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MAGNETISM AND FERROELECTRICITY

# High-Frequency Susceptibility of a Multilayered Ferromagnetic System with Two-Dimensional Inhomogeneities

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**Abstract**—This paper reports on the results of the investigation of the high-frequency susceptibility of a layered ferromagnetic structure in which, apart from a periodic change in the magnetic anisotropy parameter from layer to layer, this parameter varies along layers according to a random law (the superlattice with two-dimensional phase inhomogeneities). The evolution of the frequency dependence of the imaginary part of the averaged Green's function in the range of the energy gap (band gap) in the spectrum of waves propagating along the superlattice axis due to the change in the relative root-mean-square fluctuations of the phase $\gamma_2$ has been studied at the boundaries of the odd Brillouin zones. It has been found that, for all odd Brillouin zones, the imaginary part of the Green's function exhibits a universal behavior: the peak corresponding to the edge of the band gap with a lower frequency remains unchanged, and the peak corresponding to the edge of the band gap with a higher frequency is smoothed with an increase in the quantity $\gamma_2$. These effects, which were initially revealed at the boundary of the first Brillouin zone of the sinusoidal superlattice, have been explained, as before, by the specific features of the energy conservation laws for the incident and scattered waves in the lattice with two-dimensional inhomogeneities. It has been demonstrated that an increase in the Brillouin zone number leads to a decrease in the value of $\gamma_2$ at which the peak at the edge of the band gap with a higher frequency disappears.

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## 1. INTRODUCTION

At present, multilayered film structures (one-dimensional superlattices), which consist of periodically alternating layers of two materials with different physical properties, have been investigated extensively. In particular, among these structures are photonic and magnonic crystals that have received much attention. It is known that the spectrum of waves of any nature in periodic structures has a band structure characterized by the reciprocal lattice vector $\mathbf{q}$ ($|\mathbf{q}| \equiv q = 2\pi/l$, where $l$ is the period of the one-dimensional superlattice). At the edge of the Brillouin zones of the superlattice with the wave vector $k = nq/2$, the degeneracy is removed and gaps (band gaps) appear in the wave spectrum. The band gap $\Delta\omega_n$ is determined by the superlattice parameter $\lambda$ (the relative change in the physical parameter of the neighboring layers) and the zone number $n$. Layered structures have been frequently described using one-dimensional models that allow for the exact solution of the wave equation, such as the sinusoidal superlattice and the superlattice with a rectangular profile [1–13]. In the sinusoidal superlattice, the wave equation is brought into the Mathieu equation; in this case, the band gap in the wave spectrum is determined by the eigenvalues of this equation (see, for example, [9, 14]). Specifically, in a ferromagnetic sinusoidal superlattice with an inhomogeneous anisotropy parameter at $\lambda \ll 1$ (the limit of narrow band gaps), we have $\Delta\omega_n \sim \lambda^n$ at the Brillouin zone boundary. In the study of the wave spectrum for the sinusoidal superlattice in the vicinity of the boundary of the first Brillouin zone, the power dependence on $\lambda$ for a small value of this parameter makes it possible to ignore the band gaps at the boundaries of all the other Brillouin zones. The model of the superlattice with a rectangular profile corresponds to the layered structure with alternation of two layers with different values of any physical parameter. It should be noted that this parameter in each layer is constant and the interface has a size of the order of atomic sizes. In the continuum model, the superlattice is described by a piecewise constant function and the interface has a zero width. In a ferromagnetic superlattice with a rectangular profile formed by layers with different anisotropy parameters, at $\lambda \ll 1$, we have $\Delta\omega_n \sim \lambda/n$ at the boundaries of odd Brillouin zones and $\Delta\omega_n \sim (\lambda/n)^2$ at the boundaries of even Brillouin zones [9–12]. Therefore, in the layered structure even at $\lambda \ll 1$, it is important to investigate the wave spectrum with allowance made for its multiband structure.

In natural materials and composite systems, despite considerable progress in the preparation of the latter systems, the periodicity in the layer arrangement occurs only approximately. There are always deviations from the periodicity due to the natural or technological factors. This circumstance has stimulated the appearance of theoretical works devoted to the study of the transition from ideally periodic superlattices to partially stochastized superlattices.

The spectral properties of superlattices with random inhomogeneities will be described using the method of averaged Green's functions (see, for example, [15]), according to which a random medium is described by the correlation function $K(\mathbf{r})$ dependent on the distance between two points of the medium: $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. The correlation function of the superlattice $K(\mathbf{r})$ is determined by the random spatial modulation method developed in our earlier works [10, 16, 17], which is the generalization of the well-known method for determining the time correlation function for a random frequency (phase) modulation of a radio signal [18, 19] to the case of a spatial (in the general case, three-dimensional) modulation of the superlattice period. In our previous works [10, 16, 17, 20–22], these methods were used to investigate the spectrum of waves and their damping in the superlattice containing one-dimensional phase inhomogeneities that simulate random displacements of interfaces of the superlattice from their initial periodic positions and three-dimensional isotropic phase inhomogeneities simulating random deformations of the interfaces. In particular, it was revealed that, as the band gap in the wave spectrum decreases as a result of the one-dimensional and three-dimensional inhomogeneities, two peaks in the imaginary part of the averaged Green's function that correspond to the gap edges broaden, approach each other, and merge together into one peak. In [16, 23], the spectrum and high-frequency susceptibility of the initial sinusoidal superlattice containing two-dimensional phase inhomogeneities that simulate deformations of boundaries between layers that are identical for all layers were studied by the method of averaged Green's functions in combination with the random spatial modulation method. It was found that, for waves propagating in the direction of the superlattice axis, the imaginary part of the averaged Green's function is characterized by a specific behavior at the boundary of the first Brillouin zone: the peak corresponding to the edge of the band gap with a lower frequency remains almost unchanged with an increase in the root-mean-square fluctuation of the two-dimensional inhomogeneities $\gamma_2$, whereas the peak corresponding to the edge of the band gap with a higher frequency broadens and decreases sharply in height up to its complete disappearance with an increase in the quantity $\gamma_2$. It is of interest to investigate these effects in a layered system that is more adequate to real structures. For this purpose, in the present work, the high-frequency susceptibility (Green's function) of the superlattice with an initially rectangular profile in the presence of two-dimensional inhomogeneities is studied with due regard for a multiband structure of the wave spectrum.

## 2. THE MODEL AND CORRELATION FUNCTION

The superlattice is characterized by the dependence of a particular material parameter $A$ on the spatial coordinates $\mathbf{x} = \{x, y, z\}$. The physical nature of the parameter $A(\mathbf{x})$ can be different. This parameter can be the density of the material, force constant of an elastic medium, magnetic anisotropy, magnetization, exchange for a magnetic system, etc. The parameter $A(\mathbf{x})$ can be represented in the form

$$A(\mathbf{x}) = A + \Delta A \rho(\mathbf{x}), \quad (1)$$

where $A$ is the average parameter, $\Delta A$ is its root-mean-square deviation, and $\lambda = \Delta A/A$. The function $\rho(\mathbf{x})$ is centered ($\langle\rho(\mathbf{x})\rangle = 0$) and normalized ($\langle\rho^2(\mathbf{x})\rangle = 1$) and describes both the periodic dependence of the parameter $A(\mathbf{x})$ along the superlattice axis $z$ and the random spatial modulation of this parameter. Angle brackets indicate the averaging over an ensemble of random realizations of the function $\rho(\mathbf{x})$.

Let us consider a superlattice having a rectangular profile for which a material parameter in the initial state when random inhomogeneities are absent depends only on the coordinate $z$. According to the approach proposed in [10] to the description of superlattices with one-dimensional and three-dimensional inhomogeneities, the function $\rho(\mathbf{x})$ can be written in the form of the infinite series

$$\rho(\mathbf{x}) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} \times \cos\{(2m+1)[q(z - u_2(\mathbf{x}_\perp)) + \psi]\}, \quad (2)$$

which at $u_2(\mathbf{x}_\perp) = 0$ and $\psi = 0$ (ideal superlattice) is an expansion of a piecewise constant function into a Fourier series. The positive and negative ranges of the function $\rho(\mathbf{x})$ along the superlattice axis $z$ correspond to alternating layers of the multilayered structure, and zero points of the function $\rho(\mathbf{x})$ correspond to the boundaries of the superlattice layers. In the framework of this interpretation, the function $u_2(\mathbf{x}_\perp)$ simulates random deformations of the surfaces of these boundaries. The phase $\psi$ independent of the coordinates is characterized by a uniform distribution on the interval $(-\pi, \pi)$; $\mathbf{x}_\perp = \{x, y\}$.

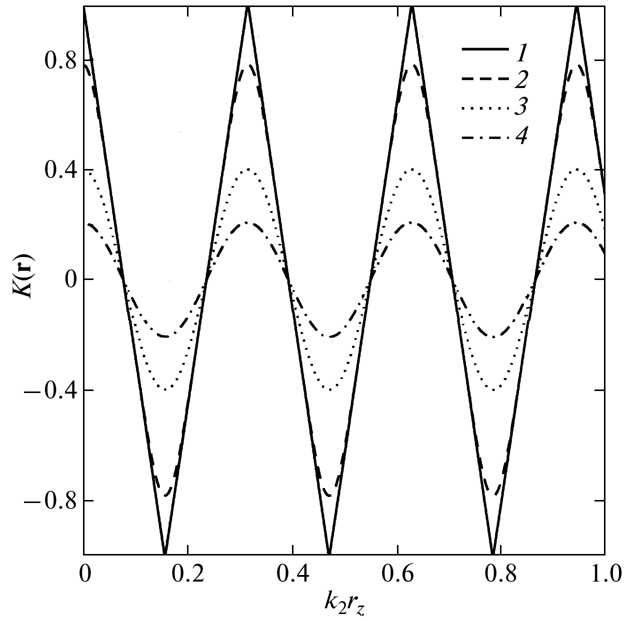


**Fig. 1.** Correlation functions of the superlattice. (*1*) Correlation function $K(\mathbf{r})$ defined by formula (11) for the ideal superlattice ($\gamma_2 = 0$). (*1–4*) Correlation functions $K(\mathbf{r})$ described by formulas (7)–(9) for the superlattice with two-dimensional inhomogeneities at $\gamma_2^2 = 0.3$, $k_2/q = 0.05$, and $k_2 r_\perp =$ (*1*) 0, (*2*) 1, (*3*) 5, and (*4*) 15.

The correlation function $K(\mathbf{r})$ depends only on the difference between the coordinates $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and is defined by the expression

$$K(\mathbf{r}) = \langle \rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle_{\psi\chi}. \tag{3}$$

Here, the averaging is performed over the homogeneous random phase $\psi$ and the random function $\chi$, where

$$\chi(\mathbf{x}_\perp, \mathbf{r}_\perp) = q[u_2(\mathbf{x}_\perp + \mathbf{r}_\perp) - u_2(\mathbf{x}_\perp)]. \tag{4}$$

We calculate the correlation function $K(\mathbf{r})$. The product of the functions $\rho(\mathbf{x})$ and $\rho(\mathbf{x} + \mathbf{r})$ can be represented in the form

$$\begin{aligned}\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r}) &= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{(-1)^{m+m'}}{(2m+1)(2m'+1)}\\ &\times\{\cos q[(2m+1)(r_z + u_2(\mathbf{x}_\perp+\mathbf{r}_\perp))\\ &-(2m'+1)u_2(\mathbf{x}_\perp) + 2m(m-m')(z+\psi/q)]\\ &+\cos q[(2m+1)(r_z - u_2(\mathbf{x}_\perp+\mathbf{r}_\perp))\\ &-(2m'+1)u_2(\mathbf{x}_\perp) + 2(m+m'+1)(z+\psi/q)]\},\end{aligned} \tag{5}$$

where $\mathbf{r}_\perp = \{r_x, r_y\}$. The second term in curly brackets after averaging over the phase $\psi$ vanishes. The expressions with $m' \neq m$ in the first term due to the averaging over $\psi$ also vanish. As a result, after this averaging, we have

$$\begin{aligned}\langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle_\psi &= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\\ &\times\cos[(2m+1)(qr_z + \chi(\mathbf{x}_\perp, \mathbf{r}_\perp))].\end{aligned} \tag{6}$$

Then, by averaging relationship (6) over $\chi$ under the assumption of the Gaussian distribution of quantities $\chi$, we obtain the expression for the correlation function $K(\mathbf{r})$ in the form

$$K(\mathbf{r}) = \sum_{m=0}^{\infty}\cos[(2m+1)qr_z]K_m(\mathbf{r}_\perp), \tag{7}$$

where $K_m(\mathbf{r}_\perp)$ is defined by the formula

$$K_m(\mathbf{r}_\perp) = \frac{8}{\pi^2(2m+1)^2}\exp\left[-\frac{(2m+1)^2}{2}Q_2(\mathbf{r}_\perp)\right]. \tag{8}$$

Here, $Q_2(\mathbf{r}_\perp) = q^2\langle[u_2(\mathbf{x}_\perp + \mathbf{r}_\perp) - u_2(\mathbf{x}_\perp)]^2\rangle$ is the dimensionless structure function of the superlattice.

The structure function in the case of two-dimensional inhomogeneities was obtained in [23] in the form

$$\begin{aligned}Q_2(\mathbf{r}_\perp) = 4\gamma_2^2\{&E_1(k_2 r_\perp) + \ln(k_2 r_\perp C)\\ &+\exp(-k_2 r_\perp) - 1\}.\end{aligned} \tag{9}$$

Here, $C \simeq 1.78$ is the Euler constant,

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt$$

is the integral exponential function, and the parameter $\gamma_2 = q\sigma_2/\sqrt{2}k_2$ determines the root-mean-square fluctuations of the random function $u_2(\mathbf{x}_\perp)$, where $\sigma_2$ and $k_2$ are the root-mean-square fluctuation and the correlation wave number of the gradient of $u_2(\mathbf{x}_\perp)$, respectively.

In the case of the ideal superlattice ($\gamma_2 = 0$), the correlation function is represented by the expression

$$K(\mathbf{r}) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos(2m+1)qr_z}{(2m+1)^2}. \tag{10}$$

By summing up this series, we obtain the periodic function that within one period has the form

$$K(\mathbf{r}) = 2\begin{cases}\dfrac{1}{2} - \dfrac{qr_z}{\pi}, & 0 \le qr_z \le \pi\\[2mm] \dfrac{qr_z}{\pi} - \dfrac{3}{2}, & \pi \le qr_z \le 2\pi.\end{cases} \tag{11}$$

This function is shown in Fig. 1. It should be noted that, in the nonideal superlattice ($\gamma_2 \neq 0$) at $r_\perp = 0$, the correlation function will have the same form. At $r_\perp \neq 0$, in the superlattice with two-dimensional inhomogeneities, the dependence of the correlation function on

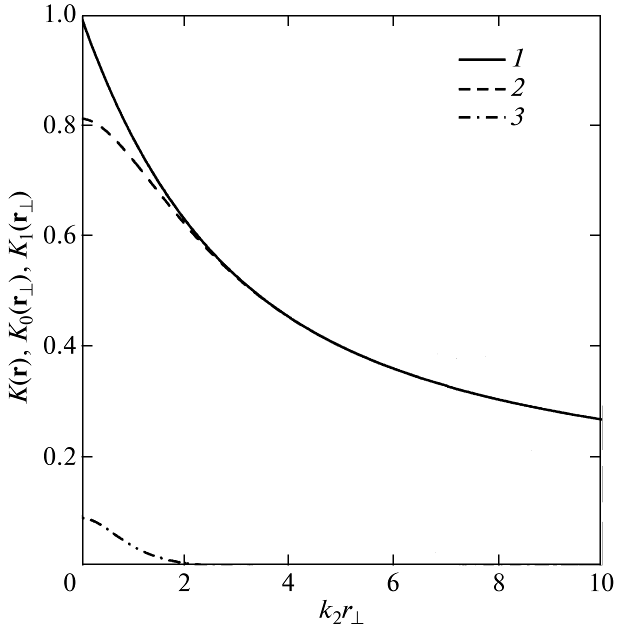


**Fig. 2.** Correlation functions of the superlattice at $r_z = 0$: (*1*) function $K(\mathbf{r})$ (relationships (7)–(9)) and (*2*, *3*) functions $K_0(\mathbf{r}_\perp)$ and $K_1(\mathbf{r}_\perp)$ (relationships (8), (9)) at $m = 0$ and 1, respectively. $\gamma_2^2 = 0.3$, $k_2/q = 0.05$.

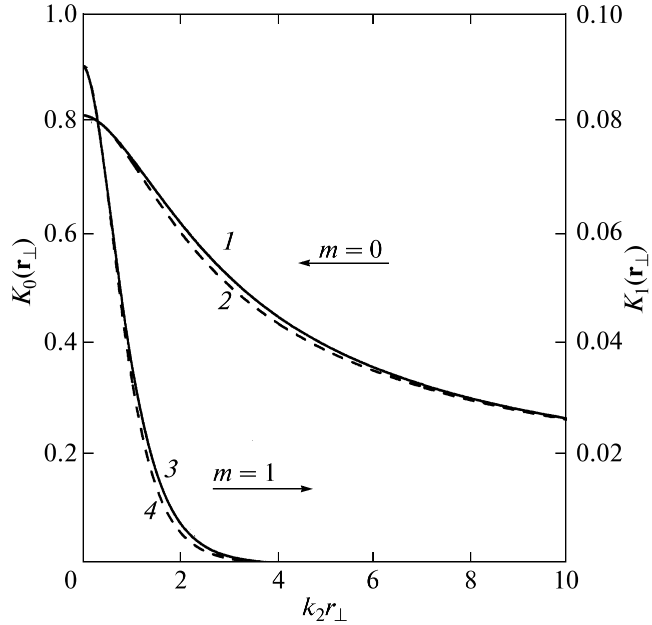


**Fig. 3.** Functions $K_m(\mathbf{r}_\perp)$ for $m$ = (*1*, *2*) 0 and (*3*, *4*) 1. Curves *1* and *3* were constructed using relationships (8) and (9), and curves *2* and *4* were obtained from the approximating function (12). $\gamma_2^2 = 0.3$, $k_2/q = 0.05$.

$r_z$ becomes smoother and remains periodic along the $z$ axis with the superlattice period. In Fig. 1, this dependence is shown for $\gamma_2^2 = 0.3$ and the ratio $q/k_2 = 20$. In the directions perpendicular to the $z$ axis, the correlation function $K(\mathbf{r})$ decreases monotonically. In Fig. 2, curve *1* shows the dependence $K(\mathbf{r})$ constructed using relationships (7)–(9) at $r_z = 0$. In Fig. 2, curves *2* and *3* shows the functions $K_0(\mathbf{r}_\perp)$ and $K_1(\mathbf{r}_\perp)$ with the structure function $Q_2(\mathbf{r}_\perp)$ represented by expression (9). It can be seen from this figure that their sum amounts to approximately 0.9 of the correlation function $K(\mathbf{r})$ at $r_\perp = 0$ and almost coincides with $K(\mathbf{r})$ at $k_2r_\perp > 1$.

The substitution of relationship (9) for the structure function $Q_2(\mathbf{r}_\perp)$ into formula (8) leads to a complex expression for the correlation function $K(\mathbf{r})$, which can be used with difficulty in further calculations. Therefore, we approximate the decreasing part of the correlation function $K_m(\mathbf{r}_\perp)$ by the simple formula

$$K_m(\mathbf{r}_\perp) = \frac{8}{\pi^2}\frac{1}{(2m+1)^2}\left(1 + \frac{C^2}{e^2}k_2^2r_\perp^2\right)^{-(2m+1)^2\gamma_2^2}, \quad (12)$$

which generalizes the modeling expression for the correlation function of the initially sinusoidal superlattice [23] to the case of the multilayered system. This expression was obtained from the limiting relationships for the function $Q_2(\mathbf{r}_\perp)$:

$$Q_2(\mathbf{r}_\perp) = \gamma_2^2\begin{cases} k_2^2r_\perp^2, & k_2r_\perp \ll 1 \\ 4\ln(k_2r_\perp C/e), & k_2r_\perp \gg 1, \end{cases} \quad (13)$$

where $e$ is the base of the natural logarithm. Figure 3 shows the decreasing parts $K_0(r_\perp)$ and $K_1(r_\perp)$ of the correlation function that are determined by expression (8) with the structure function $Q_2(\mathbf{r}_\perp)$ in the form (9) (curves *1*, *3*) and the approximating functions $K_0(r_\perp)$ and $K_1(r_\perp)$ in the form of relationship (12) (curves *2*, *4*) for $\gamma_2^2 = 0.3$. It can be seen from Fig. 3 that the model correlation function adequately describes the exact correlation function over the entire range of variation in $r_\perp$ and has asymptotics that coincides with the asymptotics of the exact function at both $k_2r_\perp \ll 1$ and $k_2r_\perp \gg 1$.

## 3. HIGH-FREQUENCY SUSCEPTIBILITY

By performing the Fourier transform with respect to the time in the wave equation describing spin waves in the ferromagnetic superlattice with the inhomogeneous magnetic anisotropy parameter $\beta(\mathbf{x})$, we obtain

$$\nabla^2 m^+ + \left[\nu - \frac{\Lambda}{\sqrt{2}}\rho(\mathbf{x})\right]m^+ = 0, \quad (14)$$

where the function $m^+$ is $m^+ = m^+(\mathbf{x}, \omega)$ and $A = \beta$ and $\Delta A = \Delta\beta$ in formula (1). Equation (14) corresponds to the situation when the directions of the external magnetic field $\mathbf{H}$, the constant component of the magnetization $\mathbf{M}_0$, and the magnetic anisotropy axis coincide with the direction of the superlattice axis $z$. In this case, we have $m^+ = M_x + iM_y$, where $M_x$ and $M_y$ are the projections of the magnetization vector onto the corresponding coordinate axes,

$$\nu = \frac{\omega - \omega_0}{\alpha g M_0}, \quad \Lambda = \frac{\sqrt{2}\Delta\beta}{\alpha},$$

$\omega$ is the frequency of the spin wave, $\omega_0 = g[H + (\beta - 4\pi)M_0]$ is the frequency of the homogeneous ferromagnetic resonance, $g$ is the gyromagnetic ratio, $M_0 = |\mathbf{M}_0|$, and $\alpha$ is the exchange constant.

The Fourier transform of the averaged Green's function for Eq. (14) has the form

$$G(\nu, \mathbf{k}) = \frac{1}{\nu - k^2 - M(\nu, \mathbf{k})}, \tag{15}$$

where $M(\nu, \mathbf{k})$ is the classical analog of the mass operator, which in the Bourret approximation [24] can be represented in the form [22]

$$M(\nu, \mathbf{k}) = -\frac{\Lambda^2}{8\pi}\int \frac{K(\mathbf{r})}{|\mathbf{r}|}\exp[-i(\mathbf{kr} + \sqrt{\nu}|\mathbf{r}|)]d\mathbf{r}. \tag{16}$$

In a thin film, the magnetic susceptibility is $\hat{\chi} \sim G$ [20]. In this respect, hereafter, in the study of the spin waves in the superlattice with the inhomogeneous anisotropy parameter, we will discuss the behavior of the Green's function $G(\nu, \mathbf{k})$.

Substituting the correlation function $K(\mathbf{r})$ in the form of expression (7) with $K_m$ in the form of relationship (12) into expression (16) and changing over to the spherical coordinate system with the polar axis along the vector $\mathbf{k}$ ($\mathbf{k} \parallel Oz$), we have

$$M(\nu, \mathbf{k}) = -\frac{\Lambda^2}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\int_0^{\infty} dr r \exp(-i\sqrt{\nu}r) \times \int_0^1 dc \frac{\sum_{j,h=0}^{1}\exp\{i(-1)^j r[(2m+1)q + (-1)^h k]c\}}{\left[1 + \frac{C^2}{e^2}k_2^2 r^2(1-c^2)\right]^{(2m+1)^2\gamma_2^2}}, \tag{17}$$

where $c = \cos\vartheta$ and $\vartheta$ is the polar angle. Here, we investigate the high-frequency susceptibility at the boundaries of odd Brillouin zones ($k = k_{rp} \equiv pq/2$). After integrating over $r$ in expression (17), substituting the obtained relationship into the Green's function (15), and introducing the dimensionless quantities, we obtain

$$\Lambda G(\nu) = \left\{X_p + \frac{4}{\pi^2}\frac{\nu_{rp}}{\Lambda\eta_{2p}^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\left(\frac{e}{C}\right)^a \times \int_0^1 \frac{dc}{(1-c^2)^a}\sum_{j,h=0}^{1} J_{jhm}(c, X_p)\right\}^{-1}, \tag{18}$$

where

$$X_p = \frac{\nu - \nu_{rp}}{\Lambda}, \quad \eta_{2p} = \frac{2k_2 k_{rp}}{\Lambda},$$

$\nu_{rp} = k_{rp}^2$, $a = (2m+1)^2\gamma_2^2$, $p = 1, 3, 5, \ldots$ is the number of the odd zone, and

$$J_{jhm}(c, X_p) = -2^{-1/2-a}\sqrt{\pi}\Gamma(1-a)\left(\frac{e^2}{C^2(1-c^2)}\right)^{1-a} \times \left[\frac{2^{-1/2+a}}{\sqrt{\pi}\Gamma(2-a)} - \frac{\mathbf{H}_{3/2-a}(iw_{jhm}^{(p)})}{(iw_{jhm}^{(p)})^{1/2-a}} + \frac{Y_{3/2-a}(iw_{jhm}^{(p)})}{(iw_{jhm}^{(p)})^{1/2-a}}\right]. \tag{19}$$

Here,

$$w_{jhm}^{(p)} = \frac{e}{\eta_{2p}C\sqrt{1-c^2}} \times \left[X_p + \frac{2\nu_{rp}}{\Lambda}\left(1 + (-1)^j c\frac{4m+2+(-1)^h p}{p}\right)\right], \tag{20}$$

$\mathbf{H}_\nu(z)$, $Y_\nu(z)$, and $\Gamma(z)$ are the Struve, Neumann, and gamma functions, respectively. However, relationship (19) holds true at $(2m+1)^2\gamma_2^2 \neq N$, where $N$ is an integer number. At $(2m+1)^2\gamma_2^2 = N$, instead of the function $J_{jhm}(c, X_p)$, simpler expressions enter into relationship (18). For example, at $(2m+1)^2\gamma_2^2 = 1$, the following function should be written under the integral sign in relationship (18):

$$F_{jhm}(c, X_p) = \frac{1}{2}[\exp(-w_{jhm}^{(p)})E_1(-w_{jhm}^{(p)}) + \exp(w_{jhm}^{(p)})E_1(w_{jhm}^{(p)})]. \tag{21}$$

At $(2m+1)^2\gamma_2^2 = 2$, the corresponding function has the form

$$P_{jhm}(c, X_p) = \frac{C^2(1-c^2)}{2e^2} \times \left\{1 - \frac{w_{jhm}^{(p)}}{2}[\exp(w_{jhm}^{(n)})E_1(w_{jhm}^{(p)}) - \exp(-w_{jhm}^{(p)})E_1(-w_{jhm}^{(p)})]\right\}. \tag{22}$$

If $j = h = 1$ is chosen in formula (20) at $m = 0$ and $p = 1$, it transforms into the expression for the quantity $w$ in [23].

The integral over $c$ in relationship (18) was calculated numerically. For large arguments of the Struve and Neumann functions, we used the asymptotics of their difference [14]:

$$\mathbf{H}_v(\xi) - Y_v(\xi) = \frac{1}{\pi}\left\{\frac{\Gamma(1/2)}{\Gamma(v+1/2)(\xi/2)^{1-v}} + \frac{\Gamma(3/2)}{\Gamma(v-1/2)(\xi/2)^{3-v}}\right\}, \quad (23)$$

where $|\xi| \gg 1$ and $|\arg\xi| < \pi$. It should be noted that, according to the used approximation of narrow band gaps, the inequality $\Lambda/\nu_{r1} \ll 1$ is valid in relationship (18). Moreover, relationship (18) was derived using the condition for the smallness of the damping associated with the two-dimensional inhomogeneities $k_2\gamma_2^2 \ll k_{r1}$.

In the absence of inhomogeneities in the superlattice and by disregarding the natural damping of waves, the band gap in the spectrum at $k = k_{rp}$ (corresponding to the spacing between the levels of the split spectrum $\nu_+(k_{rp})$ and $\nu_-(k_{rp})$) is $\Delta\nu_g = \Lambda_1/p$, $\Lambda_1 = 2\sqrt{2}\Lambda/\pi$. In this case, the dependence $G''(\nu) = \mathrm{Im}G(\nu)$ will exhibit two δ-shaped peaks with the separation $\Delta\nu_p = \Delta\nu_g$. The presence of random inhomogeneities in the superlattice leads to a modification of these peaks. As the root-mean-square fluctuations increase, the peaks broaden, decrease in the height, and approach each other until they merge together into one peak (for one-dimensional inhomogeneities) [10] or only one peak broadens and drastically decreases in the height up to the complete disappearance, whereas the second peak remains unchanged (for two-dimensional inhomogeneities) [23]. In this case, we have $\Delta\nu_p \neq \Delta\nu_g$; however, the quantity $\Delta\nu_p$ can be used for evaluating the behavior of the band gap $\Delta\nu_g$.

It can be seen from relationship (18) that the multizone scheme used for the calculations allows us to examine the frequency dependence of the magnetic susceptibility (Green's function) at the boundaries of all odd Brillouin zones. Let us consider the boundary of the first Brillouin zone ($p = 1$). Since, as follows from expressions (7)–(9) and Fig. 2, the correlation function $K(\mathbf{r})$ is described with a high accuracy by the first two terms, in numerical calculations in relationship (18), we take into account only terms with $m = 0$ and $m = 1$ in the sum over $m$; i.e., we consider the wave spectrum at the boundary of the first Brillouin zone with the inclusion of the influence of the third zone. The results of the calculations of the imaginary part of the Green's function (18) at the boundary of the first Brillouin zone with a high accuracy reproduce graphs in Fig. 2 from [23] upon change in the normalization from $\Lambda$ to $\Lambda_1$. This agreement is associated with the fact that, in the expansion of the function describing the ideal superlattice with the rectangular profile into a Fourier series, the first harmonic ($m = 0$) with the period $l$ has the largest amplitude. As a result, the main contribution to the Green's function $G(\nu)$ at the boundary of the first Brillouin zone is made by the term in relationship (18) with $j = h = 1$ that is close to the Green's function for the sinusoidal superlattice [23].

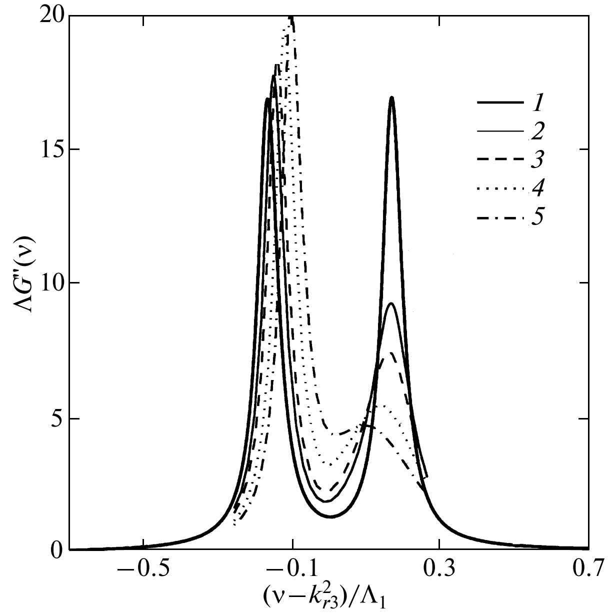


**Fig. 4.** Imaginary parts of the Green's function (18) at the boundary of the third Brillouin zone of the superlattice with two-dimensional inhomogeneities for $\eta_{21} = 4$, $\nu_{r1}/\Lambda = 20$, and $\gamma_2^2$ = (*1*) 0, (*2*) 0.03, (*3*) 0.05, (*4*) 0.10, and (*5*) 0.16.

Let us now analyze the boundary of the third Brillouin zone of the superlattice ($p = 3$). As before, we take into account only the first two terms in the sum over $m$ in relationship (18). The results of the calculations of the imaginary part of the Green's function (18) at the boundary of the third Brillouin zone are presented in Fig. 4. It can be seen from this figure that the peak at the edge of the band gap in the wave spectrum with a lower frequency remains almost unchanged with an increase in the quantity $\gamma_2$, whereas the peak at the edge of the band gap with a higher frequency broadens and decreases in the height with an increase in the quantity $\gamma_2$. Therefore, the qualitative behaviors of the peaks of the magnetic susceptibility at the boundaries of the first and third Brillouin zones coincide with each other. However, the right peak at the boundary of the third zone disappears at a considerably smaller value of $\gamma_2$. A finite height of the left

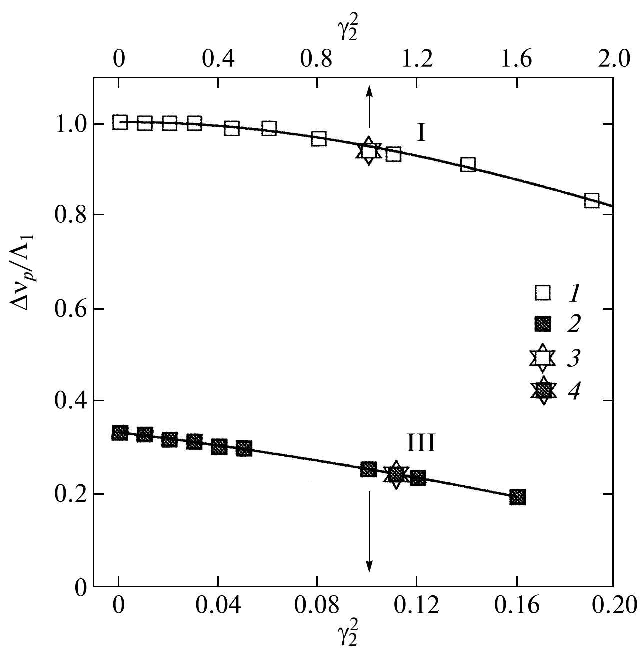


**Fig. 5.** Dependences of the separation between the peaks of the imaginary part of the Green's function $\Delta\nu_p/\Lambda_1$ on the quantity $\gamma_2^2$ at the boundaries of the (*1*) first and (*2*) third Brillouin zones. Points *3* and *4* indicate the values of $\gamma_2^2 = 1/(2m+1)^2$ for which function (21) was used in the integrand of relationship (18). Solid curves I and III were obtained by the interpolation of the sequences of points *1*, *3* and *2*, *4*, respectively.

peak in Fig. 4 and its nonzero width are due to the introduction of the "bare damping" $\Gamma_0/\Lambda = 0.03$ in the numerical calculations in order to eliminate the divergences.

The dependences of the separation between the peaks of the imaginary part of the Green's function (18) on the square of the root-mean-square fluctuation of the two-dimensional inhomogeneities $\gamma_2^2$ at the boundaries of the first and third zones are plotted in Fig. 5. It should be noted that, in this and subsequent figures, there are points additionally marked by the asterisk. At these points, the relationship $(2m+1)^2\gamma_2^2 = 1$ is satisfied and expression (21) instead of $J_{jhm}(c, X_p)$ was used in the Green's function (18). It can be seen from Fig. 5 that these points fit well the sequence corresponding to other values of $\gamma_2^2$. As can be seen from Fig. 5, one peak at the boundary of the third Brillouin zone at the value of $\gamma_2^2$ that is approximately one order of magnitude smaller than that at the boundary of the first Brillouin zone remains in the dependence of the magnetic susceptibility on the frequency. However, the band gap in the wave spectrum at the boundary of the third Brillouin zone decreases under the influence of two-dimensional inhomogeneities to a considerably smaller extent as compared to that in the presence of one-dimensional inhomogeneities. According to [10], the band gap in the wave spectrum of the superlattice with one-dimensional inhomogeneities is determined by the relationship

$$\Delta\nu_g = \frac{\Lambda_1}{p}\sqrt{1 - \frac{\pi^2}{8}\gamma_1^4 p^8 \eta_1^2}, \qquad (24)$$

where $\eta_1 = qk_1/\Lambda$ and $\gamma_1$ and $k_1$ are the relative root-mean-square fluctuation of one-dimensional inhomogeneities and their correlation wave number, respectively. It can be seen from this relationship that the closure of the band gap in the wave spectrum at the boundary of the third Brillouin zone occurs at the values of $\gamma_1^2$ that are two orders of magnitude smaller than those corresponding to the closure of the band gap at the boundary of the first Brillouin zone. Therefore, the influence of the two-dimensional inhomogeneities on the spectrum at the boundaries of the odd Brillouin zones turns out to be considerably less pronounced as compared to the one-dimensional inhomogeneities. It should be noted that, as the zone number $p$ increases, this tendency manifests itself to a greater extent.

The half-widths of the peaks at half-height of the corresponding peaks of the function $G''(\nu)$ at the boundaries of the first and third zones are presented in Fig. 6. The dependences of the half-widths of the left and right peaks on the quantity $\gamma_2^2$ for two-dimensional inhomogeneities at the boundaries of the first and third zones were constructed by subtracting the bare damping $\Gamma_0/\Lambda = 0.03$ from the calculated half-widths of the peaks. Therefore, the half-widths in Fig. 6 do not correspond to those in Fig. 4. In particular, the half-width of the left peak after this operation becomes approximately equal to zero. It can be seen from Fig. 6 that there is a sharp asymmetry in the half-widths of the left and right peaks, which for the latter peak almost linearly increases with an increase in the quantity $\gamma_2^2$. The half-width of the right peak at the boundary of the third zone is larger than that at the boundary of the first zone for the same values of $\gamma_2^2$. The slope of the straight line $\Gamma_3(\gamma_2^2)$ is also larger than the slope of the straight line $\Gamma_1(\gamma_2^2)$, even though the ratio between the slopes in Fig. 6 is visually opposite, which is associated with the difference between the scales of the upper and lower horizontal axes. These results are in agreement with the data obtained in [10] on the damping of waves $\nu_g''$ in the superlattice with

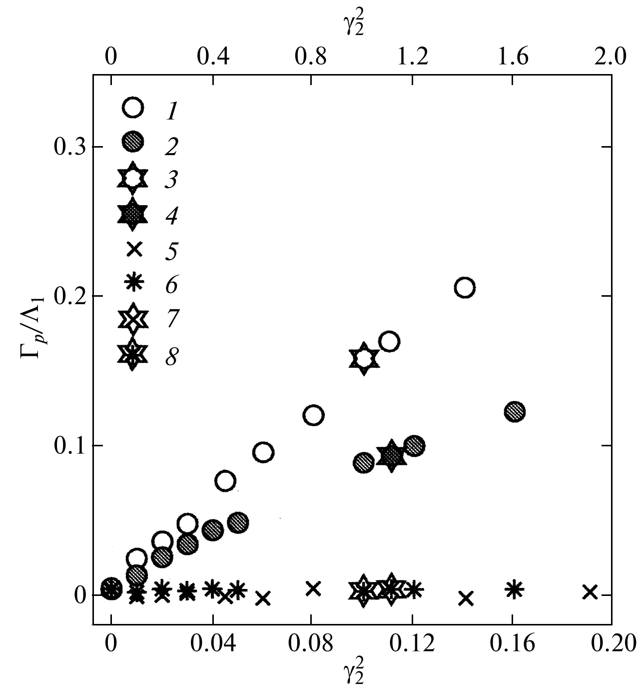


**Fig. 6.** Dependences of the half-widths of the peaks of the imaginary part of the Green's function $\Gamma_p/\Lambda_1$ on the quantity $\gamma_2^2$ at the boundaries of the first Brillouin zone for the (*1*) right and (*5*) left peaks (upper axis $\gamma_2^2$) and the third Brillouin zone for the (*2*) right and (*6*) left peaks (lower axis $\gamma_2^2$). Points *3*, *4*, *7*, and *8* indicate the values of $\gamma_2^2 = 1/(2m+1)^2$ for which expression (21) was used in the integrand of relationship (18).

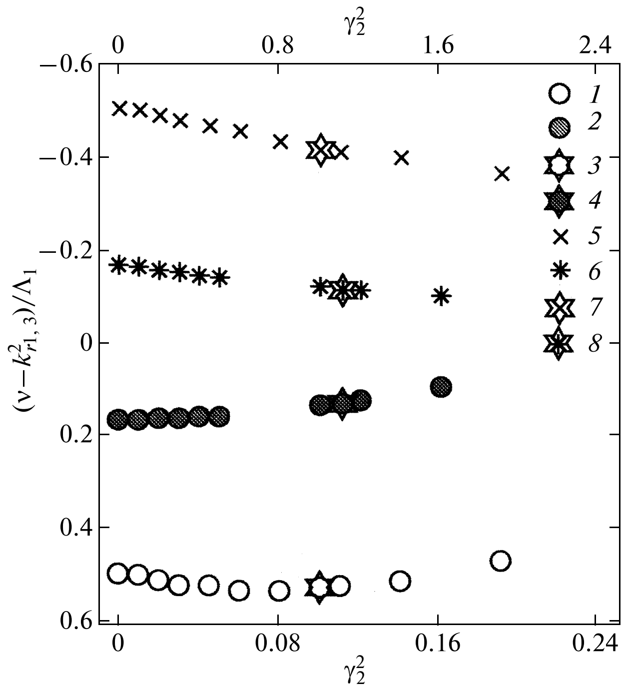


**Fig. 7.** Dependences of the positions of the peaks of the imaginary part of the Green's function on the quantity $\gamma_2^2$ at the boundaries of the first Brillouin zone for the (*1*) right and (*5*) left peaks (upper axis $\gamma_2^2$) and the third Brillouin zone for the (*2*) right and (*6*) left peaks (lower axis $\gamma_2^2$). Points *3*, *4*, *7*, and *8* indicate the values of $\gamma_2^2 = 1/(2m+1)^2$ for which expression (21) was used in the integrand of relationship (18).

the one-dimensional inhomogeneities. For the open band gap in the wave spectrum, we have

$$\nu_g'' = \gamma_1^2 k_1 q p^3. \tag{25}$$

According to this expression, the damping depends linearly on $\gamma_1^2$ at the boundary of any odd Brillouin zone and the slope of the straight line increases with an increase in the zone number $p$. It should be noted that, as follows from expression (25), the coefficient of the quantity $\gamma_1^2$ for $\nu_g''$ at $p = 3$ as compared to the coefficient for $\nu_g''$ at $p = 1$ increases by a factor of ≈30, whereas this coefficient for the superlattice with the two-dimensional inhomogeneities increases by a factor of only three.

The dependences of the positions of the maxima of the left and right peaks on the root-mean-square fluctuation at the boundaries of the first and third Brillouin zones are plotted in Fig. 7. It can be seen from this figure that there is an insignificant asymmetry in the positions of the peaks with respect to the centers of the band gaps and a shift in these centers toward the high-frequency range. It should also be noted that, when the right peak disappears, the left peak appears to be shifted from the center of the band gap in the ideal superlattice toward the low-frequency range.

Now, we discuss the revealed effect of the asymmetry of the width and amplitudes of the peaks of the Green's function due to the presence of two-dimensional inhomogeneities in the superlattice. For this purpose, the mass operator of the Green's function is conveniently expressed through the spectral density $S(\mathbf{k})$, i.e., the Fourier transform of the correlation function. In this case, within the Bourret approximation, we have

$$M(\nu, \mathbf{k}) = \frac{\Lambda^2}{2}\int \frac{S(\mathbf{k}-\mathbf{k}_s)}{\nu - k_s^2} d\mathbf{k}_s. \tag{26}$$

The spectral density of the superlattice with two-dimensional inhomogeneities that corresponds to the correlation function (7) can be written in the form

$$S(\mathbf{k}-\mathbf{k}_s) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{S_m(\mathbf{k}_{s\perp})}{(2m+1)^2} \quad (27)$$
$$\times[\delta(k_z-k_{sz}-(2m+1)q)+\delta(k_z-k_{sz}+(2m+1)q)],$$

where $\mathbf{k}_{s\perp} = \{k_{sx}, k_{sy}\}$. Substituting the spectral density $S(\mathbf{k}-\mathbf{k}_s)$ in this form into expression (26) and integrating over $k_{sz}$, we obtain

$$M(\nu, \mathbf{k}) = \frac{\Lambda^2}{4}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \times \int\frac{d\mathbf{k}_{s\perp}S_m(\mathbf{k}_{s\perp})}{\nu-(k_z-(2m+1)q)^2-k_{s\perp}^2}. \quad (28)$$

Since we consider the main branch of the wave spectrum in the range $k_z > 0$, the terms associated with the second term in square brackets in expression (27) are omitted in relationship (28). As is known, the law of conservation of the angular momentum in an inhomogeneous medium is invalid. However, constraints that follow from the law of conservation of the energy of the incident and scattered waves and the dispersion law of the scattered wave are imposed on the magnitude of the wave vector. The law of conservation of the energy in the propagation of the wave in the superlattice with the two-dimensional inhomogeneities is determined by the pole on the real axis in the integrand of expression (28); that is,

$$k_{s\perp}^2 = \nu-(k_z-(2m+1)q)^2. \quad (29)$$

The same pole determines the damping of the incident wave. Let us consider the boundary of the first Brillouin zone: $k_z = k_{r1} = q/2$. At $m = 0$, from relationship (29), we obtain the conservation law derived in [23]:

$$k_{s\perp}^2 = \nu-\nu_{r1}, \quad (30)$$

where $\nu_{r1} = k_{r1}^2$ is the frequency corresponding to the center of the band gap in the ideal superlattice at the boundary of the first Brillouin zone. The conservation law (30) is satisfied only at frequencies $\nu > \nu_{r1}$. Consequently, the damping will arise only in the range of the first peak of the unperturbed Green's function at $\nu = \nu_{r1} + \Lambda_1/2$ (Fig. 6). At frequencies $\nu < \nu_{r1}$, the wave scattering is forbidden. As a result, the damping of waves will be absent in the range of the left peak of the Green's function at $\nu = \nu_{r1} - \Lambda_1/2$. For $m > 0$ and frequencies $\nu$ in the vicinity of $\nu_{r1}$, the conservation law (29) at the boundary of the first Brillouin zone ($k_z = k_{r1}$) cannot be satisfied. This means that the terms with $m > 0$ in relationship (18) will not make a contribution to the damping of waves in the vicinity of the boundary of the first zone.

At the boundary of the third Brillouin zone ($k_z = k_{r3} = 3q/2$) at $m = 1$, from expression (29), we derive

$$k_{s\perp}^2 = \nu-\nu_{r3}, \quad (31)$$

where $\nu_{r3} = k_{r3}^2$ is the frequency that corresponds to the center of the band gap at the boundary of the third Brillouin zone. This law can hold true at frequencies $\nu > \nu_{r3}$, and the wave scattering is forbidden at frequencies $\nu < \nu_{r3}$. This implies that the damping will arise in the range of the right peak of the Green's function and be absent in the range of the left peak (Figs. 4, 6). At the boundary of the third Brillouin zone for $m = 0$, from expression (29), we also have relationship (30); however, with the difference that the frequency $\nu$ varies in the vicinity of $\nu_{r3}$. At these frequencies $\nu$, the conservation law (30) can be obeyed. This means that there is a contribution of the first term with $m = 0$ in relationship (18) to the damping of waves in the vicinity of the boundary of the third Brillouin zone. However, at the boundary of the third Brillouin zone, there arises an asymmetry of the peaks of the Green's function (Fig. 4) due to the conservation law (31). At $m > 1$ and frequencies in the vicinity of $\nu_{r3}$, the conservation law (29) at the boundary of the third Brillouin zone cannot be satisfied. Consequently, the terms with $m > 1$ in relationship (18) will not make a contribution to the damping of these waves. This circumstance additionally justifies the above choice of the first two terms in the sum over $m$ in relationship (18) in the numerical investigation of the Green's function.

Therefore, it follows from expression (29) that, at the boundary of an odd Brillouin zone with the number $p$ ($k_z = k_{rp}$), the damping of waves in relationship (18) is described by the terms with $m \le (p-1)/2$, whereas the terms with $m$ inconsistent with this relationship do not contribute to the damping.

## 4. CONCLUSIONS

Thus, the high-frequency susceptibility of a layered ferromagnetic structure (superlattice that initially has a rectangular profile) with two-dimensional inhomogeneities has been investigated. These inhomogeneities can be treated as a limiting case of defects extended along the superlattice axis when the correlation length in this direction is considerably larger than the correlation length of inhomogeneities in the $xy$ plane: $\tilde{r}_\parallel \gg \tilde{r}_\perp$. In practice, this situation can occur when the inhomogeneities of the surfaces of the superlattice layers result from the inhomogeneous deformation of the substrate surface onto which these layers are deposited. In this case, the deformations random in the $xy$ plane can be reproduced almost in phase on the surface of each new deposited layer and the superlattice can be approximately described by the correlation function that has a finite correlation length in the $xy$ plane of the layers and an infinite correlation length along the $z$ axis.

The correlation function of the layered system with two-dimensional phase inhomogeneities has been calculated. By approximating the correlation function with a simple relationship, it has been demonstrated that the effect of the asymmetry of the amplitudes and widths of the peaks of the Green's function at the edges of the band gap in the wave spectrum due to the two-dimensional inhomogeneities, which was previously revealed in [23] for the first Brillouin zone of the sinusoidal superlattice, takes place in a multilayered ferromagnetic system with an inhomogeneous magnetic anisotropy parameter at the boundaries of all odd Brillouin zones. This effect results from the law of conservation of the energy of incident and scattered waves [23] and, most likely, can be observed for superlattices of other types in the presence of two-dimensional inhomogeneities. In particular, this effect can be expected in an initially ideal one-dimensional superlattice with an arbitrary thickness of layer interfaces [11]. In practice, the effect of peak asymmetry can be used for investigating inhomogeneities in superlattices. The experimental observation of this effect for the high-frequency susceptibility would indicate the presence of two-dimensional inhomogeneities in the superlattice.

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