ISSN 1063-7761, Journal of Experimental and Theoretical Physics, 2010, Vol. 110, No. 2, pp. 345–359. © Pleiades Publishing, Inc., 2010.
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STATISTICAL, NONLINEAR, AND SOFT MATTER PHYSICS

Effects of Cross Correlations between Inhomogeneities of the Parameters of an Isotropic Medium on the Spectrum and Damping of Elastic Waves

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Received June 29, 2009

Abstract—The dispersion and damping laws have been investigated for elastic waves in an isotropic medium with one- and three-dimensional inhomogeneities of the density $p(\mathbf{x})$ of the material and the elastic force constants $\mu(\mathbf{x})$ and $\lambda(\mathbf{x})$ with allowance for the cross correlations between these inhomogeneities. It has been demonstrated that the positive cross correlations between $\mu(\mathbf{x})$ and $\lambda(\mathbf{x})$, as well as the negative cross correlations between $p(\mathbf{x})$ and $\mu(\mathbf{x})$ or $p(\mathbf{x})$ and $\lambda(\mathbf{x})$, lead to an enhancement of the modification of the dispersion law and an increase in the damping of waves. The positive cross correlations between $p(\mathbf{x})$ and $\mu(\mathbf{x})$ or $p(\mathbf{x})$ and $\lambda(\mathbf{x})$, as well as the negative cross correlations between $\mu(\mathbf{x})$ and $\lambda(\mathbf{x})$, result in the opposite effects: a weakening of the modification of the dispersion law and a decrease in the damping. An analysis of the results obtained in this paper and in our recent work [15] has made it possible to formulate the general regularity of the effects of cross correlations, irrespective of the physical nature of the waves: the effects of cross correlations between inhomogeneities of any two parameters of the material on the wave spectrum depend on whether both parameters related by the cross correlations belong to the same part of the Hamiltonian (i.e., if they both belong to either the kinetic part or the potential part of the Hamiltonian) or they belong to different parts of the Hamiltonian. The positive cross correlations lead to a greater modification of the dispersion law and to an increase in the damping of waves in the former case and to a decrease in these characteristics in the latter case. Correspondingly, the negative cross correlations in each of these cases result in the opposite effects. This regularity has been explained qualitatively.

DOI: 10.1134/S1063776110020184

1. INTRODUCTION: AUTOCORRELATIONS AND CROSS CORRELATIONS

Amorphous and nanocrystalline materials have been widely used in various modern electronic devices based on the propagation and transformation of electromagnetic, elastic, and spin waves. Theoretically, these materials are characterized by the two main properties: (1) inhomogeneity of all parameters of the Hamiltonian (density of the material; elastic force constants; exchange, magnetic anisotropy, and other parameters), and (2) extended correlations of these inhomogeneities for which the correlation length is determined by both the topological and compositional disorders and can vary over a wide range (several tens and several hundreds of interatomic distances). The existence of large correlation lengths renders it impossible to use well-developed theoretical methods that take into account effects of uncorrelated (δ-correlated) inhomogeneities for calculating a number of effects in these materials.

Effects of inhomogeneities with arbitrary correlation lengths on the spectrum and damping of spin waves in terms of the continuum model were taken into account in the first nonvanishing order of perturbation theory in our earlier works [1–3]. Later, in the same approximation, effects of correlated inhomogeneities on the spin wave spectrum were included in the lattice model of a ferromagnet [4, 5] and in the continuum model [6]. Effects of inhomogeneities with arbitrary correlation lengths on the spectrum and damping of elastic waves in the isotropic medium were taken into account in [1, 7, 8].

In these randomly inhomogeneous media, the frequency $\omega'(k)$ and damping $\omega''(k)$ of waves are functionals of the correlation functions that describe the stochastic properties of spatial functions of the parameters of the medium. In [1–6, 8], effects of each fluctuating parameter of the medium $A_i(\mathbf{x})$ (where $\mathbf{x} = \{x, y, z\}$) were considered separately: for example, it was assumed that the exchange constant $\alpha(\mathbf{x})$ is inhomogeneous and all other spin Hamiltonian parameters are constant; then, the problem of effects of inhomogeneities of the magnetic anisotropy $\beta(\mathbf{x})$ was analyzed under similar conditions, etc.

The random function $A_i(\mathbf{x})$ for each ith parameter of the medium was represented in the form

$$A_i(\mathbf{x}) = A_i[1 + \gamma_i \rho_i(\mathbf{x})], \quad (1)$$

where A_i and γ_i are the mean value and the relative root-mean-square deviation of the function $A_i(\mathbf{x})$, respectively, and $\rho_i(\mathbf{x})$ is the centered ($\langle \rho_i(\mathbf{x}) \rangle = 0$) and normalized ($\langle \rho_i^2(\mathbf{x}) \rangle = 1$) homogeneous random function of the coordinates. The stochastic characteristics of the random function $\rho_i(\mathbf{x})$ are described by the autocorrelation function $K_{ii}(\mathbf{r})$ or the spectral density $S_{ii}(\mathbf{k})$ of inhomogeneities (related to the autocorrelation function by the Fourier transform):

$$K_{ii}(\mathbf{r}) = \langle \rho_i(\mathbf{x}) \rho_i(\mathbf{x} + \mathbf{r}) \rangle,$$
$$S_{ii}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int K_{ii}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}, \quad (2)$$

where $|\mathbf{r}|$ is the distance between two points in the space and the angle brackets indicate averaging over an ensemble of realizations of the random function $\rho_i(\mathbf{x})$. From physical considerations and analysis of a number of exactly solvable models (see, for example, [9–12]), the autocorrelation functions $K_{ii}(\mathbf{r})$ were simulated by rapidly decreasing functions with arbitrary correlation lengths r_{ii} (exponential, Gaussian, and Karman functions were examined). The quantity $2r_{ii}$ determines the length of correlated (and, correspondingly, shortest) fluctuations. In the particular case of a polycrystal (nanocrystal), in the absence of any disorder inside each crystallite, the quantity $2r_{ii}$ corresponds to the mean size of the crystallite. In this case, the spectral density $S_{ii}(\mathbf{k})$ of inhomogeneities is also a monotonically decreasing function with the spectrum cut by the correlation wave number $k_{ii} = r_{ii}^{-1}$.

It should be noted that, in some specific cases, inhomogeneities can arise that are described by nonmonotonically decreasing correlation functions. Krivoglaz [13], who investigated the specific features of X-ray scattering in supersaturated solid solutions, was most likely the first to call attention to the fact that, in these media, the mean concentration profile is disturbed only in small local volumes. According to the law of conservation of atoms in each local volume, the neighboring positive and negative half-waves of deviation of the concentration from the mean concentration appear correlate with each other. Therefore, the correlation function of the concentration is also characterized by the negative half-wave after the positive half-wave and the integral of the autocorrelation function $K_{ii}(\mathbf{r})$ over the volume vanishes. In our previous works (see references in review [14]), we investigated the effects caused by these correlation functions. However, the topological disorder, which is typical of amorphous and nanocrystalline media, does not obey the local laws of conservation and leads to monotonically decreasing correlation functions [9–12], which will be studied in the present work.

Therefore, the inhomogeneity of each parameter of the medium adds two arbitrary constants to the theory: the root-mean-square deviation γ_i and the correlation length r_{ii} (or the correlation wave number k_{ii}) of the fluctuations of this parameter A_i. The corresponding quantities either should be determined from a comparison of the developed phenomenological theory with experimental data or should be calculated from microscopic models of inhomogeneities in this particular medium.

The main result of the theory developed in [1–3] is that, in the vicinity of the wave number $k = k_{ii}/2$, the laws of dispersion $\omega'(k)$ and damping $\omega''(k)$ should be changed and the corresponding change should be different for inhomogeneities of different physical parameters. In particular, the exchange inhomogeneity leads to a bending of the dispersion curve $\omega'(k)$ for spin waves toward smaller values of the frequency ω' in the range $k \sim k_{ii}/2$, whereas the inhomogeneity of the magnetic moment results in a bending toward larger values of the frequency ω'. The dispersion curve $\omega'(k)$ for elastic waves also has different bendings in this range in the case of inhomogeneities of both the elastic constants and the density of the material. The magnitudes of these effects are proportional to γ_i^2. The qualitative character of the modification of the dispersion curves $\omega'(k)$ and $\omega''(k)$ does not depend on the form of the simulating correlation function when this function is characterized by a sufficiently rapid decay of the correlations. All these effects are associated with the difference in the wave scattering from correlated ($k > k_{ii}/2$) and uncorrelated ($k \ll k_{ii}/2$) regions of inhomogeneities. This theory was used to develop the experimental method of correlation spin-wave spectroscopy, which made it possible to measure the correlation lengths of inhomogeneities r_{ii} and the root-mean-square deviations γ_i in many amorphous and nanocrystalline magnetic alloys [14] and to determine the dependences of the quantities r_{ii} and γ_i on the composition and heat treatment of the alloys.

The theory in which the inhomogeneities of each parameter are considered separately is approximately valid in a number of cases. One of them is the situation where the contribution from fluctuations of one of the parameters of the medium to the modification of the dependences $\omega'(k)$ and $\omega''(k)$ is dominant. A typical example is provided by electromagnetic waves for which the equations, as a rule, contain only one inhomogeneous parameter, i.e., the permittivity. Another case corresponds to the situation where the modifications in the wave spectrum due to the inhomogeneities of different parameters are observed in different ranges of wave numbers k (owing to the considerable difference between their correlation lengths r_{ii}) and these inhomogeneities can be treated approximately as statistically independent. In all these cases, autocorrelation functions of type (2) are usually referred to as correlation functions.

However, in the general case, the averaging of stochastic wave equations containing several inhomogeneous parameters $A_i(\mathbf{x})$ leads to the fact that the quantities $\omega'(k)$ and $\omega''(k)$, like all nonrandom characteristics of a random system, become functionals of not only the autocorrelation functions $K_{ii}(\mathbf{r})$ of each parameter A_i but also the functions of mutual correlations (cross correlations) $K_{ij}(\mathbf{r})$ between the parameters (or their spectral densities $S_{ij}(\mathbf{k})$):

$$K_{ij}(\mathbf{r}) = \langle \rho_i(\mathbf{x})\rho_j(\mathbf{x}+\mathbf{r})\rangle,$$
$$S_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int K_{ij}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}, \quad (3)$$

where $i \neq j$ and the averaging is performed over an ensemble of realizations of both random functions $\rho_i(\mathbf{x})$ and $\rho_j(\mathbf{x})$.

Thus, an analysis of averaged equations describing the wave spectrum and damping of an inhomogeneous medium should be preceded by the calculation or physically justified simulation of the functions $K_{ii}(\mathbf{r})$ and $K_{ij}(\mathbf{r})$. Consistent calculations of these functions can be performed for only a limited number of the simplest cases. In particular, in our earlier work [7], the dispersion law $\omega'(k)$ and damping $\omega''(k)$ in an isotropic medium were investigated for the case where the elastic waves $\mathbf{u}(\mathbf{x}, t)$ propagate against the background of the topological disorder described by the metastable isotropic strains $\varepsilon(\mathbf{x}) \equiv u^0_{ii}(\mathbf{x})$. In this model, the density of the material $p(\mathbf{x})$ and the elastic constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ depend on the same random function $\varepsilon(\mathbf{x})$. The function $p(\mathbf{x})$ can be explicitly written, and the elastic constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ can be represented in the form of an expansion into a power series in this function. As a result, all autocorrelation functions K_{ii} and all cross correlation functions K_{ij} that enter into the expressions for the quantities $\omega'(k)$ and $\omega''(k)$ after the averaging of the elasticity equations were expressed through the only correlation function of the random quantity $\varepsilon(\mathbf{x})$. Although this representation led to the addition of a number of arbitrary constants (coefficients of the expansion of the parameters of the medium λ and μ into a power series in terms of ε) to the theory, the total number of arbitrary constants in the theory decreased drastically because only one correlation length and one root-mean-square fluctuation of the random function $\varepsilon(\mathbf{x})$ entered into the theory instead of the correlation lengths r_{ii} and r_{ij}.

However, these simplest cases when all parameters of the medium can be represented as exact functions of the topological or compositional disorder of the alloy are more likely the exceptions. Materials with a quite complex composition that contains up to five or six different components required for imparting any special properties to the material (for example, zero average magnetostriction) have been studied and used in modern materials science. Therefore, in the general case, it has been assumed that, between each two parameters of the material, there are correlations such that a spatial fluctuation of one parameter in any neighborhood of the point $\mathbf{x}$ will favor the appearance of a spatial fluctuation of the other parameter in the same neighborhood. However, in this case, the fluctuation of the latter parameter should not reproduce exactly the form of the fluctuation of the former parameter. In the theory of random fields, these correlations are taken into account by the functions $K_{ij}(\mathbf{r})$ of mutual correlations (cross correlations) between these two parameters. Unlike the autocorrelation function $K_{ii}(\mathbf{r})$, which is equal to unity at $\mathbf{r} = 0$, the function defined by relationship (3) at $\mathbf{r} = 0$ is equal to some dimensionless coefficient κ_{ij}. This coefficient characterizes the magnitude and sign of the cross correlations between the parameters A_i and A_j and can take on arbitrary values in the range between -1 and $+1$. A particular magnitude and sign of the coefficient κ_{ij} should be determined from experiments or calculated using a microscopic model that includes real and, in some cases, complex physical relations between the parameters A_i and A_j. This formalized description makes it possible to study effects of cross correlations on the spectrum of the system at the first stage in the general form without detailed discussion of physical mechanisms that result in the appearance of these cross correlations.

In the limiting cases $\kappa_{ij} = \pm 1$, the stochastic cross correlations transform into deterministic relations between the inhomogeneities of different parameters. For $\kappa_{ij} = 1$, the random functions $\rho_i(\mathbf{x})$ and $\rho_j(\mathbf{x})$ coincide with each other. For $\kappa_{ij} = -1$, these functions are mirror images of each other: negative deviations of the function $\rho_i(\mathbf{x})$ with the same magnitude and form correspond to positive deviations of the function $\rho_j(\mathbf{x})$ and vice versa. For the isotropic elastic system, the special case $\kappa_{ij} = 1$ coincides with the model of the functional dependence between the parameters $p(\mathbf{x})$, $\lambda(\mathbf{x})$, and $\mu(\mathbf{x})$, which was investigated in [7]. In the general case, the presence of cross correlations does not change the root-mean-square deviations γ_i and γ_j of the random functions $A_i(\mathbf{x})$ and $A_j(\mathbf{x})$ and leads to a partial stochastic spatial synchronization of these functions with the degree determined by the magnitude of the cross correlation coefficient κ_{ij}.

In our previous work [15], we investigated effects of cross correlations between inhomogeneities of the exchange parameter $\alpha(\mathbf{x})$ and the magnetic anisotropy parameter $\beta(\mathbf{x})$ on the spectrum and damping of spin waves in a ferromagnet. It was demonstrated that the positive cross correlations between these parameters result in an enhancement of the modification of the dispersion law and in an increase in the damping of spin waves. The negative cross correlations lead to the opposite effects: a weakening of the modification of the dispersion law and a decrease in the damping of spin waves.

The purpose of this paper is to calculate the combined effects of inhomogeneities of the force constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ and the density of the material $p(\mathbf{x})$ on the spectrum and damping of elastic waves in the isotropic medium in the presence of cross correlations with an arbitrary magnitude and sign between the random functions $\mu(\mathbf{x})$ and $\lambda(\mathbf{x})$, $p(\mathbf{x})$ and $\mu(\mathbf{x})$, and $p(\mathbf{x})$ and $\lambda(\mathbf{x})$.

2. THE MODEL AND METHOD: ONE-DIMENSIONAL INHOMOGENEITIES

Let us consider the model of an isotropic elastic medium in which the force constants $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ and the density of the material $p(\mathbf{x})$ (where $\mathbf{x} = \{x, y, z\}$) are inhomogeneous. The equation of motion of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ has the form

$$\begin{aligned} &-p(\mathbf{x})\frac{\partial^2 u_s}{\partial t^2} + \frac{\partial}{\partial x_s}\left(\lambda(\mathbf{x})\frac{\partial u_f}{\partial x_f}\right) \\ &+ \frac{\partial}{\partial x_f}\left(\mu(\mathbf{x})\frac{\partial u_s}{\partial x_f}\right) + \frac{\partial}{\partial x_f}\left(\mu(\mathbf{x})\frac{\partial u_f}{\partial x_s}\right) = 0, \end{aligned} \tag{4}$$

where the indices s and f take on values x, y, and z and the summation over all coordinates is performed for the doubly repeated index f. The parameters $p(\mathbf{x})$, $\lambda(\mathbf{x})$, and $\mu(\mathbf{x})$ dependent on the coordinates can be represented in the form

$$\begin{aligned} p(\mathbf{x}) &= p[1 + \gamma_p\rho_p(\mathbf{x})], \quad \gamma_p = \Delta p/p, \\ \lambda(\mathbf{x}) &= \lambda[1 + \gamma_\lambda\rho_\lambda(\mathbf{x})], \quad \gamma_\lambda = \Delta\lambda/\lambda, \\ \mu(\mathbf{x}) &= \mu[1 + \gamma_\mu\rho_\mu(\mathbf{x})], \quad \gamma_\mu = \Delta\mu/\mu, \end{aligned} \tag{5}$$

where p and Δp, λ and $\Delta\lambda$, and μ and $\Delta\mu$ are the means and the root-mean-square fluctuations of the above parameters, respectively, and $\rho_p(\mathbf{x})$, $\rho_\lambda(\mathbf{x})$, and $\rho_\mu(\mathbf{x})$ are the dimensionless centered ($\langle\rho_p(\mathbf{x})\rangle = 0$, $\langle\rho_\lambda(\mathbf{x})\rangle = 0$, $\langle\rho_\mu(\mathbf{x})\rangle = 0$) and normalized ($\langle\rho_p^2(\mathbf{x})\rangle = 1$, $\langle\rho_\lambda^2(\mathbf{x})\rangle = 1$, $\langle\rho_\mu^2(\mathbf{x})\rangle = 1$) random functions of the coordinates. Braces indicate averaging over an ensemble of realizations of the corresponding random functions.

The laws of dispersion and damping of elastic waves are derived using one-dimensional inhomogeneities as an example. In this case, the displacement vector $\mathbf{u}$ is a function of only one coordinate z and the time t. For the transverse components u_t of the vector $\mathbf{u}$, Eq. (4) takes the form

$$\begin{aligned} &\frac{\partial^2 u_t}{\partial t^2} - v_t^2\frac{\partial^2 u_t}{\partial z^2} = -\frac{\Delta p}{p}\rho_p(z)\frac{\partial^2 u_t}{\partial t^2} \\ &+ \frac{\Delta\mu}{p}\left[\rho_\mu(z)\frac{\partial^2 u_t}{\partial z^2} + \frac{\partial\rho_\mu(z)}{\partial z}\frac{\partial u_t}{\partial z}\right], \end{aligned} \tag{6}$$

For the longitudinal component u_l, we have

$$\begin{aligned} &\frac{\partial^2 u_l}{\partial t^2} - v_l^2\frac{\partial^2 u_l}{\partial z^2} = -\frac{\Delta p}{p}\rho_p(z)\frac{\partial^2 u_l}{\partial t^2} \\ &+ \frac{\Delta\lambda}{p}\left[\rho_\lambda(z)\frac{\partial^2 u_l}{\partial z^2} + \frac{\partial\rho_\lambda(z)}{\partial z}\frac{\partial u_l}{\partial z}\right] \\ &+ \frac{2\Delta\mu}{p}\left[\rho_\mu(z)\frac{\partial^2 u_l}{\partial z^2} + \frac{\partial\rho_\mu(z)}{\partial z}\frac{\partial u_l}{\partial z}\right]. \end{aligned} \tag{7}$$

Here, $v_t = \sqrt{\mu/p}$ and $v_l = \sqrt{(\lambda + 2\mu)/p}$ are the velocities of the transverse and longitudinal waves, respectively.

By assuming that $\mathbf{u}(z, t) \sim e^{-i\omega t}\mathbf{u}(z)$ and performing the Fourier transform with respect to z,

$$\begin{aligned} \mathbf{u}(z) &= \int\mathbf{u}(k)e^{ikz}dk, \\ \mathbf{u}(k) &= \frac{1}{2\pi}\int\mathbf{u}(z)e^{-ikz}dz, \end{aligned} \tag{8}$$

where k is the wave vector, from Eqs. (6) and (7), we obtain the equations for the Fourier transforms of the function $\mathbf{u}(z)$:

$$\begin{aligned} &(\omega^2 - v_t^2k^2)u_t(k) = -\omega^2\gamma_p\int_{-\infty}^{\infty}\rho_p(k - k_1)u_t(k_1)dk_1 \\ &+ \frac{\mu}{p}\gamma_\mu k\int_{-\infty}^{\infty}k_1\rho_\mu(k - k_1)u_t(k_1)dk_1, \end{aligned} \tag{9}$$

$$\begin{aligned} &(\omega^2 - v_l^2k^2)u_l(k) = -\omega^2\gamma_p\int_{-\infty}^{\infty}\rho_p(k - k_1)u_l(k_1)dk_1 \\ &+ \frac{\lambda}{p}\gamma_\lambda k\int_{-\infty}^{\infty}k_1\rho_\lambda(k - k_1)u_l(k_1)dk_1 \\ &+ \frac{2\mu}{p}\gamma_\mu k\int_{-\infty}^{\infty}k_1\rho_\mu(k - k_1)u_l(k_1)dk_1. \end{aligned} \tag{10}$$

We average Eqs. (9) and (10) over random realizations of the functions $\rho_p(k - k_1)$, $\rho_\lambda(k - k_1)$, and $\rho_\mu(k - k_1)$ and decouple the formed correlators $\langle\rho u\rangle$ in the first nonvanishing order of the perturbation theory. The scheme of decoupling can be illustrated using Eq. (9) as an example. After averaging, this equation takes the form

$$(\omega^2 - v_t^2 k^2)\langle u_t(k)\rangle$$

$$= -\omega^2\gamma_p \int_{-\infty}^{\infty} \langle \rho_p(k-k_1)u_t(k_1)\rangle dk_1 \quad (11)$$

$$+ \frac{\mu}{p}\gamma_\mu k \int_{-\infty}^{\infty} k_1 \langle \rho_\mu(k-k_1)u_t(k_1)\rangle dk_1.$$

At the first step, we decouple the mean of the products of the functions ρ_i ($i = p$, μ) and u_t in Eq. (11) according to the general rules into the product of the means and the correlator of the product of the centered values of these functions; that is,

$$\langle \rho_i u_t\rangle = \langle \rho_i\rangle\langle u_t\rangle + \langle \rho_i \mathring{u}_t\rangle, \quad (12)$$

where

$$\mathring{u}_t(k) = u_t(k) - \langle u_t(k)\rangle. \quad (13)$$

The product of the means in relationship (12) vanishes because the functions ρ_i are centered. Therefore, the averaged equation (11) should not contain terms proportional to the first powers of the quantities γ_i and correlators of the type $\langle \rho_i \mathring{u}_t\rangle$ are retained in the integral terms of this equation.

At the second step, the quantity $u_t(k)$ is formally expressed from Eqs. (9) with an increase in the indices on the wave number k by unity in this relationship:

$$u_t(k_1) = -\omega^2\gamma_p \int_{-\infty}^{\infty} \frac{\rho_p(k_1-k_2)u_t(k_2)dk_2}{\omega^2 - v_t^2 k_1^2}$$
$$+ \frac{\mu}{p}\gamma_\mu k_1 \int_{-\infty}^{\infty} \frac{k_2\rho_\mu(k_1-k_2)u_t(k_2)dk_2}{\omega^2 - v_t^2 k_1^2}. \quad (14)$$

By subtracting the quantity $\langle u_t(k_1)\rangle$ from this expression, we substitute the centered value $\mathring{u}_t$ into the correlators $\langle \rho_i \mathring{u}_s\rangle$ of Eq. (11). In this case, the terms $\langle \rho_i(u_t)\rangle$ vanish because the functions ρ_i are centered. Therefore, the performed operation is equivalent to the direct substitution of expression (14) into Eq. (11), which takes the form

$$(\omega - v_t^2 k^2)\langle u_t(k)\rangle = \omega^4\gamma_p^2$$

$$\times \iint \frac{\langle \rho_p(k-k_1)\rho_p(k_1-k_2)u_t(k_2)\rangle dk_1 dk_2}{\omega^2 - v_t^2 k_1^2}$$

$$-\omega^2\gamma_p\gamma_\mu\frac{\mu}{p}$$

$$\times \iint \frac{kk_1\langle \rho_\mu(k-k_1)\rho_p(k_1-k_2)u_t(k_2)\rangle dk_1 dk_2}{\omega^2 - v_t^2 k_1^2} \quad (15)$$

$$-\omega^2\gamma_p\gamma_\mu\frac{\mu}{p}$$

$$\times \iint \frac{k_1 k_2\langle \rho_p(k-k_1)\rho_\mu(k_1-k_2)u_t(k_2)\rangle dk_1 dk_2}{\omega^2 - v_t^2 k_1^2}$$

$$+\gamma_\mu^2\frac{\mu^2}{p^2}\iint \frac{kk_2k_1^2\langle \rho_\mu(k-k_1)\rho_\mu(k_1-k_2)u_t(k_2)\rangle dk_1 dk_2}{\omega^2 - v_t^2 k_1^2}.$$

By carrying out the same transformations with Eq. (10) for the longitudinal waves, we bring it into a form similar to Eq. (15) but more cumbersome because, apart from the terms proportional to γ_p^2, γ_μ^2, and $\gamma_p\gamma_\mu$, this equation contains the terms proportional to γ_λ^2, $\gamma_\rho\gamma_\lambda$, and $\gamma_\lambda\gamma_\mu$. In these equations, the means of the products of three random functions under the integral sign are decoupled in the first nonvanishing order of perturbation theory (the Bourret approximation [16]); that is,

$$\langle u_s(k_2)\rho_i(k-k_1)\rho_j(k_1-k_2)\rangle$$
$$\approx \langle u_s(k_2)\rangle\langle \rho_i(k-k_1)\rho_j(k_1-k_2)\rangle, \quad (16)$$

where $s = t$, l and each of the subscripts i and j takes on values p, λ, and μ. In this relationship, the correlator $\langle \mathring{u}_s(k_2)\mathring{P}_{ij}\rangle$ (where $\mathring{P}_{ij} = \rho_i\rho_j - \langle \rho_i\rho_j\rangle$) on the right-side is rejected. The substitution of expression (14) into this correlator with an increase in the indices on the wave number k by unity would lead to the next term of the expansion of the perturbation theory, etc.

Since $\rho_i(z)$ and $\rho_j(z)$ are homogeneous random functions, they satisfy the relationship

$$\langle \rho_i(k')\rho_j^*(k'')\rangle = S_{ij}(k')\delta(k'-k''), \quad (17)$$

where $S_{ij}(k)$ are the components of the spectral density matrix of the random functions $\rho_i(k)$) and $\rho_j(k)$. The components of the correlation matrix of the random functions $\rho_i(z)$ and $\rho_j(z)$ are defined by the relationship

$$K_{ij}(r) = \langle \rho_i(z)\rho_j(z+r)\rangle, \quad (18)$$

where r is the distance between two points. The diagonal components ($i = j$) of the correlation matrix are autocorrelation functions of the ith inhomogeneous parameter, and the off-diagonal components ($i \neq j$) describe the cross correlations between the ith and jth parameters. The components $K_{ij}(r)$ and $S_{ij}(r)$ are related by the Fourier transform (the Wiener–

Khintchin theorem for homogeneous random functions)

$$K_{ij}(r) = \int S_{ij}(k)e^{ikr}dk,$$
$$S_{ij}(k) = \frac{1}{2\pi}\int K_{ij}(r)e^{-ikr}dr. \tag{19}$$

Taking into account expressions (16) and (17), we perform the integration in Eq. (15) and the corresponding equation for $\langle u_l\rangle$ over k_2. Then, the quantities $\langle u_t(k)\rangle$ and $\langle u_l(k)\rangle$ can be removed from the integral sign, and we obtain the complex dispersion relations in the following general form:
for the transverse waves,

$$\omega^2 - v_t^2k^2 = \omega^4\gamma_p^2L_{pp}^t - 2\omega^2\gamma_p\gamma_\mu v_t^2kL_{p\mu}^t + \gamma_\mu^2 v_t^4k^2L_{\mu\mu}^t; \tag{20}$$

and for the longitudinal waves,

$$\begin{aligned}\omega^2 - v_l^2k^2 &= \omega^4\gamma_p^2L_{pp}^l - 2\omega^2\gamma_p\gamma_\lambda(v_l^2 - 2v_l^2)kL_{p\lambda}^l \\ &- 4\omega^2\gamma_p\gamma_\mu v_t^2k^2L_{p\mu}^l + \gamma_\lambda^2(v_l^2 - 2v_t^2)^2k^2L_{\lambda\lambda}^l \\ &+ 4\gamma_\lambda\gamma_\mu v_t^2(v_l^2 - 2v_t^2)k^2L_{\lambda\mu}^l + 4\gamma_\mu^2v_t^4L_{\mu\mu}^l.\end{aligned} \tag{21}$$

Here,

$$\begin{aligned}L_{pp}^t &= \int_{-\infty}^{\infty}\frac{S_{pp}(k-k_1)}{\omega^2 - v_t^2k_1^2}dk_1,\\ L_{p\mu}^t &= \int_{-\infty}^{\infty}\frac{k_1S_{p\lambda}(k-k_1)}{\omega^2 - v_t^2k_1^2}dk_1,\\ L_{\mu\mu}^t &= \int_{-\infty}^{\infty}\frac{k_1^2S_{\lambda\mu}(k-k_1)}{\omega^2 - v_t^2k_1^2}dk_1,\\ L_{pp}^l &= \int_{-\infty}^{\infty}\frac{S_{pp}(k-k_1)}{\omega^2 - v_l^2k_1^2}dk_1,\\ L_{p\mu}^l = L_{p\lambda}^l &= \int_{-\infty}^{\infty}\frac{k_1S_{p\lambda}(k-k_1)}{\omega^2 - v_l^2k_1^2}dk_1,\\ L_{\lambda\lambda}^l = L_{\mu\mu}^l = L_{\lambda\mu}^l &= \int_{-\infty}^{\infty}\frac{k_1^2S_{\lambda\mu}(k-k_1)}{\omega^2 - v_l^2k_1^2}dk_1.\end{aligned} \tag{22}$$

In these relations, the terms proportional to γ_p^2, γ_λ^2, and γ_μ^2 account for effects of the inhomogeneities of the density of the material and the force constants. The terms proportional to the products $\gamma_p\gamma_\lambda$, $\gamma_p\gamma_\mu$, and $\gamma_\lambda\gamma_\mu$ include effects of cross correlations between the corresponding inhomogeneities.

It is assumed that the correlations exponentially decay for the autocorrelation functions of the density of the material $K_{pp}(r)$ and the force constants $K_{\lambda\lambda}(r)$ and $K_{\mu\mu}(r)$, as well as for the cross correlation functions between the fluctuations of the density and force constants $K_{p\lambda}(r)$ and $K_{p\mu}(r)$ and between the force constants $K_{\lambda\mu}(r)$:

$$K_{ii} = \exp(-k_{ii}r), \quad K_{ij} = \kappa_{ij}\exp(-k_{ij}r), \tag{23}$$

where $r = |x - x'|$, κ_{ij} are the dimensionless correlation coefficients lying in the range $-1 < \kappa_{ij} < 1$, $k_{ii} = r_{ii}^{-1}$ and $k_{ij} = r_{ij}^{-1}$ are the correlation wave numbers, and r_{ii} and r_{ij} are the correlation lengths.

In the general case, the correlation length r_{ii} for inhomogeneities of each parameter i can be different. Moreover, the cross correlation lengths r_{ij} between inhomogeneities of different parameters i and j can also be different. For simplicity, we restrict ourselves to the case where all correlation lengths are identical to each other. This situation can be encountered, for example, in nanocrystalline alloys in which the material in the volume of each grain (crystallite) is homogeneous, but the parameters of each grain differ from each other due to random deviations in the alloy's composition from average. In this case, all correlation lengths are approximately identical: $r_{ii} \approx r_{ij} \approx r_c$ ($k_{ii} \approx k_{ij} \approx k_c$), where the quantity $2r_c$ corresponds to the mean grain size in the nanocrystalline alloy. Generalization of the results to the case of different correlation lengths does not involve fundamental problems but leads to cumbersome expressions.

According to formula (19), the following spectral densities correspond to correlation functions (23):

$$S_{ii}(k) = \frac{1}{\pi}\frac{k_c}{k_c^2 + k^2}, \quad S_{ij}(k) = \frac{\kappa_{ij}}{\pi}\frac{k_c}{k_c^2 + k^2}. \tag{24}$$

The calculations of integrals (22) with the use of the theory of residues with these spectral densities lead to the following relationships for the integrals entering into relation (20):

$$\begin{aligned}L_{pp}^t &= A\left[a_- + v_t^2k_c^2 + \frac{v_tk_c}{\omega}i(a_+ + v_t^2k_c^2)\right],\\ L_{p\mu}^t &= kA(a_- - v_t^2k_c^2 + 2v_t^2i\omega k_c),\\ L_{\mu\mu}^t &= A\{[k^2a_- - k_c^2(a_+ + v_t^2(k^2 + k_c^2))]\\ &+ \frac{i\omega k_c}{v_t}(a_+ + v_t^2k_c^2)\},\end{aligned} \tag{25}$$

where

$$a_\pm = \omega^2 \pm v_t^2 k^2,$$
$$A = \{[(\omega - v_t k)^2 + v_t^2 k_c^2][(\omega + v_t k)^2 + v_t^2 k_c^2]\}^{-1}. \quad (26)$$

For the integrals entering into relation (21), we find

$$L_{pp}^l = B\left[b_- + v_l^2 k_c^2 + \frac{v_l k_c}{\omega} i(b_+ + v_l^2 k_c^2)\right],$$

$$L_{p\mu}^l = L_{p\lambda}^l = kB(b_- - v_l^2 k_c^2 + 2v_l^2 i\omega k_c),$$
$$L_{\lambda\lambda}^l = L_{\mu\mu}^l = L_{\lambda\mu}^l = B\{[k^2 b_- - k_c^2(b_+ + v_l^2(k^2 + k_c^2))] + \frac{i\omega k_c}{v_l}(b_+ + v_l^2 k_c^2)\}, \quad (27)$$

where

$$b_\pm = \omega^2 \pm v_l^2 k^2,$$
$$B = \{[(\omega - v_l k)^2 + v_l^2 k_c^2][(\omega + v_l k)^2 + v_l^2 k_c^2]\}^{-1}. \quad (28)$$

Relations (20) and (21) can be represented in the form

$$\omega - v_t k = \frac{R_t(\omega, k)}{\omega + v_t k}, \quad \omega - v_l k = \frac{R_l(\omega, k)}{\omega + v_l k}, \quad (29)$$

where R_t and R_l are the right-hand sides of relations (20) and (21), respectively. We consider the complex dispersion law in the first order of perturbation theory by setting $\omega \approx v_t k$ and $\omega \approx v_l k$ on the right-hand sides of expressions (29) and representing ω in the form $\omega = \omega' + i\omega''$ on the left-hand sides of these expressions. By introducing the dimensionless quantities $u = k/k_c$ and $\beta = v_t/v_l$, we derive the laws of dispersion and damping of transverse and longitudinal elastic waves with allowance for mutual correlations between inhomogeneities of different parameters in the first nonvanishing order of perturbation theory in the final form. These laws for the transverse waves have the form

$$\omega' = v_t k\left\{1 + \frac{1}{2(1+4u^2)} \times [(\gamma_p^2 + 2\kappa_{p\mu}\gamma_p\gamma_\mu)u^2 - \gamma_\mu^2(1+3u^2)]\right\}, \quad (30)$$

$$\frac{\omega''}{v_t k_c} = \frac{u^2}{2(1+4u^2)} \times [(1+2u^2)(\gamma_p^2 + \gamma_\mu^2) - 4\kappa_{p\mu}\gamma_p\gamma_\mu u^2]. \quad (31)$$

For the longitudinal waves, we obtain more complex expressions, because, in this case, they include the ratio β between the velocities of transverse and longitudinal waves:

$$\omega' = v_l k\left\{1 + \frac{u^2}{2(1+4u^2)} \times [\gamma_p^2 + 2\kappa_{p\lambda}\gamma_p\gamma_\lambda(1-2\beta^2) + 4\kappa_{p\mu}\gamma_p\gamma_\mu\beta^2] - \frac{1+3u^2}{2(1+4u^2)}[\gamma_\lambda^2(1-2\beta^2)^2 + 4\kappa_{\lambda\mu}\gamma_\lambda\gamma_\mu\beta^2(1-2\beta^2) + 4\gamma_\mu^2\beta^4]\right\}, \quad (32)$$

$$\frac{\omega''}{v_l k_c} = \frac{u^2(1+2u^2)}{2(1+4u^2)}[\gamma_p^2 + \gamma_\lambda^2(1-2\beta^2)^2 + 4\kappa_{\lambda\mu}\gamma_\lambda\gamma_\mu\beta^2(1-2\beta^2) + 4\gamma_\mu^2\beta^4] - \frac{2u^4}{1+4u^2}[\kappa_{p\lambda}\gamma_p\gamma_\lambda(1-2\beta^2) + 2\kappa_{p\mu}\gamma_p\gamma_\mu\beta^2]. \quad (33)$$

The denominator $1 + 4u^2$, which enters into all relationships (30)–(33), leads to the bending of curves $\omega'(k)$ and $\omega''(k)$ in the vicinity of $k = k_c/2$, because these curves are characterized by different asymptotics at $2u \ll 1$ and $2u \gg 1$. This is a well-known effect, which was first revealed in our earlier work [1] for both spin and elastic waves. We consider the dispersion law and damping of transverse elastic waves. The expression in braces in dispersion law (30) represents the coefficient (dependent on the wave number k) so that the difference of this coefficient from unity characterizes a change in the wave velocity due to the inhomogeneity. It can be seen that, in the absence of cross correlations, inhomogeneities γ_p of the density of the material lead to an increase in this coefficient and inhomogeneities γ_μ of the elastic constant result in its decrease. However, the quantities γ_p^2 and γ_μ^2 enter into relationship (30) with the coefficients u^2 and $1 + 3u^2$, respectively. Therefore, for identical relative fluctuations γ_p and γ_μ, the curve $\omega'(k)$ will deviate from the unperturbed dispersion law toward lower frequencies. The appearance of positive and negative cross correlations leads to a decrease and increase in this deviation, respectively. The damping of transverse waves (expression (31)) in the absence of cross correlations is proportional to the sum of γ_p^2 and γ_μ^2. The appearance of the positive mutual correlations $\kappa_{p\lambda}$ results in a decrease in the damping of waves, and the appearance of the negative mutual correlations leads to an increase in the damping of waves. This effect seems to be the most interesting, because, from general physical considerations, it can be expected that positive correlations arise between the inhomogeneities of the density and the elastic constants, which lead to the decrease in the damping of waves.

Now, we consider relationships (32) and (33) for the dispersion law and damping of longitudinal waves. In this case, the coefficient of change in the wave velocity in braces in relationship (32) also consists of the difference between two terms: the positive part contains the density fluctuations γ_p and the negative part involves fluctuations of the elastic constants γ_λ and γ_μ. The cross correlations $\kappa_{p\lambda}$, $k_{p\mu}$, and $\kappa_{\lambda\mu}$ can enter into both parts depending on their sign. The inhomogeneities of all elastic parameters γ_p, γ_λ, and γ_μ lead to an increase in the damping of waves (relationship (33)). The effects associated with the cross correlations for the longitudinal waves depend on the type of parameters for which the cross correlations between inhomogeneities manifest themselves. The difference of the component $\kappa_{p\lambda}$ or $\kappa_{p\mu}$ (or both these components) from zero leads to a decrease in the damping at $\kappa_{p\lambda} > 0$ and $\kappa_{p\mu} > 0$ and to an increase in the damping at negative cross correlations. However, the difference of the component $k_{\lambda\mu}$ from zero (for $\kappa_{p\mu} = \kappa_{p\lambda} = 0$) radically changes the situation: the damping increases at positive values of $k_{\lambda\mu}$ and decreases at $k_{\lambda\mu} < 0$.

3. THREE-DIMENSIONAL INHOMOGENEITIES

In this case, the displacement vector $\mathbf{u}$ is a function of all three coordinates and time. By assuming that $\mathbf{u}(\mathbf{x}, t) \sim \exp(-i\omega t)\mathbf{u}(\mathbf{x})$ and performing the Fourier transform of the function $\mathbf{u}(\mathbf{x})$

$$\mathbf{u}(\mathbf{x}) = \int \mathbf{u}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k},$$
$$\mathbf{u}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^3 \int \mathbf{u}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}, \tag{34}$$

from Eq. (4), we obtain the vector equation for the Fourier transform of the function $\mathbf{u}(\mathbf{x})$:

$$\begin{aligned}
&(\omega^2 - v_t^2k^2)\mathbf{u}(\mathbf{k}) - (v_l^2 - v_t^2)\mathbf{k}(\mathbf{k}\cdot\mathbf{u}(\mathbf{k})) \\
&= -\gamma_p\omega^2\int \rho_p(\mathbf{k}-\mathbf{k}_1)\mathbf{u}(\mathbf{k}_1)d\mathbf{k}_1 \\
&+ \gamma_\mu v_t^2\int (\mathbf{k}\cdot\mathbf{k}_1)\rho_\mu(\mathbf{k}-\mathbf{k}_1)\mathbf{u}(\mathbf{k}_1)d\mathbf{k}_1 \\
&+ \gamma_\lambda(v_l^2 - 2v_t^2)\mathbf{k}\int \rho_\lambda(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_1))d\mathbf{k}_1 \\
&+ \gamma_\mu v_t^2\int \mathbf{k}_1\rho_\mu(\mathbf{k}-\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{u}(\mathbf{k}_1))d\mathbf{k}_1.
\end{aligned} \tag{35}$$

With the aforementioned method, it is easy to show that the right-hand side of the averaged equation should not contain terms proportional to the first powers of the quantities γ_i. Then, we increase the indices on $\mathbf{k}$ in Eq. (35) by unity:

$$\begin{aligned}
&(\omega^2 - v_t^2k_1^2)\mathbf{u}(\mathbf{k}_1) - (v_l^2 - v_t^2)\mathbf{k}_1(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_1)) \\
&= -\gamma_p\omega^2\int \rho_p(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2 \\
&+ \gamma_\mu v_t^2\int (\mathbf{k}_1\cdot\mathbf{k}_2)\rho_\mu(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2 \\
&+ \gamma_\lambda(v_l^2 - 2v_t^2)\mathbf{k}_1\int \rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2 \\
&+ \gamma_\mu v_t^2\int \mathbf{k}_2\rho_\mu(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2.
\end{aligned} \tag{36}$$

According to this method, the function $\mathbf{u}(\mathbf{k}_1)$ should be expressed from this relationship. However, unlike the case of one-dimensional inhomogeneities, this cannot be carried out directly. Therefore, we use the following procedure. By scalarly multiplying Eq. (36) by $\mathbf{k}_1$, we find

$$\begin{aligned}
&(\omega^2 - v_l^2k_1^2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_1)) \\
&= -\gamma_p\omega^2\int \rho_p(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2 \\
&+ \gamma_\lambda(v_l^2 - 2v_t^2)k_1^2\int \rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2 \\
&+ 2\gamma_\mu v_t^2\int (\mathbf{k}_1\cdot\mathbf{k}_2)\rho_\mu(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2.
\end{aligned} \tag{37}$$

We express the scalar product $\mathbf{k}_1 \cdot \mathbf{u}(\mathbf{k}_1)$ from Eq. (37) and substitute it into the left-hand side of Eq. (36). As a result, Eq. (36) takes the form from which the function $\mathbf{u}(\mathbf{k}_1)$ can be formally expressed as follows:

$$\begin{aligned}
\mathbf{u}(\mathbf{k}_1) &= -\gamma_p\omega^2\int \frac{\rho_p(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2}{\omega^2 - v_t^2k_1^2} \\
&- \gamma_p\omega^2(v_l^2 - v_t^2)\int \frac{\mathbf{k}_1\rho_p(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2 - v_t^2k_1^2)(\omega^2 - v_l^2k_1^2)} \\
&+ \gamma_\lambda(v_l^2 - 2v_t^2)\int \frac{\mathbf{k}_1\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{\omega^2 - v_t^2k_1^2} \\
&+ \gamma_\mu v_t^2\int \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)\mathbf{u}(\mathbf{k}_2)d\mathbf{k}_2}{\omega^2 - v_t^2k_1^2} \\
&+ \gamma_\lambda(v_l^2 - 2v_t^2)(v_l^2 - v_t^2) \\
&\times \int \frac{k_1^2\mathbf{k}_1\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2 - v_t^2k_1^2)(\omega^2 - v_l^2k_1^2)} \\
&+ \gamma_\mu v_t^2\int \frac{\mathbf{k}_2\rho_\lambda(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{\omega^2 - v_t^2k_1^2}
\end{aligned} \tag{38}$$

$$+ 2\gamma_\mu v_t^2(v_l^2 - v_t^2)$$
$$\times \int \frac{\mathbf{k}_1(\mathbf{k}_1 \cdot \mathbf{k}_2)\rho_\lambda(\mathbf{k}_1 - \mathbf{k}_2)(\mathbf{k}_1 \cdot \mathbf{u}(\mathbf{k}_2))d\mathbf{k}_2}{(\omega^2 - v_t^2 k_1^2)(\omega^2 - v_l^2 k_1^2)}.$$

We substitute this relationship into the right-hand side of Eq. (35) and average the derived equation over random realizations of the random functions $\rho_p(\mathbf{k})$, $\rho_\mu(\mathbf{k})$, and $\rho_\lambda(\mathbf{k})$. Since the general relationships are complex, we initially demonstrate further calculations for the simplest situation where only the density of the material is inhomogeneous and both force constants are homogeneous ($\gamma_p \neq 0$, $\gamma_\mu = \gamma_\lambda = 0$). In this case, the averaged vector equation is equivalent to the following system of equations for the averaged transverse $\langle u_t(\mathbf{k})\rangle$ and longitudinal $\langle u_l(\mathbf{k})\rangle$ waves:

$$\{(\omega^2 - v_t^2 k^2)\delta_{sf} - (v_l^2 - v_t^2)k_s k_f\}\langle u_f(\mathbf{k})\rangle$$
$$= \gamma_p^2\omega^4\delta_{sf}\iint \frac{\langle \rho_p(\mathbf{k}-\mathbf{k}_1)\rho_p(\mathbf{k}_1-\mathbf{k}_2)u_f(\mathbf{k}_2)\rangle d\mathbf{k}_1 d\mathbf{k}_2}{\omega^2 - v_t^2 k_1^2}$$
$$+ \gamma_p^2\omega^4(v_l^2 - v_t^2) \tag{39}$$
$$\times \iint \frac{k_{1s}k_{1f}\langle \rho_p(\mathbf{k}-\mathbf{k}_1)\rho_p(\mathbf{k}_1-\mathbf{k}_2)u_f(\mathbf{k}_2)\rangle d\mathbf{k}_1 d\mathbf{k}_2}{(\omega^2 - v_t^2 k_1^2)(\omega^2 - v_l^2 k_1^2)},$$

where the indices s and f take on the values t and l and δ_{sf} is the Kronecker symbol.

The means of the products of three random functions are decoupled according to the approximate expression (16). The components of the spectral density matrix $S_{ij}(\mathbf{k})$ and the correlation matrix $K_{ij}(\mathbf{r})$ are defined by three-dimensional analogs of relationships (17) and (18), and they are related to each other by three-dimensional analogs of the Fourier transforms (19).

After the approximate decoupling of correlations (similar to expression (16)) and integration over $\mathbf{k}_2$, we have the system of linear homogeneous equations for the quantities $\langle u_t(\mathbf{k})\rangle$ and $\langle u_l(\mathbf{k})\rangle$. The determinant of this system has the form

$$P_{sf}(\omega, \mathbf{k}) = (\omega^2 - v_t^2 k^2 - \gamma_p^2\omega^4 L_{tpp})\delta_{sf}$$
$$- (v_l^2 - v_t^2)k_s k_f - \gamma_p^2\omega^4(v_l^2 - v_t^2)L_{ltppsf}, \tag{40}$$

where

$$L_{tpp} = \int \frac{S_{pp}(\mathbf{k}-\mathbf{k}_1)d\mathbf{k}_1}{\omega^2 - v_t^2 k_1^2},$$
$$L_{ltppsf} = \int \frac{k_{1s}k_{1f}S_{pp}(\mathbf{k}-\mathbf{k}_1)d\mathbf{k}_1}{(\omega^2 - v_t^2 k_1^2)(\omega^2 - v_l^2 k_1^2)}. \tag{41}$$

In the coordinate system with the z axis coinciding with the direction of the vector $\mathbf{k}$, in the absence of inhomogeneities ($\gamma_p = 0$), the tensor P_{sf} has a diagonal form and the equation $P_{sf}(\omega, \mathbf{k}) = 0$ leads to the independent dispersion laws for the transverse and longitudinal waves.

It is assumed that the decay of correlations is characterized by exponential isotropic functions for both the autocorrelation functions of all inhomogeneous parameters and the cross correlations between inhomogeneities of these parameters,

$$K_{ii}(r) = \exp(-k_c r), \quad K_{ij}(r) = \kappa_{ij}\exp(-k_c r), \tag{42}$$

where $r = |\mathbf{r}|$, as well as by the corresponding components of the spectral density matrix,

$$S_{ii}(\mathbf{r}) = \frac{1}{\pi^2}\frac{k_c}{(k_c^2 + k^2)^2},$$
$$S_{ij}(\mathbf{k}) = \frac{\kappa_{ij}}{\pi^2}\frac{k_c}{(k_c^2 + k^2)^2}. \tag{43}$$

We substitute the expression for the quantity $S_{pp}(k)$ into the integrals in expression (41) and change over in these integrals to the spherical coordinate system. After replacement of $x = \cos\theta$ and integration over the azimuthal angle φ, we find

$$L_{tpp} = \frac{2}{\pi k_c^2}\int_0^\infty\int_{-1}^1 \frac{u_1^2 du dx}{(u_\omega^2 - v_t^2 u_1^2)(1 + u^2 + u_1^2 - 2uu_1 x)^2},$$

$$L_{ltppxx} = L_{ltppyy} = \frac{1}{\pi k_c^2}\int_0^\infty\int_{-1}^1 \frac{u_1^4(1-x^2)du dx}{(u_\omega^2 - v_t^2 u_1^2)(u_\omega^2 - v_l^2 u_1^2)(1 + u^2 + u_1^2 - 2uu_1 x)^2},$$
$$L_{ltppzz} = \frac{2}{\pi k_c^2}\int_0^\infty\int_{-1}^1 \frac{u_1^4 x^2 du dx}{(u_\omega^2 - v_t^2 u_1^2)(u_\omega^2 - v_l^2 u_1^2)(1 + u^2 + u_1^2 - 2uu_1 x)^2}, \tag{44}$$

$$L_{ltppxy} = L_{ltppyx} = L_{ltppxz} = L_{ltppzx} = L_{ltppyz} = L_{ltppzy} = 0,$$

where we introduce the dimensionless quantities $u = k/k_c$, $u_1 = k_1/k_c$, and $u_\omega = \omega/k_c$.

It follows from relationships (44) that the tensor $P_{sk}(\omega, k)$ remains diagonal at $\gamma_p \neq 0$. Therefore, by equating determinant (40) to zero, we obtain the independent equations for the complex laws of dispersion of the transverse and longitudinal waves:

$$\omega^2 - v_t^2 k^2 = \gamma_p^2 \omega^4 [L_{tpp}^t + (v_l^2 - v_t^2) L_{ltpp}^t], \quad (45)$$

$$\omega^2 - v_l^2 k^2 = \gamma_p^2 \omega^4 [L_{tpp}^l + (v_l^2 - v_t^2) L_{ltpp}^l], \quad (46)$$

where $L_{tpp}^t = L_{tpp}^l = L_{tpp}$, $L_{ltpp}^t = L_{ltppx} = L_{ltppyy}$, and $L_{ltpp}^l = L_{ltppzz}$.

As in the case of one-dimensional inhomogeneities, Eqs. (45) and (46) were represented in a form similar to relationships (29) and $\omega \approx v_t k$ and $\omega \approx v_l k$ were approximately set on the right-hand sides of these equations. In integrals (44), we changed the limits of integration by using the relationship $\int_0^\infty \int_{-1}^1 \longrightarrow \int_{-\infty}^\infty \int_0^1$, which is valid for these integrands. The integrals over the variable u_1 were calculated with the theory of residues. Then, integration over the variable x was performed exactly with the tables of integrals [17]. This results in cumbersome expressions, which are somewhat simplified in the first order of perturbation theory for transverse ($u_\omega \approx v_t u$) and longitudinal ($u_\omega \approx v_l u$) waves. As an illustration, we write one of these expressions in the form

$$\begin{aligned} L_{ltpp}^t = \frac{1}{4 v_t^2 (v_l^2 - v_t^2) u^5 k_c^2} \Bigg\{ & (1 + 2u^2) \\ & + \arctan \frac{u}{1 + 2u^2} - (1 + q_+) \arctan \frac{u(1 + q_-)}{1 + q_+} \\ & + 2q_- \arctan u + \frac{i}{2} \Bigg[4(1 - \beta) u^2 - (1 + 2u^2) \\ & \times \ln(1 + 4u^2) + (1 + q_+) \ln \frac{1 + p_+}{1 + p_-} \Bigg] \Bigg\}, \end{aligned} \quad (47)$$

where $q_\pm = (1 \pm \beta^2) u^2$, $p_\pm = (1 \pm \beta)^2 u^2$, and $\beta = v_t / v_l$. It can be seen that, in the three-dimensional case, the dispersion and damping laws contain not only rational functions of the argument u but also transcendental functions of rational functions of this argument.

By representing ω in the form $\omega = \omega' + i\omega''$, we obtain the dispersion and damping laws in the case of inhomogeneity of the density of the material in the following form:

for transverse elastic waves,

$$\begin{aligned} \omega' = {} & v_t k \\ & \times \{1 + \gamma_p^2 v_t^2 u^2 [(L_{tpp}^t)' + (v_l^2 - v_t^2)(L_{ltpp}^t)']\}, \end{aligned} \quad (48)$$

$$\frac{\omega''}{v_t k_c} = \gamma_p^2 v_t^2 u^2 [(L_{tpp}^t)'' + (v_l^2 - v_t^2)(L_{ltpp}^t)'']; \quad (49)$$

and for longitudinal elastic waves,

$$\begin{aligned} \omega' = {} & v_l k \\ & \times \{1 + \gamma_p^2 v_l^2 u^2 [(L_{tpp}^l)' + (v_l^2 - v_t^2)(L_{ltpp}^l)']\}, \end{aligned} \quad (50)$$

$$\frac{\omega''}{v_l k_c} = \gamma_p^2 v_l^2 u^2 [(L_{tpp}^l)'' + (v_l^2 - v_t^2)(L_{ltpp}^l)'']. \quad (51)$$

Here, L' and L'' are the real and imaginary parts of expressions (44), respectively. It can be seen from these relationships that, as a result of the inhomogeneities of the density of the material, the dispersion and damping laws for the longitudinal and transverse waves depend not only on the intrinsic velocities but also on the ratio β between the velocities of the transverse and longitudinal waves. This ratio enters into relationships (48)–(51) directly and via the parameters $p_\pm$ and $q_\pm$.

Now, we turn back to the complete expression (38) for the function $u(\mathbf{k}_1)$ with the inhomogeneities of both the density of the material and the force constants. After the substitution of this expression into the right-hand side of Eq. (35), we obtain the cumbersome relationship in which the correlations are decoupled using the procedure described above for the case of the inhomogeneity in density. As in the last case, we derive the independent equations for the complex dispersion laws for the transverse and longitudinal waves. These equations account for the inhomogeneities of all parameters of the material and the cross correlations between these inhomogeneities. From these equations, we derive the complex dispersion relations $\omega(k)$ in the following form:

for the transverse waves,

$$\begin{aligned} \omega = {} & v_t k \\ & \times \Bigg\{ 1 - \frac{v_t^2}{2} [\gamma_p^2 u^2 (2 L_{tpp}^{t20} - (v_l^2 - v_t^2)(L_{ltpp}^{t40} - L_{ltpp}^{t42})) \\ & - 4\gamma_p \gamma_\mu u (L_{tp\mu}^{t31} - (v_l^2 - v_t^2)(L_{ltp\mu}^{t51} - L_{ltp\mu}^{t53})) \\ & + \gamma_\mu^2 (L_{t\mu\mu}^{t40} + L_{t\mu\mu}^{t42} - 4(v_l^2 - v_t^2)(L_{tk\mu\mu}^{t62} - L_{tl\mu\mu}^{t64}))] \Bigg\}; \end{aligned} \quad (52)$$

and for the longitudinal waves,

$$\begin{aligned} \omega = {} & v_l k \{ 1 - v_l^2 [\gamma_p^2 u^2 (L_{tpp}^{l20} - (v_l^2 - v_t^2) L_{ltpp}^{l42}) \\ & - 2\gamma_p \gamma_\lambda u (1 - 2\beta^2)(L_{tp\lambda}^{l31} - (v_l^2 - v_t^2) L_{ltp\lambda}^{l51}) \end{aligned}$$

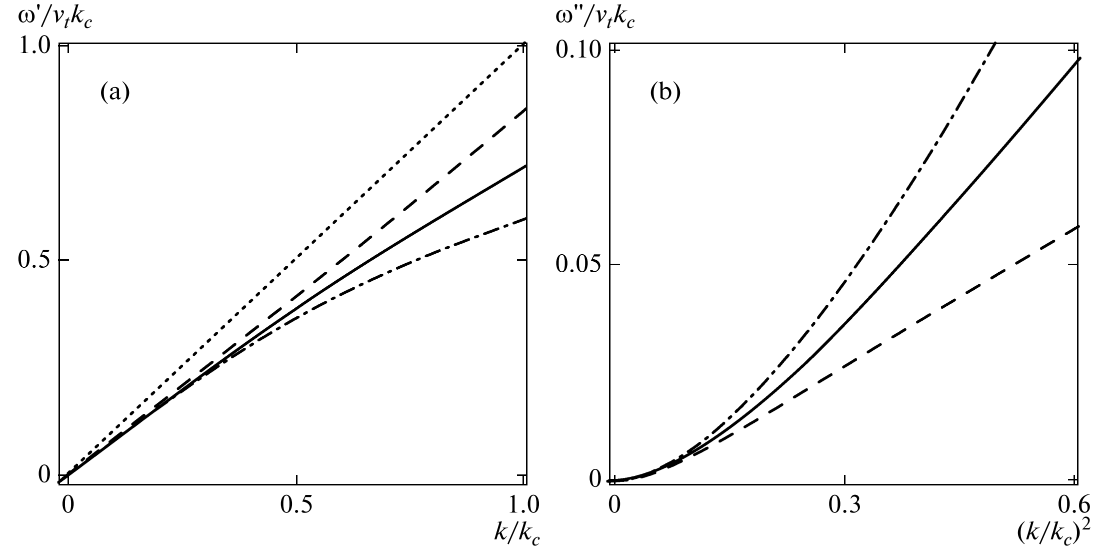

Fig. 1. (a) Dispersion and (b) damping laws for transverse elastic waves in the medium with different coefficients of cross correlations $\kappa_{p\mu}$ between three-dimensional inhomogeneities of the corresponding parameters of the material: $\kappa_{p\mu} = 0$ (solid curves), $\kappa_{p\mu} = 0.9$ (dashed curves), and –0.9 (dot-dashed curves). The dotted straight line represents the dispersion law in a homogeneous medium.

$$\begin{aligned}
&-4\gamma_p\gamma_\mu\beta^2(L_{tp\mu}^{l31} - (v_l^2 - v_t^2)L_{ltp\mu}^{l53}) \\
&+\gamma_\lambda^2(1-2\beta^2)^2(L_{t\lambda\lambda}^{l40} - (v_l^2 - v_t^2)L_{lt\lambda\lambda}^{l60}) \\
&+4\gamma_\lambda\gamma_\mu\beta^2(1-2\beta^2)(L_{t\lambda\mu}^{l42} - (v_l^2 - v_t^2)L_{lt\lambda\mu}^{l62}) \\
&+4\gamma_\mu^2\beta^4(L_{t\mu\mu}^{l42} - (v_l^2 - v_t^2)L_{lt\mu\mu}^{l64})]\}.
\end{aligned} \tag{53}$$

These relations contain 22 complex integral expressions, which can be written in the generalized form

$$\begin{aligned}
L_{tij}^{lmn} &= \frac{1}{v_t^2\pi}\int_{-\infty}^{\infty}\int_0^1 \frac{u_1^m x^n S_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_1}du_1dx, \\
L_{ltij}^{lmn} &= \frac{1}{v_l^2 v_t^2\pi}\int_{-\infty}^{\infty}\int_0^1 \frac{u_1^m x^n S_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_2}du_1dx, \\
L_{ltpp}^{tmn} &= \frac{1}{v_l^2 v_t^2\pi}\int_{-\infty}^{\infty}\int_0^1 \frac{u_1^m x^n S_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_4}du_1dx, \\
L_{tij}^{tmn} &= \frac{1}{v_t^2\pi}\int_{-\infty}^{\infty}\int_0^1 \frac{u_1^m x^n S_{ij}(\mathbf{u}-\mathbf{u}_1)}{Z_3}du_1dx,
\end{aligned} \tag{54}$$

where

$$Z_1 = u_1^2 - \frac{u^2}{\beta^2},$$

$$Z_2 = \left(u_1^2 - \frac{u^2}{\beta^2}\right)(u_1^2 - u^2),$$

$$Z_3 = u_1^2 - u^2,$$

$$Z_4 = (u_1^2 - \beta^2 u^2)(u_1^2 - u^2).$$

In relationships (54), we can single out two groups of integrals that describe processes of different physical natures: the integrals L_{tij}^{tmn} containing no β determine the contribution of scattering processes for waves of the same type to the modification of the dispersion law, and all the other integrals containing β determine the contribution of scattering processes with a change in wave type. Integrals (54) over the variable u_1 were calculated with the theory of residues. Then, integration over the variable x could be exactly performed with the tables of integrals [17].

By representing ω in the form $\omega = \omega' + i\omega''$ and all relationships for L as $L = L' + iL''$, the dispersion and damping laws for the transverse and longitudinal waves can be easily separated from expressions (52) and (53). The cumbersome relationships obtained were analyzed graphically. Furthermore, the limiting expressions corresponding to small and large wave numbers were examined analytically. As for the one-dimensional inhomogeneities, the main changes in both curves $\omega'(k)$ and $\omega''(k)$ are observed in the vicinity of the quantity $k = k_c/2$, which separates the range of waves scattering from uncorrelated ($k < k_c/2$) and correlated ($k > k_c/2$) regions of fluctuations. For trans-

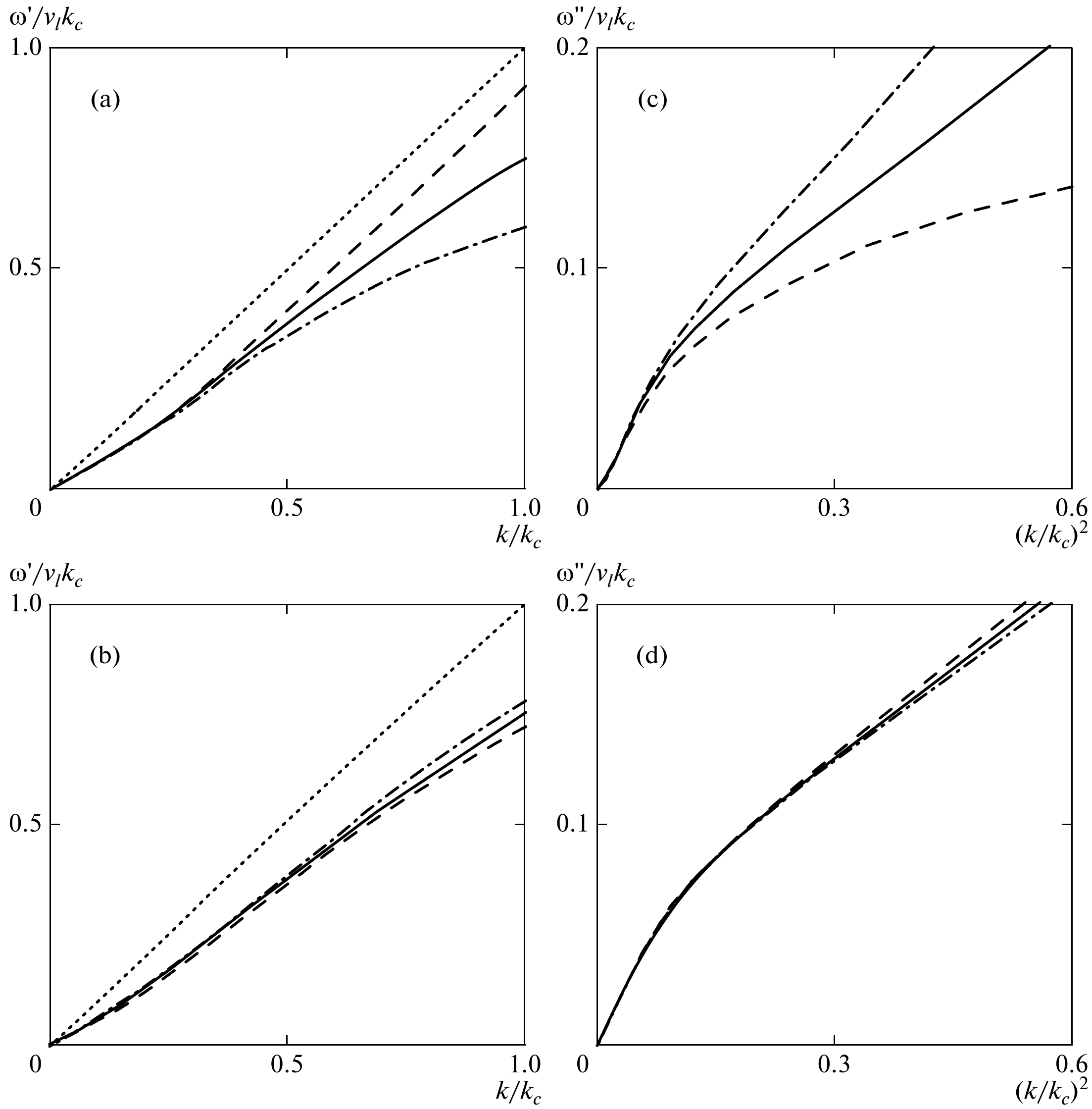

Fig. 2. (a, b) Dispersion and (c, d) damping laws for longitudinal elastic waves in the medium for different coefficients of cross correlations κ_{ij} between three-dimensional inhomogeneities of different parameters of the material. (a–d) Solid curves correspond to the laws for the cross correlation coefficients $\kappa_{ij} = 0$. Dashed curves indicate the laws for the cross correlation coefficients (a, c) $\kappa_{p\lambda} = \kappa_{p\mu} = 0.9$, $\kappa_{\lambda\mu} = 0$ and (b, d) $\kappa_{p\lambda} = \kappa_{p\mu} = 0$, $\kappa_{\lambda\mu} = 0.9$. Dot-dashed curves represent the laws for (a, c) $\kappa_{p\lambda} = \kappa_{p\mu} = -0.9$, $\kappa_{\lambda\mu} = 0$ and (b, d) $\kappa_{p\lambda} = \kappa_{p\mu} = 0$, $\kappa_{\lambda\mu} = -0.9$. Dotted straight lines correspond to the dispersion law in a homogeneous medium.

verse waves, the dispersion law in the limiting cases is written in the form

$$\frac{\omega'}{v_t k_c} \approx \begin{cases} \left(1 - \frac{\gamma_\mu^2}{3}\right)u - \left(\gamma_p^2 - \frac{2}{3}\kappa_{p\mu}\gamma_p\gamma_\mu + \frac{11\gamma_\mu^2}{15}\right)\frac{u^3}{2}, & 2u \ll 1, \\ \left(1 - \frac{1}{8}(\gamma_p^2 - 6\kappa_{p\mu}\gamma_p\gamma_\mu + 5\gamma_\mu^2)\right)u + A, & 2u \gg 1, \end{cases} \quad (55)$$

and the damping is described by the expressions

$$\frac{\omega''}{v_t k_c} \approx \begin{cases} [\gamma_p^2(2 + \beta^3) + 2\gamma_\mu^2]\frac{u^4}{3}, & 2u \ll 1, \\ (\gamma_p^2 - 2\kappa_{p\mu}\gamma_p\gamma_\mu + \gamma_\mu^2)\frac{u^2}{4} + B, & 2u \gg 1, \end{cases} \quad (56)$$

where A and B are some constants dependent on the parameters γ_i and κ_{ij}. Since the limiting relationships for the longitudinal waves are cumbersome owing to the complex dependences in their coefficients on the parameters, they are not presented in this paper. The damping of both longitudinal and transverse waves obeys universal relationships different for scattering from uncorrelated and correlated regions of the fluctuations:

$$\omega'' \propto \begin{cases} k^4, & k \ll k_c/2, \\ k^2, & k \gg k_c/2. \end{cases} \tag{57}$$

The changeover from the Rayleigh law $\omega'' \propto k^4$ to the law $\omega'' \propto k^2$ in the vicinity of the crossover point $k_c/2$ was investigated in our previous works [1, 7].

The dependences $\omega'(k)$ and $\omega''(k^2)$ for different cross correlation coefficients κ_{ij} between inhomogeneities are plotted in Figs. 1 and 2. These dependences were calculated using identical root-mean-square fluctuations γ_i: $\gamma_p^2 = \gamma_\mu^2 = \gamma_\lambda^2 = 0.8$. It can be seen from Fig. 1a that, in the absence of cross correlations, the dispersion curve of the transverse elastic waves (solid curve) deviates from the unperturbed dispersion law (dotted curve) toward low frequencies with further bending in the same direction in the vicinity of the point $k/k_c = 0.5$. With the appearance of positive cross correlations between the inhomogeneities of the quantities p and μ, the dispersion curve (dashed curve) becomes close to the unperturbed dispersion law and its bending decreases. The negative cross correlations enhance the modification of the dispersion law (dot-dashed curve). The damping ω'' of the transverse elastic waves as a function of k^2 is presented in Fig. 1b. In these coordinates, the function $\omega''(k^2)$ according to the limiting relationships (56) has the form of a parabola to the left of the crossover point $(k/k_c)^2 = 0.25$ and a straight line to the right of this point. The appearance of the positive and negative cross correlations between the inhomogeneities of the quantities p and μ leads to a decrease (dashed curve) and an increase (dot-dashed curve) of the damping. Figure 2 shows the dependences $\omega'(k)$ and $\omega''(k^2)$ for the longitudinal elastic waves with due regard for the cross correlations between the inhomogeneities of different parameters of the material. In this case, in the absence of cross correlations, the dispersion law $\omega'(k)$ has an inflection point rather than the bending point in the vicinity of the point $k/k_c = 0.5$ (solid curves in Figs. 2a, 2b). This is associated with the appearance of one more crossover point at $k/k_c \approx \beta$ in the dependence $\omega'(k)$ for the longitudinal waves. In order to separate this crossover from the crossover at $k/k_c = 0.5$, the value of β was chosen as $\beta = 0.2$ in constructing the graphs (as a rule, the parameter β lies in the range $0.2 \le \beta \le 0.5$). The existence of this second crossover was first revealed in [7]. As a result of the positive cross correlations between the inhomogeneities of the density and elastic constants ($\kappa_{p\mu}$, $\kappa_{p\lambda}$), the curve $\omega'(k)$ comes close to the unperturbed dispersion law (dashed curve in Fig. 2a). The negative cross correlations $\kappa_{p\mu}$ and $\kappa_{p\lambda}$ enhance the modification of the dispersion law in the range $k/k_c > 0.5$ (dot-dashed curve in Fig. 2a). The cross correlations between the elastic constants μ and λ lead to directly opposite effects: the positive and negative cross correlations $\kappa_{\lambda\mu}$ increase and decrease the modification of the dispersion law, respectively (Fig. 2b).

The dependences of the damping $\omega''(k^2)$ (Figs. 2c, 2d) also exhibit both crossover points. The additional crossover in these coordinates corresponds to the point $(k/k_c)^2 \approx \beta^2 = 0.04$. Therefore, the Rayleigh law $\omega'' \propto k^4$ manifests itself only in a narrow range to the left of this point. As for the transverse waves, the damping law $\omega'' \propto k^2$ manifests itself to the right of the point of the main crossover $(k/k_c)^2 = 0.25$. As in the case of one-dimensional inhomogeneities, the appearance of the positive cross correlations between inhomogeneities of the density and elastic constants results in a decrease in the damping (dotted line in Fig. 2c) and the appearance of the negative cross correlations leads to an increase in the damping (dot-dashed curve). The cross correlations between the inhomogeneities of the elastic constants lead to the opposite effects (Fig. 2d): the positive and negative cross correlations increase and decrease the damping, respectively.

4. DISCUSSION OF THE RESULTS AND CONCLUSIONS

At first glance, we obtain a paradoxical result: the positive cross correlations ($\kappa_{ij} > 0$) between inhomogeneities of some parameters of the material lead to an enhancement of the modification of the dispersion law and an increase in the damping of elastic waves, whereas the cross correlations with the same sign between inhomogeneities of other parameters of the material result in a weakening of this modification and a decrease in the damping of waves. Since the negative cross correlations ($\kappa_{ij} < 0$) always lead to effects opposite to those observed for the positive cross correlations, their effects are characterized by the same ambiguity.

This ambiguity is eliminated if we take into account the fact that parameter p corresponds to the kinetic part of the Hamiltonian and the parameters μ and λ belong to the potential part. This allows us to assume that the character of the effects of cross correlations changes depending on whether both parameters belong to the same part of the Hamiltonian (both parameters belong to the potential or kinetic part) or these parameters correspond to different parts of the Hamiltonian. Indeed, parameters μ and λ belong to the potential part of the Hamiltonian and the positive cross correlations between their inhomogeneities lead to the enhancement of the modification of the dispersion law and to the increase in the damping of elastic waves. Parameter p characterizes the inertness of the elastic system and corresponds to the kinetic part of the Hamiltonian, and the positive cross correlations between inhomogeneities of p and μ or p and λ result in the weakening of the modification of the dispersion law and the decrease in the damping of elastic waves. Let us analyze the results obtained in our earlier work [15] from this point of view. Both inhomogeneous parameters considered in [15] (the exchange parameter α and the magnetic anisotropy β) belong to the potential part of the spin Hamiltonian. Correspondingly, the positive cross correlations between them according to the above assumptions should lead to an enhancement of the modification of the dispersion law and an increase in the damping of spin waves. This result was actually obtained in the calculations carried out in [15].

The physical mechanism responsible for the differences between the effects of cross correlations can be understood in terms of the following simplified model of an inhomogeneous medium. The dispersion law for transverse elastic waves in a homogeneous isotropic medium is defined by the expression

$$\omega = \sqrt{\frac{\mu}{p}}k. \tag{58}$$

In the case of very smooth inhomogeneities with a characteristic size of $2r_c$ that is considerably larger than the wavelength ($k_c \ll k$), the medium can be approximately represented as consisting of a set of homogeneous regions with a size $2r_c$, so that the parameters of the medium p and μ are constant within these regions but are different in different regions (the model of independent grains (crystallites)). When the fluctuations of the quantities p and μ do not correlate with each other, the frequency at specific k can differ substantially in different regions, because the parameter μ increases and the parameter p decreases (or vice versa) with respect to the mean values of these quantities and the frequency of waves is determined by the ratio between μ and p. The positive cross correlations lead to a spatial synchronization of fluctuations of two random functions without a change in the root-mean-square deviations of each function. Therefore, in each our region, the deviation (with any sign) of the quantity μ from its mean value corresponds to the deviation (with the same sign) of the quantity p from its mean value. As a result, a random scatter in the frequencies of waves in different regions decreases. We can even imagine a hypothetical limiting situation where the frequency of elastic waves will be almost identical over the entire space, despite strong deviations of the parameters μ and p in different spatial regions. The negative cross correlations result in the spatial synchronization of deviations of the quantities μ and p with opposite signs and, correspondingly, in a larger scatter in the frequencies of waves in different regions as compared to the case $\kappa_{p\mu} = 0$.

A similar analysis for the cross correlations between the quantities p and μ or p and λ can be performed for longitudinal waves for which the dispersion law is determined by the expression

$$\omega = \sqrt{\frac{\lambda + 2\mu}{p}}k. \tag{59}$$

However, this example can also be used to qualitatively examine the effects of cross correlations between the quantities λ and μ. In the absence of these cross correlations, apart from the cases of a simultaneous increase or decrease in the parameters λ and μ, there are situations where an increase (decrease) in the quantity λ is accompanied by a decrease (increase) in the quantity μ. The presence of positive cross correlations excludes the latter situations: in this case, the parameters λ and μ increase or decrease in a synchronous manner, which enhances the scatter of the frequencies ω in different regions of the material.

The same qualitative examination of the effects of cross correlations between different parameters of the Hamiltonian can be carried out for a simplified model of the inhomogeneous medium for spin waves. However, in this case, it should be kept in mind that the role of the inertial parameter entering into the kinetic part of the Hamiltonian is played not by the gyromagnetic ratio but by the inverse quantity, i.e., the magnetomechanical parameter $p_m = g^{-1}$. It is this parameter that is contained ahead of the kinetic term $\partial\mathbf{M}(\mathbf{x}, t)/\partial t$ in the Landau–Lifshitz equation. The expression for the frequency of spin waves takes the form

$$\omega = \frac{(\alpha k^2 + \beta)M + H}{p_m}, \tag{60}$$

where M is the magnetization and H is the magnetic field. In this expression, as in relationships (58) and (59), the frequency is determined by the ratio between the parameters of the potential part of the Hamiltonian and the parameters of its kinetic part. Therefore, all the above considerations for the elastic waves can be repeated for the spin waves. In particular, the positive cross correlations between the inhomogeneities of the exchange α and the anisotropy β will lead to an increase in the damping and the same correlations between α and p_m or β and p_m will result in a decrease in the damping.

The analysis of the results obtained in this paper and in our previous work [15], as well as the aforementioned qualitative consideration of the model of independent grains, allows us to formulate the general regularity of the effects of cross correlations, irrespective of the physical nature of waves: the effects of cross correlations between inhomogeneities of any two parameters of the material on the wave spectrum depend on whether both parameters related by the cross correlations belong to the same part of the Hamiltonian (i.e., both belong to either the kinetic or the potential part of the Hamiltonian) or they belong to different parts of the Hamiltonian. The positive cross correlations lead to an enhancement of the modification of the dispersion law and to an increase in the damping of waves in the former case and to a decrease in these characteristics in the latter case. Correspondingly, the negative cross correlations in each of these cases result in the opposite effects.

These regularities can be used in designing and developing a technology for preparing amorphous and nanocrystalline alloys for creating conditions favorable for useful cross correlations (to decrease the damping of waves) and for limiting adverse cross correlations that increase damping. The specific effects revealed in our work due to the cross correlations can be experimentally observed in situations in which we can expect considerable changes in the cross correlation coefficients κ_{ij}, for example, for a series of samples of alloys or solid solutions with different ratios between components upon the transition of an amorphous material to a nanocrystalline state during annealing, etc. The separation of the effects associated with these cross correlations in processing the results of these experiments is complicated by the fact that the change in the cross correlation coefficients can be accompanied by a change in the root-mean-square fluctuations of the parameters of the medium. However, it can be seen from formulas (30)–(33), (55), and (56) and Figs. 1 and 2 that there exist situations in which the effects determined by the root-mean-square fluctuations and the effects caused by the cross correlations affect the form of the curves $\omega'(k)$ and $\omega''(k)$ differently.

ACKNOWLEDGMENTS

This study was supported in part by the Council on Grants from the President of the Russian Federation for the Support of Leading Scientific Schools (grant no. 3818.2008.3) and the Presidium of the Russian Academy of Sciences (program no. 27.1) and performed within the framework of the Federal Target Program (State Contract no. 02.740.11.0220).

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Translated by N. Wadhwa