogeneous medium. The dispersion relation for longitudinal elastic waves in a homogeneous isotropic medium is

$$\omega=\left(\frac{\lambda+2\mu}{p}\right)^{1/2}k. \tag{21}$$

In the case of very smooth inhomogeneities with a characteristic size $2r_c$ much greater than the wavelength ($k_c \ll k$), the medium can be regarded approximately as consisting of a set of homogeneous regions of size $2r_c$ within which the parameters p, μ, and λ of the medium are constant, while they are different in different regions (a model of independent grains, or crystallites). If the fluctuations in p, μ, and λ are uncorrelated among themselves, then the frequency for a given k can be substantially different in the different regions, since there will be regions in which a simultaneous increase in μ (or λ) and decrease in p (or vice versa) occur relative to the average values of the quantities, while the frequency of the wave is determined by the ratio of $\lambda+2\mu$ to p. Positive cross correlations lead to spatial synchronization of the fluctuations in the two random functions without changing the magnitudes of the mean square deviations of either function. Thus, now in each of our regions the deviation (with arbitrary sign) of μ (or λ) from its average value corresponds to a deviation of the same sign of p from its average. As a result, the random spread in the frequencies of the waves in different regions of the material decreases. We can even imagine a hypothetical limiting case in which the elastic wave frequency is essentially the same throughout all space, despite large deviations in μ (or λ) and p in different regions of space. Negative cross correlations lead to spatial synchronization of the deviations in μ (or λ) and p with opposite signs and, accordingly, to an even greater spread in the frequencies of the waves in different regions than for the case of $\kappa_{p\mu}=0$. This example can also be used in a qualitative examination of the effect of cross correlations between λ and μ. When these cross correlations are absent, besides cases of simultaneous increases or decreases in λ and μ, there are cases where μ decreases (increases) when λ increases (decreases). Including positive cross correlations eliminates these last cases: now λ and μ increase or decrease synchronously; this enhances the spread in the values of ω in different regions of the material. A similar qualitative examination of the effect of cross correlations between different parameters of the hamiltonian can be made using a simplified model of an inhomogeneous medium for the case of spin waves.

In Eq. (21), as in any expression for the frequency of a wave, the parameters corresponding to the potential part of the hamiltonian (here λ and μ) appear in the numerator while the parameters corresponding to the kinetic part of the hamiltonian (here p) appear in the denominator. Accordingly, an analysis of the results obtained in this paper, along with the qualitative treatment based on a model of independent grains, makes it possible to formulate a general rule for the effect of cross correlations that is independent of the physical nature of the waves: the way cross correlations between inhomogeneities in any two parameters of a substance affect the wave spectrum is determined by whether the two parameters coupled by the cross correlations belong to the same part of the hamiltonian (i.e., both belong to the kinetic, or to the potential part) or they belong to different parts of the hamiltonian. In the first case positive cross correlations lead to greater modifications in the dispersion relation and wave damping, and in the second, to reductions in these characteristics. Correspondingly, negative cross correlations lead to the opposite effects in both these cases.

This work was partially supported by Program No. 27 of the Presidium of the Russian Academy of Sciences, State contract No. 02.740.11.0220 as part of the Federal targeted program and the State program “Development of the scientific potential of higher education,” project No. 2.1.1/3498.

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Translated by D. H. McNeill