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Crossing Resonance of Stochastically Interacting Wave Fields

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Abstract—The dynamic susceptibilities (Green's functions) of the system of two interacting wave fields of different physical natures with a stochastically inhomogeneous coupling parameter between them with zero mean value have been examined. The well-known self-consistent approximation taking into account all diagrams with noncrossing correlation/interaction lines has been generalized to the case of stochastically interacting wave fields. The analysis has been performed for spin and elastic waves. The results obtained taking into account the processes of multiple scattering of waves from inhomogeneities are significantly different from those obtained for this situation earlier in the Bourret approximation [R.C. Bourret, Nuovo Cimento **26**, 1 (1962)]. Instead of frequencies degeneracy removal in the wave spectrum and the splitting of resonance peaks of dynamic susceptibilities, a wide single-mode resonance peak should be observed at the crossing point of the unperturbed dispersion curves. The fine structure appears at vertices of these wide peaks in the form of a narrow resonance on the Green's-function curve of one field and a narrow antiresonance on the vertex of the Green's-function curve of the other field.

1. INTRODUCTION

It is well known that the interaction between two wave fields of different physical natures in a solid is responsible for the existence of coupled oscillations of these fields (magnetoelastic waves, polaritons, etc.). At the crossing point of the dispersion curves (if it exists), this interaction results in crossing resonance at which the features of coupled oscillations are most pronounced. For definiteness, we consider in this work crossing resonance between spin and elastic waves, which is usually called magnetoelastic resonance. Magnetoelastic resonance was predicted by A.I. Akhiezer at the International Conference on the Physics of Magnetic Phenomena [1]. The phenomenological theory of this phenomenon was developed in [2–4]. These fundamental works stimulated intensive theoretical and experimental investigations of effects caused by the interaction between spin and elastic waves in ferro-, ferri-, and antiferromagnets, which were reported in original papers, as well as in reviews and books [5–7]. As any crossing resonance in a uniform medium, the magnetoelastic resonance is manifested both in degeneracy removal of the interacting wave field frequencies at the crossing point of their unperturbed dispersion curves and in the appearance of two resonance peaks in the frequency dependences of the Green's functions $G_m''(\omega)$ and $G_u''(\omega)$ of spin and elastic waves, respectively. The gap between the energy levels in the spectrum, as well as the spacing between the maxima of each Green's function, is determined by the coupling parameter ε between the wave fields.

Crossing resonance in an inhomogeneous medium was studied in the Bourret approximation [8, 9] (single scattering of waves from inhomogeneities) in [10]. The investigation was performed for the extremely inhomogeneous model of the interaction between two wave fields, where the coupling parameter between the fields is a random function of the coordinates with zero mean value. In this case, the interaction between the fields is due only to spatial fluctuations of this parameter. There are two reasons for the investigation of crossing resonance in this model. First, such materials are now produced artificially because a number of physical properties are improved owing to zero mean value of the coupling parameter (e.g., the initial susceptibility in alloys with zero magnetostriction increases [11]). Second, the theory of the spectrum of waves in the extremely inhomogeneous medium should be developed for the subsequent transition to a more complex case of an arbitrary relation between the mean and rms values of the coupling parameter.

Consideration performed in [10] for the scalar model of acoustic and optical phonons predicted disorder-induced crossing resonance, i.e., degeneracy removal and the formation of a gap in the spectrum at the crossing point of dispersion curves of wave fields. In contrast to crossing resonance in the homogeneous medium, the gap in this case is determined by rms fluctuation of the coupling parameter $\Delta\varepsilon$. Other significant differences from the homogeneous case were also obtained; they will be discussed in Section 4. In the sequel, disorder-induced crossing resonance was also

examined for coupled magnetoelastic waves [12] and polaritons [13] (see also [14] and references therein).

In [15, 16], we considered the same model with the approximate inclusion of the multiple scattering of waves from inhomogeneities and obtained preliminary results fundamentally differing from those obtained in [10, 12−14]. The main result of those works remains valid: crossing resonance caused only by spatial fluctuations of the coupling parameter occurs at zero mean value of this parameter. However, physical phenomena that should be observed at this resonance are significantly different from those predicted in the Bourret approximation.

To take into account the effect of the multiple scattering of waves from inhomogeneities on the spectral properties of the waves, several variants of the self-consistent approximation are used. In this work, we use the variant of the self-consistent approximation proposed by Migdal [17] in order to take into account the electron−phonon interaction in the homogeneous medium and by Kraichnan [18] in order to examine the spectral properties of waves in inhomogeneous media. This variant of the self-consistent approximation in its development is discussed in Section 3, where the relation between this variant of self-consistent approximation and another variant, coherent potential approximation, proposed by Soven [19] and Taylor [20] is also briefly discussed.

The aim of this work is to generalize this variant of the self-consistent approximation to the case of two stochastically interacting wave fields of different physical natures and to use this generalization to develop the theory of crossing resonance in a medium with an inhomogeneous coupling parameter with zero mean value.

The paper is organized as follows. The system of coupled equations for the Green's functions of spin and elastic waves is derived in Section 2. The self-consistent approximation for one wave field is discussed in Section 3, where the self-consistent approximation for two stochastically coupled wave fields is also derived. The theory of crossing resonance in a medium with an inhomogeneous coupling parameter with zero mean value is developed in Section 4. The results are summarized and discussed in Section 5.

2. SYSTEM OF EQUATIONS FOR GREEN'S FUNCTIONS

We consider the model of a ferromagnet where only the magnetostriction parameter $\varepsilon(\mathbf{x})$, $(\mathbf{x} = \{x, y, z\})$ is inhomogeneous. The equations of motion for this medium include the Landau−Lifshitz equation for the magnetization vector $\mathbf{M}$ and the equation of motion of the theory of elasticity for the elastic displacement vector $\mathbf{u}$:

$$\dot{\mathbf{M}} = -g[\mathbf{M} \times \mathbf{H}^e], \qquad (1)$$

$$p\ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad (2)$$

where g is the gyromagnetic ratio, p is the density of the medium, and σ_{ij} is the stress tensor, where $i, j = x, y, z$. The effective magnetic field $\mathbf{H}^e$ and stress tensor σ_{ij} have the form

$$\mathbf{H}^e = -\frac{\partial \mathcal{H}}{\partial \mathbf{M}} + \frac{\partial}{\partial \mathbf{x}} \frac{\partial \mathcal{H}}{\partial (\partial \mathbf{M}/\partial \mathbf{x})}, \qquad (3)$$

$$\sigma_{ij} = \frac{\partial \mathcal{H}}{\partial u_{ij}}, \qquad (4)$$

where $u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ is the elastic strain tensor. The energy density $\mathcal{H}$ is chosen in the form

$$\mathcal{H} = \frac{\alpha}{2}\left(\frac{\partial \mathbf{M}}{\partial \mathbf{x}}\right)^2 - \mathbf{M}\mathbf{H} + \frac{\lambda u_{ii}^2}{2} + \mu u_{ij}^2$$
$$+ \varepsilon(\mathbf{x})M_i M_j u_{ij}, \qquad (5)$$

where $\mathbf{H}$ is the external static magnetic field, α is the exchange parameter, and λ and μ are the elastic force constants. The magnetoelastic parameter $\varepsilon(\mathbf{x})$ can be represented in the form

$$\varepsilon(\mathbf{x}) = \varepsilon + \Delta\varepsilon\rho(\mathbf{x}), \qquad (6)$$

where ε and $\Delta\varepsilon$ are the average value and rms fluctuation of the magnetoelastic parameter and $\rho(\mathbf{x})$ is a centered ($\langle\rho\langle\mathbf{x}\rangle\rangle = 0$) and normalized ($\langle\rho^2(\mathbf{x})\rangle = 1$) random function of coordinates. Angle brackets stand for the average over the ensemble of the realizations of the corresponding random function.

We assume that the mean value of the magnetoelastic parameter is $\varepsilon = 0$. The stochastic properties of $\rho(\mathbf{x})$ are characterized by the correlation function depending on the difference $\mathbf{r} = \mathbf{x} - \mathbf{x}'$,

$$K(\mathbf{r}) = \langle\rho(\mathbf{x})\rho(\mathbf{x} + \mathbf{r})\rangle \qquad (7)$$

or by the Fourier transform of the correlation function, i.e., the spectral density

$$S(\mathbf{k}) = \frac{1}{(2\pi)^d}\int K(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}, \qquad (8)$$

where d is the dimensionality of the space.

The substitution of energy density (5) into equations of motion (1) and (2) provides the following coupled system of equations for the magnetization $\mathbf{M}$ and displacement $\mathbf{u}$:

$$-\frac{\dot{\mathbf{M}}}{g} = \left[\mathbf{M} \times \left\{\alpha\Delta\mathbf{M} + \mathbf{H} - \varepsilon(\mathbf{x})M_j\left(\frac{\partial \mathbf{u}}{\partial x_j} + \frac{\partial u_j}{\partial \mathbf{x}}\right)\right\}\right], \qquad (9)$$

$$\ddot{\mathbf{u}} = v_t^2\Delta\mathbf{u} + (v_l^2 - v_t^2)\text{grad div}\mathbf{u}$$
$$+ \frac{1}{p}\frac{\partial}{\partial x_j}(\varepsilon(\mathbf{x})M_j\mathbf{M}), \qquad (10)$$

where $v_l = \sqrt{(\lambda + 2\mu)/p}$ and $v_t = \sqrt{\mu/p}$ are the longitudinal and transverse components of the velocity of elastic waves, respectively.

We linearize Landau–Lifshitz equation (9) in the usual way ($M_z \approx M$; M_x, $M_y \ll M$) and consider a model problem for elastic waves, where the condition $u_z = 0$, as well as the condition $v_l = v_t = v_u$ (v_u is the velocity of the elastic wave), is imposed. Supposing M_x, $M_y \propto e^{i\omega t}$) and introducing circular projections

$$m^{\pm} = M_x \pm iM_y,$$
$$u^{\pm} = u_x \pm iu_y,$$
(11)

we arrive at the following coupled system of two scalar equations for resonance projections m^+ and u^+ (below, the superscript + will be omitted):

$$\nabla^2 m + \nu_m m - \frac{\Delta\varepsilon}{\alpha} M\rho(\mathbf{x})\frac{\partial u}{\partial z} = 0,$$
(12)

$$\nabla^2 u + \nu_u u + \frac{\Delta\varepsilon}{\mu} M\frac{\partial}{\partial z}(\rho(\mathbf{x})m) = 0.$$
(13)

Here,

$$\nu_m = \frac{\omega - \omega_0}{\alpha g M}, \quad \nu_u = \frac{\omega^2}{v_u^2},$$
(14)

where ω_0 is the frequency of the homogeneous ferromagnetic resonance, which depends on the magnetic field and the demagnetizing factors of the sample and $v_u = \sqrt{\mu/p}$.

The system of equations for the Green's functions $G_m(\mathbf{x}, \mathbf{x}_0)$ and $G_u(\mathbf{x}, \mathbf{x}_0)$ for spin and elastic waves, respectively, correspond to the system of equations (12) and (13) has the form

$$\nabla^2 G_m(\mathbf{x}, \mathbf{x}_0) + \nu_m G_m(\mathbf{x}, \mathbf{x}_0)$$
$$- \frac{\Delta\varepsilon}{\alpha} M\rho(\mathbf{x})\frac{\partial G_u(\mathbf{x}, \mathbf{x}_0)}{\partial z} = \delta(\mathbf{x} - \mathbf{x}_0),$$
(15)

$$\nabla^2 G_u(\mathbf{x}, \mathbf{x}_0) + \nu_u G_u(\mathbf{x}, \mathbf{x}_0)$$
$$+ \frac{\Delta\varepsilon}{\mu} M\frac{\partial}{\partial z}(\rho(\mathbf{x})G_m(\mathbf{x}, \mathbf{x}_0)) = \delta(\mathbf{x} - \mathbf{x}_0).$$
(16)

We represent the Green's functions G_m and G_u in the form

$$G_{m, u} = G_{m, u}^0 + G_{m, u}',$$
(17)

where G_m^0 and G_u^0 are the initial Green's functions of the noninteracting spin and elastic waves, respectively, and G_m' and G_u' are the respective corrections caused by the inhomogeneous interaction parameter.

The substitution of Eqs. (17) into Eqs. (15) and (16) gives the following system of two independent equations for G_m^0 and G_u^0:

$$\nabla^2 G_m^0(\mathbf{x}, \mathbf{x}_0) + \nu_m G_m^0(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0),$$
(18)

$$\nabla^2 G_u^0(\mathbf{x}, \mathbf{x}_0) + \nu_u G_u^0(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0),$$
(19)

and the following coupled system of two equations for G_m' and G_u':

$$\nabla^2 G_m'(\mathbf{x}, \mathbf{x}_0) + \nu_m G_m'(\mathbf{x}, \mathbf{x}_0) = \Phi_u(\mathbf{x}, \mathbf{x}_0),$$
(20)

$$\nabla^2 G_u'(\mathbf{x}, \mathbf{x}_0) + \nu_u G_u'(\mathbf{x}, \mathbf{x}_0) = \Phi_m(\mathbf{x}, \mathbf{x}_0),$$
(21)

where

$$\Phi_u(\mathbf{x}, \mathbf{x}_0) = \frac{\Delta\varepsilon}{\alpha} M\rho(\mathbf{x})\frac{\partial G_u(\mathbf{x}, \mathbf{x}_0)}{\partial z},$$
(22)

$$\Phi_m(\mathbf{x}, \mathbf{x}_0) = -\frac{\Delta\varepsilon}{\mu} M\frac{\partial}{\partial z}(\rho(\mathbf{x})G_m(\mathbf{x}, \mathbf{x}_0)).$$
(23)

Formal solutions of Eqs. (20) and (21) can generally be represented in the form of integrals of the products of unperturbed Green's functions and right-hand sides of these equations. Therefore, generating integral equations for series of the Green's functions G_m and G_u have the form

$$G_m(\mathbf{x}, \mathbf{x}_0) = G_m^0(\mathbf{x}, \mathbf{x}_0)$$
$$+ \int G_m^0(\mathbf{x}, \mathbf{x}')\Phi_u(\mathbf{x}', \mathbf{x}_0)d\mathbf{x}',$$
(24)

$$G_u(\mathbf{x}, \mathbf{x}_0) = G_u^0(\mathbf{x}, \mathbf{x}_0)$$
$$+ \int G_u^0(\mathbf{x}, \mathbf{x}')\Phi_m(\mathbf{x}', \mathbf{x}_0)d\mathbf{x}'.$$
(25)

It is inconvenient that Eq. (25) contains the derivatives of the random function $\rho(\mathbf{x})$. For this reason, we transform this equation through integration by parts, as was made for a similar case in [21]:

$$\int_V G_u^0(\mathbf{x}, \mathbf{x}')\frac{\partial P(\mathbf{x}', \mathbf{x}_0)}{\partial z'}d\mathbf{x}'$$
$$= \int_S G_u^0(\mathbf{x}, \mathbf{x}')P(\mathbf{x}', \mathbf{x}_0)dxdy\big|_{z=-L}^{z=L}$$
$$- \int_V \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'}P(\mathbf{x}', \mathbf{x}_0)d\mathbf{x},$$
(26)

where $P(\mathbf{x}', \mathbf{x}_0) = \rho(\mathbf{x}')G_m(\mathbf{x}', \mathbf{x}_0)$. Setting the surface S at infinity, where the Green's functions vanish, we obtain the following form for the system of two coupled generating integral equations for Green's functions G_m and G_u:

$$G_m(\mathbf{x}, \mathbf{x}_0) = G_m^0(\mathbf{x}, \mathbf{x}_0) + \frac{\Delta\varepsilon}{\alpha} M\int G_m^0(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')$$
$$\times \frac{\partial G_u(\mathbf{x}', \mathbf{x}_0)}{\partial z'}d\mathbf{x}',$$
(27)

$$G_u(\mathbf{x}, \mathbf{x}_0) = G_u^0(\mathbf{x}, \mathbf{x}_0) + \frac{\Delta\varepsilon}{\mu} M\int \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'}$$
$$\times \rho(\mathbf{x}')G_m(\mathbf{x}', \mathbf{x}_0)d\mathbf{x}'.$$
(28)

The substitution of Eq. (27) into the right-hand side of Eq. (28) and Eq. (28) into the right-hand side

of Eq. (27) provides the following two independent integrodifferential equations for the Green's functions G_m and G_u:

$$G_m(\mathbf{x}, \mathbf{x}_0) = G_m^0(\mathbf{x}, \mathbf{x}_0)$$

$$+ \frac{\Delta\varepsilon}{\alpha} M \int G_m^0(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\frac{\partial G_u^0(\mathbf{x}', \mathbf{x}_0)}{\partial z'}d\mathbf{x}'$$

$$+ \frac{(\Delta\varepsilon)^2}{\alpha\mu} M^2 \iint G_m^0(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\frac{\partial^2 G_u^0(\mathbf{x}', \mathbf{x}'')}{\partial z'\partial z''}$$

$$\times \rho(\mathbf{x}'')G_m(\mathbf{x}'', \mathbf{x}_0)d\mathbf{x}''d\mathbf{x}', \tag{29}$$

$$G_u(\mathbf{x}, \mathbf{x}_0) = G_u^0(\mathbf{x}, \mathbf{x}_0)$$

$$+ \frac{\Delta\varepsilon}{\mu} M \int \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'}\rho(\mathbf{x}')G_m^0(\mathbf{x}', \mathbf{x}_0)d\mathbf{x}'$$

$$+ \frac{(\Delta\varepsilon)^2}{\alpha\mu} M^2 \iint \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'}\rho(\mathbf{x}')G_m^0(\mathbf{x}', \mathbf{x}'')$$

$$\times \rho(\mathbf{x}'')\frac{\partial G_u(\mathbf{x}'', \mathbf{x}_0)}{\partial z''}d\mathbf{x}'d\mathbf{x}''. \tag{30}$$

Using the usual procedure of successive iterations of each of these equations, we obtain the series for the Green's functions $G_m(\mathbf{x}, \mathbf{x}_0)$ and $G_u(\mathbf{x}, \mathbf{x}_0)$. Averaging these series over the ensemble of random functions $\rho(\mathbf{x})$ and decoupling correlation functions using the Gauss formula, we obtain the following perturbation series for the averaged Green's functions:

$$\bar{G}_m(\mathbf{x}, \mathbf{x}_0) = G_m^0(\mathbf{x}, \mathbf{x}_0) + \gamma^2 \iint G_m^0(\mathbf{x}, \mathbf{x}_1)$$

$$\times \frac{\partial^2 G_u^0(\mathbf{x}_1, \mathbf{x}_2)}{\partial z_1\partial z_2}G_m^0(\mathbf{x}_2, \mathbf{x}_0)K(\mathbf{x}_1, \mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2$$

$$+ \gamma^4 \iiiint G_m^0(\mathbf{x}, \mathbf{x}_1)\frac{\partial^2 G_u^0(\mathbf{x}_1, \mathbf{x}_2)}{\partial z_1\partial z_2}G_m^0(\mathbf{x}_2, \mathbf{x}_3)$$

$$\times \frac{\partial^2 G_u^0(\mathbf{x}_3, \mathbf{x}_4)}{\partial z_3\partial z_4}G_m^0(\mathbf{x}_4, \mathbf{x}_0) \tag{31}$$

$$\times [K(\mathbf{x}_1, \mathbf{x}_2)K(\mathbf{x}_3, \mathbf{x}_4) + K(\mathbf{x}_1, \mathbf{x}_3)K(\mathbf{x}_2, \mathbf{x}_4)$$

$$+ K(\mathbf{x}_1, \mathbf{x}_4)K(\mathbf{x}_2, \mathbf{x}_3)]d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 + \dots,$$

$$\bar{G}_u(\mathbf{x}, \mathbf{x}_0) = G_u^0(\mathbf{x}, \mathbf{x}_0) + \gamma^2 \iint \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}_1)}{\partial z_1}$$

$$\times G_m^0(\mathbf{x}_1, \mathbf{x}_2)\frac{\partial G_u^0(\mathbf{x}_2, \mathbf{x}_0)}{\partial z_2}K(\mathbf{x}_1, \mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2$$

$$+ \gamma^4 \iiiint \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}_1)}{\partial z_1}G_m^0(\mathbf{x}_1, \mathbf{x}_2) \tag{32}$$

$$\times \frac{\partial^2 G_u^0(\mathbf{x}_2, \mathbf{x}_3)}{\partial z_2\partial z_3}G_m^0(\mathbf{x}_3, \mathbf{x}_4)\frac{\partial G_u^0(\mathbf{x}_4, \mathbf{x}_0)}{\partial z_3}$$

$$\times [K(\mathbf{x}_1, \mathbf{x}_2)K(\mathbf{x}_3, \mathbf{x}_4) + K(\mathbf{x}_1, \mathbf{x}_3)K(\mathbf{x}_2, \mathbf{x}_4)$$

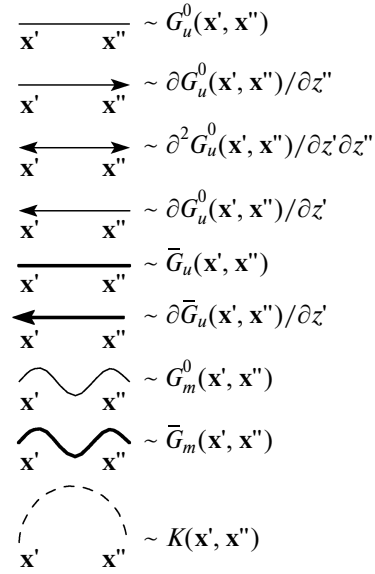$$+ K(\mathbf{x}_1, \mathbf{x}_4)K(\mathbf{x}_2, \mathbf{x}_3)]d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 + \dots,$$

where $K(\mathbf{x}_l, \mathbf{x}_n) \equiv K(\mathbf{x}_l - \mathbf{x}_n)$ are the correlation functions given by Eq. (7) and

$$\gamma^2 = \frac{(\Delta\varepsilon)^2}{\alpha\mu}M^2. \tag{33}$$

Fig. 1. Diagram notation. Arrows indicate the points at which derivatives are calculated.

It is well known that the series for the averaged Green's functions $\bar{G}$ in the presence of inhomogeneities in the local terms of the Hamiltonian contain the products of initial Green's functions G^0. In [21], it was shown that the series for $\bar{G}$ in the presence of inhomogeneities in the nonlocal terms contain the products of the first and second derivatives of the initial Green's functions G^0 and do not contain these functions themselves. On the contrary, the series of the Green's functions given by Eqs. (31) and (32) corresponding to inhomogeneities of the magnetoelastic coupling parameter contain the products of both the derivatives of initial elastic Green's functions G_u^0 and the initial spin Green's functions G_m^0. The Green's functions G_m^0 and derivatives of the Green's functions G_u^0 in each term of series (31) and (32) alternate with each other.

Figure 1 shows diagrams, where the notation introduced in [21] for the elastic Green's functions and their derivatives is used and wavy lines corresponding to the spin Green's functions are added.

The diagram representation of Eqs. (31) and (32) is shown in Fig. 2. The system of Dyson equations is
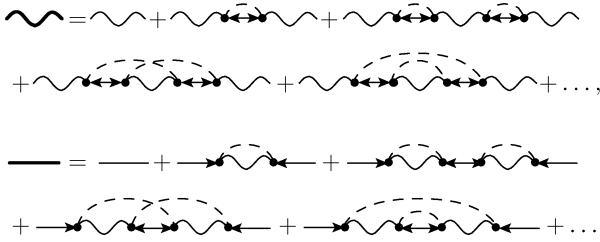
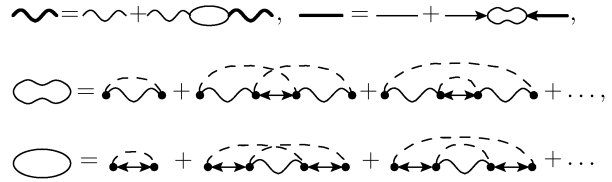**Fig. 2.** Diagrammatic representation of Eqs. (31) and (32).

Fig. 3. Diagrammatic representation of Eqs. (34)−(37).

derived from Eqs. (31) and (32) in the standard way and has the form

$$\bar{G}_m(\mathbf{x}, \mathbf{x}_0) = G_m^0(\mathbf{x}, \mathbf{x}_0)$$
$$+ \iint G_m^0(\mathbf{x}, \mathbf{x}')Q_u(\mathbf{x}', \mathbf{x}'')\bar{G}_m(\mathbf{x}'', \mathbf{x}_0)d\mathbf{x}'d\mathbf{x}'', \quad (34)$$

$$\bar{G}_u(\mathbf{x}, \mathbf{x}_0) = G_u^0(\mathbf{x}, \mathbf{x}_0) + \iint \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'}$$
$$\times Q_m(\mathbf{x}', \mathbf{x}'')\frac{\partial \bar{G}_u(\mathbf{x}'', \mathbf{x}_0)}{\partial z''}d\mathbf{x}'d\mathbf{x}'', \quad (35)$$

where the mass operators (self-energies) Q_m and Q_u are given by the series

$$Q_m(\mathbf{x}', \mathbf{x}'') = \gamma^2 G_m^0(\mathbf{x}', \mathbf{x}'') \times K(\mathbf{x}', \mathbf{x}'')$$
$$+ \gamma^4 \iint G_m^0(\mathbf{x}', \mathbf{x}_1)\frac{\partial^2 G_u^0(\mathbf{x}_1, \mathbf{x}_2)}{\partial z_1 \partial z_2}G_m^0(\mathbf{x}_2, \mathbf{x}'')$$
$$\times [K(\mathbf{x}', \mathbf{x}_2)K(\mathbf{x}_1, \mathbf{x}'') + K(\mathbf{x}', \mathbf{x}'')K(\mathbf{x}_1, \mathbf{x}_2)] \quad (36)$$
$$\times d\mathbf{x}_1 d\mathbf{x}_2 + ...,$$

$$Q_u(\mathbf{x}', \mathbf{x}'') = \gamma^2 \frac{\partial^2 G_u^0(\mathbf{x}', \mathbf{x}'')}{\partial z' \partial z''}K(\mathbf{x}', \mathbf{x}'')$$
$$+ \gamma^4 \iint \frac{\partial^2 G_u^0(\mathbf{x}', \mathbf{x}_1)}{\partial z' \partial z_1}G_m^0(\mathbf{x}_1, \mathbf{x}_2)$$
$$\times \frac{\partial^2 G_u^0(\mathbf{x}_2, \mathbf{x}'')}{\partial z_2 \partial z''}[K(\mathbf{x}', \mathbf{x}_2)K(\mathbf{x}_1, \mathbf{x}'') \quad (37)$$
$$+ K(\mathbf{x}', \mathbf{x}'')K(\mathbf{x}_1, \mathbf{x}_2)]d\mathbf{x}_1 d\mathbf{x}_2 + ...$$

Figure 3 shows the diagrams corresponding to the system of Dyson equations (34) and (35) and mass operators (36) and (37). The main feature of the system of Dyson equations is that the elastic mass operator Q_u enters into the equation for the spin Green's function $\bar{G}_m$ and the spin mass operator Q_m appears in the equation for the elastic Green's function $\bar{G}_u$.

All quantities for a randomly homogeneous medium depend on the coordinate difference $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. Performing the Fourier transforms

$$\bar{G}_{m,u}(\mathbf{r}) = \int \bar{G}_{m,u}(\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{r}}d\mathbf{k},$$
$$G_{m,u}^0(\mathbf{r}) = \int G_{m,u}^0(\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{r}}d\mathbf{k}, \quad (38)$$

$$Q_{m,u}(\mathbf{r}) = \int Q_{m,u}(\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{r}}d\mathbf{k},$$

we obtain the system of Dyson equations for the Fourier transforms of the Green's functions from Eqs. (35) and (36) and express $\bar{G}_m(\mathbf{k})$ and $\bar{G}_u(\mathbf{k})$ from this system as

$$\bar{G}_m(\mathbf{k}) = \frac{1}{[G_m^0(\mathbf{k})]^{-1} - (2\pi)^{2d}Q_u(\mathbf{k})}, \quad (39)$$

$$\bar{G}_u(\mathbf{k}) = \frac{1}{[G_u^0(\mathbf{k})]^{-1} - (2\pi)^{2d}k_z^2 Q_m(\mathbf{k})}. \quad (40)$$

The following expressions for the Fourier transforms $G_{m,u}^0(\mathbf{k})$ of the initial Green's functions are obtained from Eqs. (18) and (19):

$$G_m^0(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu_m - k^2}, \quad (41)$$

$$G_u^0(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu_u - k^2}. \quad (42)$$

The substitution of Eqs. (41) and (42) into Eqs. (39) and (40) gives the following final form of the system of Dyson equations in the $\mathbf{k}$ space:

$$\bar{G}_m(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu_m - k^2 - (2\pi)^d Q_u(\mathbf{k})}, \quad (43)$$

$$\bar{G}_u(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu_u - k^2 - (2\pi)^d k_z^2 Q_m(\mathbf{k})}. \quad (44)$$

Thus, all exact equations for the Green's functions $\bar{G}_m$ and $\bar{G}_u$ are represented in the $\mathbf{r}$ and $\mathbf{k}$ spaces.

3. SELF-CONSISTENT APPROXIMATION

3.1. One Wave Field

Before deriving the self-consistent approximation for two stochastically interacting fields, we briefly recall the main stages of the derivation of a similar approximation for one wave field. This section is necessary because the concepts and formulas presented here are used below and for some terminological specifications. This variant of the self-consistent approximation was used at the end of the 1950s and the beginning of the 1960s for a problem that does not concern

inhomogeneous media. It was proposed by Migdal [17] for studying the electron–phonon interaction in the homogeneous medium and was analyzed in detail by Pines [22, 23], Puff and Whitfield [24], and Abrikosov, Gor'kov, and Dzyaloshinskii [25].

We briefly present the key results of those works. The system of equations for electrons and photons is approximately reduced to one equation for the electron Green's function $G(x_i, x_j)$, where the points x_i and x_j are related by the electron–phonon interaction operator $D(x_i, x_j)$. As a result, the Dyson equation for the electron Green's function is obtained in the standard form

$$\bar{G}(\mathbf{x}, \mathbf{x}_0) = G^0(\mathbf{x}, \mathbf{x}_0)$$
$$+ \iint G^0(\mathbf{x}, \mathbf{x}')Q(\mathbf{x}', \mathbf{x}'')\bar{G}(\mathbf{x}'', \mathbf{x}_0)d\mathbf{x}'d\mathbf{x}'', \quad (45)$$

where both the Green's function $G^0(\mathbf{x}, \mathbf{x}')$ and mass operator $Q(\mathbf{x}', \mathbf{x}'')$ depend on the parameters of the electron–phonon interaction. The Migdal self-consistent approximation is based on the representation of the mass operator $Q(\mathbf{x}', \mathbf{x}'')$ in the form

$$Q(\mathbf{x}', \mathbf{x}'') \approx \bar{G}(\mathbf{x}', \mathbf{x}'')D(\mathbf{x}', \mathbf{x}''). \quad (46)$$

After the Fourier transform of Eqs. (45) and (46), the Green's function $\bar{G}(\mathbf{k})$ is expressed from the first equation in terms of the mass operator $Q(\mathbf{k})$ in the standard form

$$\bar{G}(\mathbf{k}) = \frac{1}{[G^0(\mathbf{k})]^{-1} - (2\pi)^{2d}Q(\mathbf{k})}, \quad (47)$$

and the second equation has the form

$$Q(\mathbf{k}) \approx \int \bar{G}(\mathbf{k}_1)D(\mathbf{k} - \mathbf{k}_1)d\mathbf{k}. \quad (48)$$

The initial Green's function for this problem in the **k** space has the form [22]

$$G^0(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu - \nu_0 - k^2}, \quad (49)$$

where ν and ν_0 are the normalized frequencies of electrons and phonons, respectively. Substituting Eq. (49) into Eq. (47), changing **k** by $\mathbf{k}_1$ in Eq. (47), and substituting the resulting equation into Eq. (48), we obtain the following final equations of this variant of the self-consistent approximation:

$$\bar{G}(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu - \nu_0 - k^2 - M_\mathbf{k}}, \quad (50)$$

$$M_\mathbf{k} \approx \int \frac{D(\mathbf{k} - \mathbf{k}_1)d\mathbf{k}_1}{\nu - \nu_0 - k_1^2 - M_{\mathbf{k}_1}}, \quad (51)$$

where $M_\mathbf{k} = (2\pi)^d Q(\mathbf{k})$ is the renormalized mass operator.

To examine the properties of the introduced approximation, Eq. (46) in [22−24] was substituted into Eq. (45) and the successive iteration of the obtained closed nonlinear equation for the Green's function in the **x** space was performed. As a result, a
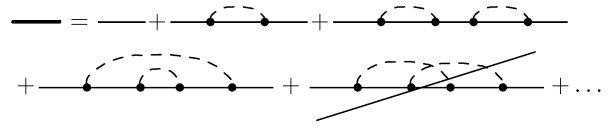

Fig. 4. Series for the approximate Green's function of one wave field.

series for the approximate Green's function $\bar{G}(\mathbf{x}, \mathbf{x}_0)$, whose diagrammatic representation is shown in Fig. 4, where the solid and dashed lines correspond to the initial electron Green's function and phonon interaction, respectively, in contrast to all other diagrams in this work. This series contains all diagrams entering into the exact expressions for $\bar{G}(\mathbf{x}, \mathbf{x}_0)$, except for the diagrams with crossing lines of the phonon interaction (the first of such diagrams in Fig. 4 is crossed out). Puff and Whitfield [24] showed that the initial self-consistent approximation corresponds to the limit $T \longrightarrow 0$ in the Hartree−Fock approximation. The variant of the self-consistent approximation specified by Eqs. (50) and (51) was widely used in the calculations of various effects of the electron–phonon interaction (see, e.g., [26, 27]).

In the same years, a similar variant of the self-consistent approximation was independently proposed by Kraichnan [18], who studied the effect of inhomogeneities on the dynamic susceptibility of disordered systems. The derivation of this self-consistent approximation from the expansion of the vertex part of the Green's function can be found in [28]. The Dyson equation has the form of Eq. (45) and the mass operator is approximately represented in the form

$$Q(\mathbf{x}', \mathbf{x}'') \approx \gamma^2 \bar{G}(\mathbf{x}', \mathbf{x}'')K(\mathbf{x}', \mathbf{x}''), \quad (52)$$

where γ and $K(\mathbf{x}', \mathbf{x}'')$ are the rms fluctuation and normalized correlation function of inhomogeneities, respectively ($K(\mathbf{x}', \mathbf{x}') = 1$). In [18], this approximate equation was substituted into Dyson equation (45) and the following closed integral equation nonlinear in $\bar{G}$ is obtained:

$$\bar{G}(\mathbf{x}, \mathbf{x}_0) \approx G^0(\mathbf{x}, \mathbf{x}_0) + \gamma^2 \iint G^0(\mathbf{x}, \mathbf{x}')K(\mathbf{x}', \mathbf{x}'')$$
$$\times \bar{G}(\mathbf{x}', \mathbf{x}'')\bar{G}(\mathbf{x}'', \mathbf{x}_0)d\mathbf{x}'d\mathbf{x}''. \quad (53)$$

The successive iteration of Eq. (53) leads to a series for Green's functions whose diagrammatic representation formally coincides with the representation shown in Fig. 4, but the dashed lines denote stochastic correlation between the points rather than the interaction. The Fourier transform of Eq. (53) gives the following simpler form of the Kraichnan equation in the **k** space:

$$\bar{G}(\mathbf{k}) \approx G^0(\mathbf{k}) + \gamma^2 (2\pi)^{2d}G^0(\mathbf{k})\bar{G}(\mathbf{k})$$
$$\times \int S(\mathbf{k} - \mathbf{k}_1)\bar{G}(\mathbf{k}_1)d\mathbf{k}_1, \quad (54)$$

where $S(\mathbf{k})$ is the Fourier transform of the correlation function. The Kraichnan approximation in this form was analyzed in many works (see, e.g., [29–31]) and its approximate solutions were used when discussing various problems of stochastic hydrodynamics and stochastic radiophysics. The Kraichnan approximation can be represented in another form. Using the same method applied to derive the Migdal approximation, we obtain

$$\bar{G}(\mathbf{k}) = \frac{1}{(2\pi)^d}\frac{1}{\nu - k^2 - M_{\mathbf{k}}}, \qquad (55)$$

$$M_{\mathbf{k}} \approx \gamma^2 \int \frac{S(\mathbf{k} - \mathbf{k}_1)d\mathbf{k}_1}{\nu - k_1^2 - M_{\mathbf{k}_1}}. \qquad (56)$$

The Kraichnan approximation in this form was used, e.g., in [32] to calculate the effect of inhomogeneities of magnetic anisotropy on the width and shape of magnetic resonances in a ferromagnet.[1]

The comparison of Eqs. (55) and (56) with Eqs. (50) and (51) shows that the Migdal approximation [17] and Kraichnan approximation [18], which were proposed for different problems, are mathematically identical and have the common constraint: the expansion of the Green's function obtained in these approximations contain all diagrams except for the diagrams with crossing interaction/correlation lines between different points. For this reason, this variant will be called below the noncrossing correlation approximation, where correlations are treated in a wide sense as stochastic correlations and averaged physical interactions. A similar noncrossing correlation approximation called the self-consistent Born approximation is also used in the theory of the scattering of electrons from impurities [25, 33, 34]. Owing to the specific shape of noncrossing interaction lines of electrons with impurity, the corresponding diagrams are sometimes called wigwam diagrams. In view of isolation of different fields of physics, mathematical results obtained with the Kraichnan noncrossing correlation approximation in the stochastic hydrodynamics and radiophysics are usually not cited and used in the works where the noncrossing correlation approximation is developed and used in the theory of condensed matter. The same is referred to the works where the Kraichnan noncrossing correlation approximation is developed and used.

The applicability condition of the noncrossing correlation approximation for the problem of the scattering of electrons from impurities was approximately estimated in [25, 33, 34]. It was shown that the ratios of the contributions from two diagrams (crossed out and included in the noncrossing correlation approxi-

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[1] Formula (21) in [32] contains an extra term with the product $K(\mathbf{x}_1, \mathbf{x}_3)K(\mathbf{x}_2, \mathbf{x}_4)$. This does not affect the results of that work, because Eq. (21) was not used.

mation) in the second line in Fig. 4 is small under the condition

$$(k_F l)^{-1} \ll 1, \qquad (57)$$

where k_F is the Fermi momentum and l is the electron mean free path. For the problem of the scattering of waves from inhomogeneities of a continuous medium, where waves with wavenumbers near a certain resonance wavenumber k_r play the main role, condition (57) can be transformed to the form

$$k_c/k_r \ll 1, \qquad (58)$$

where k_c is the correlation wavenumber ($k_c^{-1} = r_c$ is the radius of correlations of inhomogeneities).

Conditions (57) and (58) are derived from the comparison of two second-order diagrams in the expansion of the Green's function. However, the ratio of the number of the rejected diagrams to the number of the diagrams included in the noncrossing correlation approximation increases rapidly with the order of diagrams. To supplement the estimate of the applicability of the noncrossing correlation approximation, we consider the limiting case $k_c = 0$, where the series for the Green's function can be summed exactly. In this case, random functions $\rho(\mathbf{x})$ become random values whose stochastic properties are described by a certain distribution function $f(\rho)$ (a similar model of independent grains in polycrystal was introduced in [35] to calculate the lineshape of the ferromagnetic resonance). The averaged Green's function is specified by the expression

$$\bar{G}(\nu, \mathbf{k}) = \int G(\nu, \mathbf{k}; \rho)f(\rho)d\rho, \qquad (59)$$

where $G(\nu, \mathbf{k}; \rho)$ is one of the realizations of the random ensemble of the Green's functions, which is a solution of a differential equation with the constant coefficients. For example, the Green's function for spin waves in a ferromagnet with the random value of uniaxial magnetic anisotropy (and the unchanged orientation of the anisotropy axis) has the form [32]

$$G(\nu, \mathbf{k}; \rho) = \frac{1}{(2\pi)^d}\frac{1}{\nu - k^2 - \sigma\rho}. \qquad (60)$$

Here, $\sigma = \Delta\beta/\alpha$, where $\Delta\beta$ is the rms fluctuation of the anisotropy magnitude, and ρ is a centered ($\langle\rho\rangle = 0$) and normalized ($\langle\rho^2\rangle = 1$) random value specific for each grain of the polycrystal. The distribution function $f(\rho)$ for the exact averaged Green's function in Eq. (59) is the Gaussian [34]

$$f(\rho) = \frac{1}{(2\pi)^{1/2}}\exp\left(-\frac{\rho^2}{2}\right). \qquad (61)$$

Since the imaginary part of function (60) is a Dirac delta function multiplied by π/σ, the imaginary part of integral (59) is easily calculated:

$$G''(\nu, \mathbf{k}) = \frac{\pi}{(2\pi)^{7/2}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad (62)$$

where $x = \nu - k^2$.

At the same time, the integral self-consistency equation for the noncrossing correlation approximation in the limit $k_c \longrightarrow 0$ is also solved exactly and leads to the following expression for the averaged Green's function [32]:

$$G''_{\text{NCA}}(\nu, \mathbf{k}) = \begin{cases} \dfrac{\sqrt{(2\sigma)^2 - x^2}}{(2\pi)^3 \cdot 2\sigma^2}, & |x| \leq 2\sigma, \\[2mm] 0, & |x| > 2\sigma. \end{cases} \qquad (63)$$

Substituting Eqs. (60) and (63) into Eq. (59), we easily determine that the averaging of the Green's function in the noncrossing correlation approximation corresponds to the distribution function

$$f(\rho) = \begin{cases} \dfrac{\sqrt{4 - \rho^2}}{2\pi}, & |\rho| \leq 2, \\[2mm] 0 & |\rho| > 2. \end{cases} \qquad (64)$$

Green's functions (62) and (63) are shown in Fig. 5. The shapes of the functions are different and repeat the shapes of respective distribution functions (61) and (64). The denominators of both Green's functions have no poles and the characteristics of the spectrum are determined from the condition of zero denominator of unaveraged Green's function (60). The ensemble of random frequencies in independent grains is determined by the expression

$$\nu = k^2 - \sigma\rho, \qquad (65)$$

and the averaged characteristics are determined by averaging of this ensemble with the corresponding distribution function. Both distribution functions, exact (61) and approximate (64), provide the same results for the mean frequency and its standard deviation characterizing the half-width of the resonance line:

$$\langle \nu \rangle = k^2, \quad \sqrt{\langle (\nu - \langle \nu \rangle)^2 \rangle} = \sigma, \qquad (66)$$

and the imaginary parts of the frequencies vanish at $k_c = 0$. Damping-induced tails in the approximate Green's function appear at $k_c \neq 0$ [32] and the shape of the function $\overline{G}_{\text{NCA}}(\nu)$ becomes more similar to the shape of the exact function (if $\langle \nu \rangle \geqslant \sigma$; otherwise, damping leads to the strong asymmetry of the resonance line [32]). It is noteworthy that the noncrossing correlation approximation specified by Eqs. (52)–(56) can be used to take into account the effect of inhomogeneities of the parameters of only local terms of the Hamiltonian of the continuous medium model (or, correspondingly, the diagonal terms of the lattice Hamiltonian) such as magnetic anisotropy for spin waves, density of the substance for elastic waves, and the dielectric constant for electromagnetic waves. The off-diagonal terms of the lattice Hamiltonian, which describes the interactions with atoms of the environment (exchange in ferro- and ferrimagnets, elastic
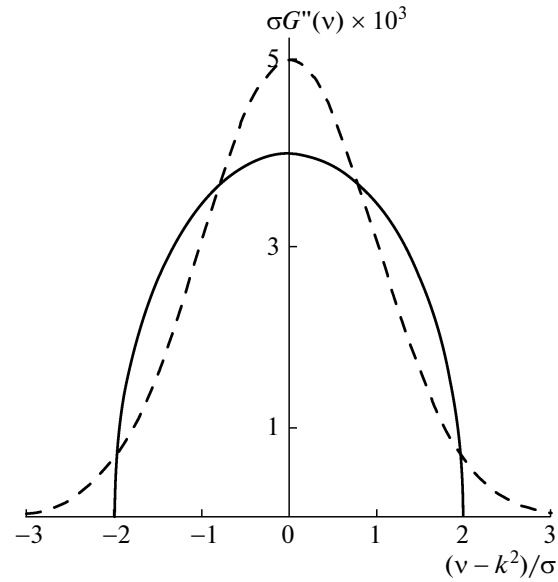

Fig. 5. Imaginary part of the Green's function for the case $k_c = 0$ calculated (solid curve) in the noncrossing correlation approximation and (dashed curve) by the accurate summation of all diagrams.

interactions between atoms, etc.) correspond to non-local terms containing the derivatives of the generalized coordinates of the system in the Hamiltonian of a continuous medium. The noncrossing correlation approximation was generalized in [21] to the case of inhomogeneities of the parameters of the nonlocal terms of the Hamiltonian. The integral term of the Dyson equation in this case contains the spatial derivatives of the Green's functions rather than the functions themselves:

$$\overline{G}(\mathbf{x}, \mathbf{x}_0) = G_0(\mathbf{x}, \mathbf{x}_0) + \iint \frac{\partial G_0(\mathbf{x}, \mathbf{x}')}{\partial x_i'} Q_{i'j''}(\mathbf{x}', \mathbf{x}'') \\ \times \frac{\partial \overline{G}(\mathbf{x}'', \mathbf{x}_0)}{\partial x_j''} d\mathbf{x}' d\mathbf{x}'', \qquad (67)$$

where the mass operator is a matrix whose components contain the second derivatives of the initial Green's functions and have the following form for the exact problem:

$$Q_{i'j''}(\mathbf{x}', \mathbf{x}'') = \gamma^2 \frac{\partial^2 G_0(\mathbf{x}', \mathbf{x}'')}{\partial x_i' \partial x_j''} K(\mathbf{x}', \mathbf{x}'')$$

$$+ \gamma^4 \iint \frac{\partial^2 G_0(\mathbf{x}', \mathbf{x}_1)}{\partial x_i' \partial x_1^k} \frac{\partial^2 G_0(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_1^k \partial x_2^l} \frac{\partial^2 G_0(\mathbf{x}_2, \mathbf{x}'')}{\partial x_2^l \partial x_j''} \qquad (68)$$

$$\times [K(\mathbf{x}', \mathbf{x}_2) K(\mathbf{x}_1, \mathbf{x}'') + K(\mathbf{x}', \mathbf{x}'') K(\mathbf{x}_1, \mathbf{x}_2)]$$

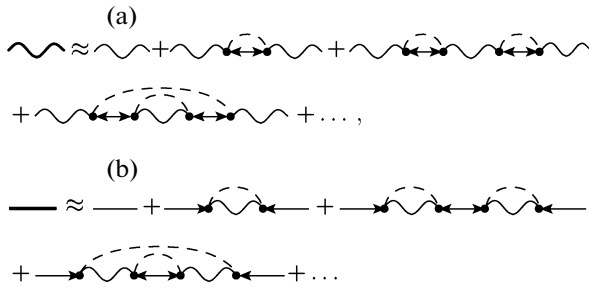$$\times d\mathbf{x}_1 d\mathbf{x}_2 + \dots .$$

Fig. 6. Diagrammatic representation of approximate functions (a) $\overline{G}_m(\mathbf{x}, \mathbf{x}_0)$ and (b) $\overline{G}_u(\mathbf{x}, \mathbf{x}_0)$.

In order to introduce the noncrossing correlation approximation, each component of this matrix is represented in the form

$$Q_{i'j''}(\mathbf{x}', \mathbf{x}'') \approx \gamma^2 \frac{\partial^2 \overline{G}(\mathbf{x}', \mathbf{x}'')}{\partial x_i' \partial x_j''} K(\mathbf{x}', \mathbf{x}''). \qquad (69)$$

This representation of the mass operator for the non-local terms of the Hamiltonian is equivalent to representation (52) for local terms. Details of the subsequent derivation can be found in [21], where simple final expressions of the noncrossing correlation approximation for inhomogeneities of the nonlocal terms of the Hamiltonian were obtained:

$$\overline{G}_{\mathbf{k}} = \frac{1}{(2\pi)^d} \frac{1}{\nu - k^2 - T_{\mathbf{k}}}, \qquad (70)$$

$$T_{\mathbf{k}} = \frac{\gamma^2}{(2\pi)^d} \int \frac{(\mathbf{k} \cdot \mathbf{k}_1)^2 S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu - k_1^2 - T_{\mathbf{k}_1}}. \qquad (71)$$

Here, the scalar interaction potential $T_{\mathbf{k}}$ is the sum of all components of the matrix of the products $k^i k^j Q_{i'j''}(\mathbf{k})$:

$$T_{\mathbf{k}} = (2\pi)^d k^i k^j Q_{i'j''}(\mathbf{k}). \qquad (72)$$

In addition to the noncrossing correlation approximation considered in this work, we briefly mention another variant of the self-consistent approximation—coherent potential approximation, which is widely used for lattice models of solids. The classical form of the coherent potential approximation developed by Soven [19] and Taylor [20] in 1967 was based on the requirement of zero mean value of the scattering matrix; in the simplest case, this requirement leads to the following integral equation for the mass operator:

$$M_k \approx (\sigma^2 - M_k^2) \int \frac{dk_1}{\nu - \nu_0 - k_1^2 - M_{k_1}}, \qquad (73)$$

where ν_0 is the average frequency of vibrations of ions.

Numerous advantages of the coherent potential approximation were manifested in calculations of the density of states of disordered alloys. However, its classical form has two fundamental constraints: stochastic independence of inhomogeneities and applicability only to inhomogeneities of the diagonal terms of the Hamiltonian. The coherent potential approximation was subsequently developed in numerous works, where various cluster schemes of the generalization of the coherent potential approximation were proposed, correlation of inhomogeneities was approximately taken into account, etc. (see, e.g., [36–42]). These improvements of the coherent potential approximation made it possible to obtain a number of important physical results. Since the coherent potential approximation was widely used and integral equations (73) and (56) exhibited some similarity, the equations of the noncrossing correlation approximation were incorrectly considered in some works (see, e.g., [21, 32]) as a variant of the coherent potential approximation. In reality, the noncrossing correlation approximation, rather than the coherent potential approximation, was used in [32] to calculate the effect of inhomogeneities of magnetic anisotropy on the shape and width of the line of magnetic resonances, and the noncrossing correlation approximation, rather than the coherent potential approximation, was generalized in [21] to inhomogeneities of the nonlocal parameters of the Hamiltonian.

3.2. Two Wave Fields

We now derive the approximate self-consistent equations for the mass operators $Q_m(\mathbf{k})$ and $Q_u(\mathbf{k})$ entering into Eqs. (43) and (44). To this end, the mass operators $Q_{m,u}(\mathbf{x}', \mathbf{x}'')$ are approximately expressed in terms of the desired Green's functions $\overline{G}_{m,u}(\mathbf{x}', \mathbf{x}'')$:

$$Q_m(\mathbf{x}', \mathbf{x}'') \approx \gamma^2 \overline{G}_m(\mathbf{x}', \mathbf{x}'') K(\mathbf{x}', \mathbf{x}''), \qquad (74)$$

$$Q_u(\mathbf{x}', \mathbf{x}'') \approx \gamma^2 \frac{\partial^2 \overline{G}_u(\mathbf{x}', \mathbf{x}'')}{\partial z' \partial z''} K(\mathbf{x}', \mathbf{x}''). \qquad (75)$$

Representation (74) for the spin operator formally corresponds to the noncrossing correlation approximation for the local terms of Hamiltonian (52) and representation (75) for the elastic operator corresponds to the noncrossing correlation approximation for the nonlocal terms (69). The substitution of Eqs. (74) and (75) into Dyson equations (34) and (35), respectively, provides the closed system of nonlinear integrodifferential equations of the noncrossing correlation approximation for the functions $\overline{G}_m$ and $\overline{G}_u$:

$$\overline{G}_m(\mathbf{x}, \mathbf{x}_0) \approx G_m^0(\mathbf{x}, \mathbf{x}_0) + \gamma^2 \iint G_m^0(\mathbf{x}, \mathbf{x}') K(\mathbf{x}', \mathbf{x}'')$$
$$\times \frac{\partial^2 \overline{G}_u(\mathbf{x}', \mathbf{x}'')}{\partial z' \partial z''} \overline{G}_m(\mathbf{x}'', \mathbf{x}_0) d\mathbf{x}' d\mathbf{x}'', \qquad (76)$$

$$\overline{G}_u(\mathbf{x}, \mathbf{x}_0) \approx G_u^0(\mathbf{x}, \mathbf{x}_0)$$
$$+ \gamma^2 \iint \frac{\partial G_u^0(\mathbf{x}, \mathbf{x}')}{\partial z'} K(\mathbf{x}', \mathbf{x}'') \qquad (77)$$

$$\times \frac{\partial \overline{G}_u(\mathbf{x}'', \mathbf{x}_0)}{\partial z''} d\mathbf{x}' d\mathbf{x}''.$$

Successive iterations of these equations provide series of the functions $\overline{G}_m$ and $\overline{G}_u$, which are shown in Fig. 6 in the diagrammatic representation. Comparison of these diagrams with the diagrams for exact Green's functions (see Fig. 2) shows that Eqs. (76) and (77) do not contain terms with crossing correlation lines. Thus, the representation of mass operators in the form of Eqs. (74) and (75) and the closed system of integrodifferential equations (76) and (77) following from this representation really correspond to the noncrossing correlation approximation for two (spin and elastic) stochastically interacting wave fields.

The system of Eqs. (76) and (77) in the $\mathbf{k}$ space has the form

$$\overline{G}_m(\mathbf{k}) \approx G_m^0(\mathbf{k}) + \gamma^2 (2\pi)^{2d} G_m^0(\mathbf{k}) \overline{G}_m(\mathbf{k})$$
$$\times \int k_{1z}^2 S(\mathbf{k} - \mathbf{k}_1) \overline{G}_u(\mathbf{k}_1) d\mathbf{k}_1, \quad (78)$$

$$\overline{G}_u(\mathbf{k}) \approx G_u^0(\mathbf{k}) + \gamma^2 k_z^2 (2\pi)^{2d} G_u^0(\mathbf{k}) \overline{G}_u(\mathbf{k})$$
$$\times \int S(\mathbf{k} - \mathbf{k}_1) \overline{G}_m(\mathbf{k}_1) d\mathbf{k}_1. \quad (79)$$

It can be seen that the resulting closed system of equations of the noncrossing correlation approximation is too complex for analysis in the $\mathbf{r}$ space given by Eqs. (76) and (77) and in the $\mathbf{k}$ space specified by Eqs. (78) and (79). For this reason, we pass to another representation of this system corresponding to Eqs. (55), (56), (70), and (71) for the case of one wave field. To this end, we take the Fourier transform of Eqs. (74) and (75):

$$Q_m(\mathbf{k}) \approx \gamma^2 \int \overline{G}_m(\mathbf{k}_1) S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1, \quad (80)$$

$$Q_u(\mathbf{k}) \approx \gamma^2 \int k_{1z}^2 \overline{G}_u(\mathbf{k}_1) S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1. \quad (81)$$

The substitution of exact expressions (43) and (44) into Eqs. (80) and (81) gives the following system of approximate coupled integral equations for the mass operators in the $\mathbf{k}$ space:

$$Q_m(\mathbf{k}) \approx \frac{\gamma^2}{(2\pi)^d} \int \frac{S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu_m - k_1^2 - (2\pi)^d Q_u(\mathbf{k}_1)}, \quad (82)$$

$$Q_u(\mathbf{k}) \approx \frac{\gamma^2}{(2\pi)^d} \int \frac{k_{1z}^2 S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu_u - k_1^2 - (2\pi)^d k_{1z}^2 Q_m(\mathbf{k}_1)}. \quad (83)$$

In terms of the quantities

$$M_{\mathbf{k}} = (2\pi)^d k_z^2 Q_m(\mathbf{k}), \quad U_{\mathbf{k}} = (2\pi)^d Q_u(\mathbf{k}), \quad (84)$$

expressions (43) and (44) for the Green's functions, as well as self-consistency equations (82) and (83), can be represented in the final compact simple form

$$\overline{G}_m(\mathbf{k}) = \frac{1}{(2\pi)^d} \frac{1}{\nu_m - k^2 - U_{\mathbf{k}}}, \quad (85)$$

$$\overline{G}_u(\mathbf{k}) = \frac{1}{(2\pi)^d} \frac{1}{\nu_u - k^2 - M_{\mathbf{k}}}, \quad (86)$$

$$M_{\mathbf{k}} \approx \gamma^2 k_z^2 \int \frac{S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu_m - k_1^2 - U_{\mathbf{k}_1}}, \quad (87)$$

$$U_{\mathbf{k}} \approx \gamma^2 \int \frac{k_{1z}^2 S(\mathbf{k} - \mathbf{k}_1) d\mathbf{k}_1}{\nu_u - k_1^2 - M_{\mathbf{k}_1}}. \quad (88)$$

The system of Eqs. (85)−(88) is the generalization of the noncrossing correlation approximation for the case of two wave fields of different physical natures that are coupled through the inhomogeneous interaction parameter with zero mean value. Below, we will use the equations of the noncrossing correlation approximation in the form of Eqs. (85)−(88) and, for simplicity, $M_{\mathbf{k}}$ and $U_{\mathbf{k}}$ will be called spin and elastic mass operators, respectively. This will not provide misunderstanding because the true mass operators $Q_m(\mathbf{k})$ and $Q_u(\mathbf{k})$ are not used in the self-consistent scheme of Eqs. (85)−(88).

4. DISORDER-INDUCED CROSSING RESONANCE

In this section, we use the noncrossing correlation approximation generalized in Section 3 to examine phenomena appearing at the crossing point of dispersion curves of two wave fields coupled by the inhomogeneous parameter with zero mean value.

We consider a one-dimensional model. In this case, in Eqs. (85)−(88), $d = 1$ and the vector $\mathbf{k}$ has one component $k_z = k$. To analyze self-consistency equations (87) and (88), we represent these equations in several forms. Substituting the first equation into the right-hand side of the second equation and vice versa, we obtain two independent nonlinear integral equations for the mass operators M_k and U_k:

$$M_k = \gamma^2 k^2 \int \frac{S(k - k_1) dk_1}{\nu_m - k_1^2 - \gamma^2 \int \frac{k_1^2 S(k_1 - k_2) dk_2}{\nu_u - k_2^2 - M_{k_2}}}, \quad (89)$$

$$U_k = \gamma^2 \int \frac{k_1^2 S(k - k_1) dk_1}{\nu_u - k_1^2 - \gamma^2 k_1^2 \int \frac{S(k_1 - k_2) dk_2}{\nu_m - k_2^2 - U_{k_2}}}. \quad (90)$$

The successive iteration of each of these equations provides the representations of M_k and U_k in the form of the infinite chain fractions

$$M_k = \gamma^2 k^2 \int \cfrac{S(k-k_1)dk_1}{\nu_m - k_1^2 - \gamma^2 \int \cfrac{k_2^2 S(k_1-k_2)dk_2}{\nu_u - k_2^2 - \gamma^2 k_2^2 \int \cfrac{S(k_2-k_3)dk_3}{\nu_m - k_3^2 - \gamma^2 \int \ldots}}}, \qquad (91)$$

$$U_k = \gamma^2 \int \cfrac{k_1^2 S(k-k_1)dk_1}{\nu_u - k_1^2 - \gamma^2 k_1^2 \int \cfrac{S(k_1-k_2)dk_2}{\nu_m - k_2^2 - \gamma^2 \int \cfrac{k_3^2 S(k_2-k_3)dk_3}{\nu_u - k_3^2 - \gamma^2 k^2 \int \ldots}}}. \qquad (92)$$

For the numerical analysis, it is convenient to represent Eqs. (87) and (88) in the form of the recurrent formulas

$$M_k^{(n)} = \gamma^2 k^2 \int \frac{S(k-k_1)dk_1}{\nu_m - k_1^2 - U_{k_1}^{(n-1)}}, \qquad (93)$$

$$U_k^{(n)} = \gamma^2 \int \frac{k_1^2 S(k-k_1)dk_1}{\nu_u - k_1^2 - M_{k_1}^{(n-1)}}, \qquad (94)$$

where the superscript n is the number of included links of chain fractions (91) and (92).

Simulating the correlation properties of the random function $\rho(x)$ by an exponential correlation function, we obtain the expressions

$$K(r) = \exp(-k_c r), \quad S(k) = \frac{1}{\pi} \frac{k_c}{k_c^2 + k^2}, \qquad (95)$$

where $r = |x - x'|$ and k_c is the correlation wavenumber of inhomogeneities ($r_c = k_c^{-1}$ is the correlation radius). We first consider the first link of a chain fraction and set $n = 1$ in Eqs. (93) and (94). In this case, these equations have the following form corresponding to the Bourret approximation [8]:

$$M_k^{(1)} = \gamma^2 k^2 \int \frac{S(k-k_1)dk_1}{\nu_m - k_1^2}, \qquad (96)$$

$$U_k^{(1)} = \gamma^2 \int \frac{k_1^2 S(k-k_1)dk_1}{\nu_u - k_1^2}. \qquad (97)$$

Substituting $S(k)$ in form (95) and performing integration using the residue theory, we obtain M_k and U_k in the first approximation:

$$M_k^{(1)} = \gamma^2 k^2 \frac{1}{\left(\sqrt{\nu_m} - ik_c\right)^2 - k^2}\left(1 - \frac{ik_c}{\sqrt{\nu_m}}\right), \qquad (98)$$

$$U_k^{(1)} = \gamma^2 \frac{k^2 + ik_c\sqrt{\nu_u} + k_c^2}{\left(\sqrt{\nu_u} - ik_c\right)^2 - k^2}. \qquad (99)$$

The substitution of these expressions into Eqs. (85) and (86) gives the Green's functions $G_m^{(1)}$ and $G_u^{(1)}$ in the first approximation. Figure 7 shows the dependences of the imaginary parts of these Green's functions at the crossing resonance point $k = k_r$ on the frequency ω and correlation wavenumber k_c (the dimensionless quantities ηG_m and ηG_u, where $\eta = \gamma k_r$, are plotted).
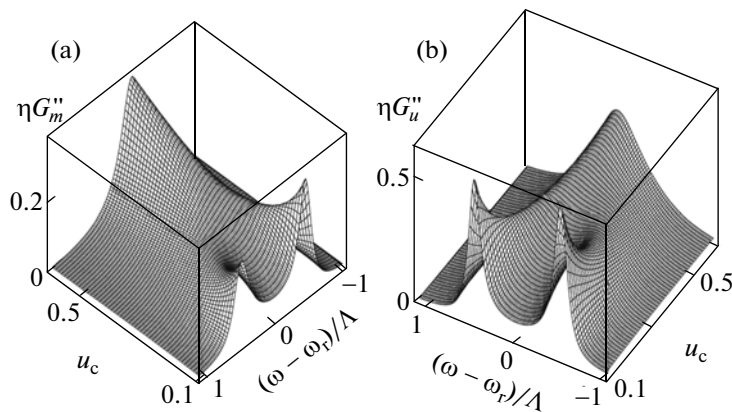

Fig. 7. Imaginary parts of the first-approximation Green's functions $G_m^{(1)}(\omega)$ and $G_u^{(1)}(\omega)$ at $k = k_r$ versus the normalized correlation wavenumber of inhomogeneities $u_c = k_c/\sqrt{\eta}$.
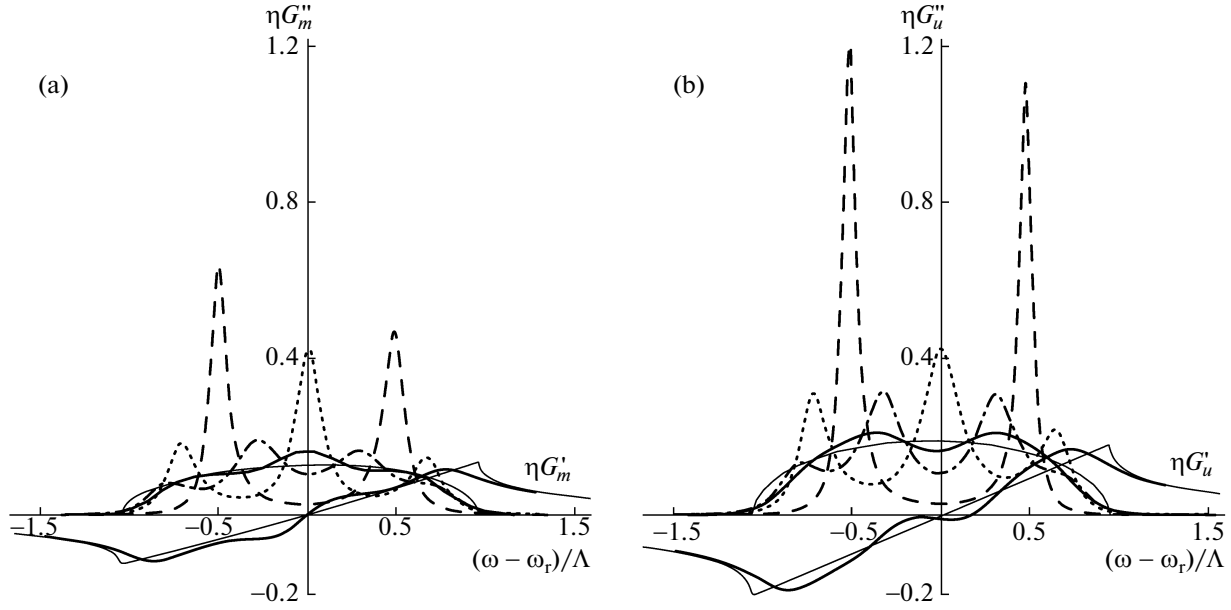
Fig. 8. Imaginary parts of the Green's functions of (a) spin, $G_m''(\omega)$, and (b) elastic, $G_u''(\omega)$, waves at $k = k_r$ and $u_c = 0.05$ as calculated in the approximations (dashed lines) $n = 1$ (Bourret approximation), (dotted lines) $n = 2$, (dash–dotted lines) $n = 3$, (thick solid lines) $n \longrightarrow \infty$, and (thin solid lines) exact solutions (114) and (115) for $u_c = 0$. The real parts of the Green's functions $G_m'(\omega)$ and $G_u'(\omega)$ are also shown for $n \longrightarrow \infty$ and exact solutions.

The following equations of the first approximation for the complex eigenfrequencies ν_m and ν_u are obtained from the condition of zero denominators of the functions $G_m^{(1)}$ and $G_u^{(1)}$:

$$(\nu_u - k^2)[(\sqrt{\nu_m} - ik_c)^2 - k^2] \\ - \gamma^2 k^2 (1 - ik_c / \sqrt{\nu_m}) = 0, \tag{100}$$

$$(\nu_m - k^2)[(\sqrt{\nu_u} - ik_c)^2 - k^2] \\ - \gamma^2 (k^2 + ik_c \sqrt{\nu_u} + k_c^2) = 0. \tag{101}$$

Such (and similar) equations were examined in [10, 12–14]. Equations (100) and (101) with $k_c = 0$ coincide with each other and describe frequency degeneracy removal and the appearance of a gap in the spectrum at the crossing point of the initial dispersion curves. The width of this gap is determined by the expression

$$\Lambda = \zeta \sqrt{2\omega_M \omega_r}, \tag{102}$$

where $\zeta = (\Delta\varepsilon)M/\sqrt{\mu}$, $\omega_M = gM$, and ω_r is the frequency corresponding to the crossing of the dispersion curves of unperturbed spin and elastic waves. Damping different for spin and elastic waves appears at $k_c \neq 0$,

$$\omega_m'' = 2\sqrt{(\omega_r - \omega_0)\omega_M \alpha k_c^2}, \tag{103}$$

$$\omega_u'' = v_u k_c, \tag{104}$$

and the widths of the gaps in the spectrum of spin and elastic waves become different,

$$\Delta_m = \sqrt{\Lambda^2 - \omega_u''^2}, \quad \Delta_u = \sqrt{\Lambda^2 - \omega_m''^2}. \tag{105}$$

At small k_c values, ω_m'', $\omega_u'' < \Lambda$ (open gap) and the picture is similar to the magnetoelastic resonance, which appears under the action of the nonrandom coupling parameter in a homogeneous medium. Difference is that the width of the gap in our case is determined by the rms fluctuation of the coupling parameter rather than by this parameter itself (its mean value is zero). An increase in k_c is accompanied by an increase in the damping frequencies ω_m'', ω_u'' and by a decrease in the widths of the gaps Δ_m and Δ_u. The existence of different dispersion equations (100) and (101) for spin and elastic waves and differences in the dependences of Δ_m and Δ_u on k_c were explained in [10, 12–14] by the fact that Eq. (100) describes the interaction of coherent spin waves with fluctuation elastic waves and Eq. (101) describes the interaction of coherent elastic waves with fluctuation spin waves.

We now demonstrate that all these effects are unreal and are due to the Bourret approximation [8] used in [10, 12–14]. We consider the form of the imaginary parts of the Green's functions $G_m''(\omega)$ and $G_u''(\omega)$ successively in the first, second, and third approximations for a small parameter k_c corresponding to $u_c \equiv k_c / \sqrt{\eta} = 0.05$. In the numerical calculation in each approximation n, the preceding approximation for the elastic mass operator $U_k^{(n-1)}$ is substituted into the

integrand in recurrence formula (93) for the mass operator $M_k^{(n)}$, and the preceding approximation for $M_k^{(n-1)}$ is substituted into the integrand in Eq. (94) for $U_k^{(n)}$. The first approximations (Bourret approximations) lead to the expressions for $G_m^{(1)}(\omega)$ and $G_u^{(1)}(\omega)$ that are presented in Fig. 8 by dashed lines having two peaks at the points

$$\omega \approx \omega_r \pm \Lambda/2. \tag{106}$$

However, this picture is destroyed in the next ($n = 2$) approximation. The corresponding functions $G_m^{(2)}(\omega)$ and $G_u^{(2)}(\omega)$ are shown in Fig. 8 by dotted lines. Each of these functions has three peaks and is not similar to crossing resonance in the homogeneous medium. Each of the functions $G_m^{(3)}(\omega)$ and $G_u^{(3)}(\omega)$ obtained with the inclusion of the next term in continued fractions has four maxima (dash−dotted lines in Fig. 8).

We temporarily interrupt the derivation of successive approximations and consider the limit $k_c \longrightarrow 0$ corresponding to the model of independent grains in the polycrystal with random values of the magnetostriction parameter in grains. In this case,

$$S(k_i - k_j) \longrightarrow \delta(k_i - k_j), \tag{107}$$

where $\delta(k)$ is the Dirac delta function and all integrals in Eqs. (89)−(94) are calculated exactly. The system of Eqs. (89) and (90) or Eqs. (93) and (94) is reduced to equations quadratic in M_k or U_k. Substituting the solutions of these equations into Eqs. (85) and (86), we obtain the exact expressions for the Green's functions in the case $k_c = 0$ in the form

$$G_m(\omega) = \frac{1}{\pi} \frac{1}{x - i\sqrt{4\gamma^2 k^2 \frac{x}{y} - x^2}}, \tag{108}$$

$$G_u(\omega) = \frac{1}{\pi} \frac{1}{y - i\sqrt{4\gamma^2 k^2 \frac{y}{x} - x^2}}, \tag{109}$$

where $x = \nu_m - k^2$ and $y = \nu_u - k^2$. The condition of zero denominators of these expressions provides the dispersion laws for the real frequencies of spin and elastic waves, which coincide with the initial dispersion laws,

$$\nu_m = k^2, \quad \nu_u = k^2. \tag{110}$$

The imaginary parts of the frequencies are absent, because damping caused by inhomogeneities appears only at $k_c \neq 0$. However, the dynamic susceptibilities given by Eqs. (108) and (109) have both real and imaginary parts. Multiplying the numerators and denominators of Eqs. (108) and (109) by the complex conjugate quantities, we obtain

$$G_m(\omega) = \frac{1}{4\pi\gamma^2 k^2} \frac{y}{x}\left(x + i\sqrt{4\gamma^2 k^2 \frac{x}{y} - x^2}\right), \tag{111}$$

$$G_u(\omega) = \frac{1}{4\pi\gamma^2 k^2} \frac{x}{y}\left(y + i\sqrt{4\gamma^2 k^2 \frac{y}{x} - x^2}\right). \tag{112}$$

We pass from normalized frequencies to ω in these expressions:

$$x = \frac{\omega - \omega_m(k)}{\alpha g M}, \quad y = \frac{\omega^2 - \omega_u^2(k)}{v_u^2}, \tag{113}$$

where $\omega_m(k) = \omega_0 + \alpha g M k^2$ and $\omega_u(k) = v_u k$. We consider Eqs. (111) and (112) at $k = k_r$. In this case, $\omega_m(k_r) = \omega_u(k_r) = \omega_r$ and Eqs. (111) and (112) after the corresponding algebra become

$$G_m(\omega) = \alpha \frac{\omega + \omega_r}{4\pi\zeta^2 \omega_r^2}$$
$$\times \left[\omega - \omega_r + i\sqrt{4\zeta^2 \frac{\omega_M \omega_r^2}{\omega + \omega_r} - (\omega - \omega_r)^2}\right], \tag{114}$$

$$G_u(\omega) = \frac{1}{4\pi\zeta^2 \omega_M k_r^2}$$
$$\times \left[\omega - \omega_r + i\sqrt{4\zeta^2 \frac{\omega_M \omega_r^2}{\omega + \omega_r} - (\omega - \omega_r)^2}\right]. \tag{115}$$

The real and imaginary parts of the Green's functions corresponding to these expressions are shown by thin solid lines in Fig. 8. It can be seen that $G_m''(\omega)$ and $G_u''(\omega)$ for $k_c = 0$ have wide single-mode peaks with the width approximately equal to Λ. Since damping in the system is absent, the width of the peak is completely determined by the stochastic distribution of the frequencies of δ-like peaks forming this wide peak.

According to Eqs. (114) and (115), the ratio of the amplitudes of these wide peaks at $\omega = \omega_r$ is

$$G_m''(\omega_r)/G_u''(\omega_r) = v_m/v_u, \tag{116}$$

where $v_m = 2\alpha g M k_r$ is the velocity of spin waves at $k = k_r$. The inequalities $v_m < v_u$ and $v_m > v_u$ are always satisfied at the first and second crossing points of the dispersion curves of spin and elastic waves, respectively.

We now return to the derivation of successive approximations for the Green's functions $G_m''(\omega)$ and $G_u''(\omega)$ at $k_c \neq 0$. As n increases, the number of peaks on the plots of the Green's functions $G_m''(\omega)$ and $G_u''(\omega)$ increases as $n + 1$ and the amplitude of the peaks decreases. The area under each of the $G_m''(\omega)$ and $G_u''(\omega)$ curves remains unchanged and these curves approach their limiting positions corresponding to the exact solutions of integral equations (89) and (90) (Fig. 8, thick solid lines). As can be seen in this figure,
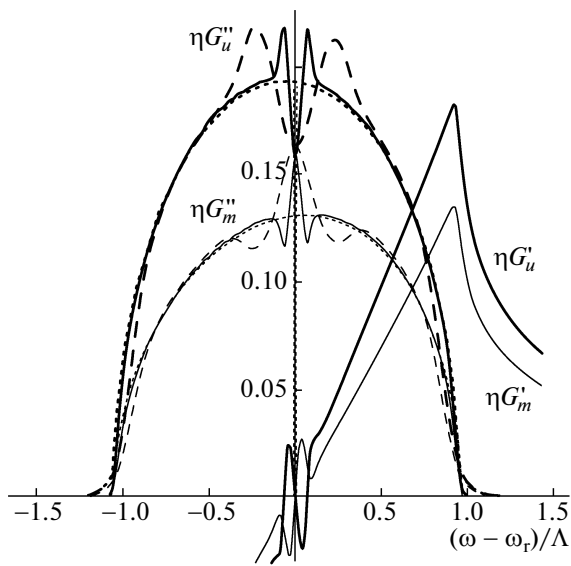
Fig. 9. Green's functions of (thin lines) spin, $G_m''(\omega)$ and $G_m'(\omega)$, and (thick lines) elastic, $G_u''(\omega)$ and $G_u'(\omega)$, waves at $u_c =$ (dashed lines) 1.6×10^{-2} and (solid lines) 1.6×10^{-3} and (dotted lines) exact solutions (108) and (109) at $u_c = 0$.

the curves corresponding to a small k_c value ($u_c = 0.05$) differ only slightly from the curves for $k_c = 0$ (thin solid lines in Fig. 8) and differ strongly from the dashed lines calculated in the Bourret approximation for the
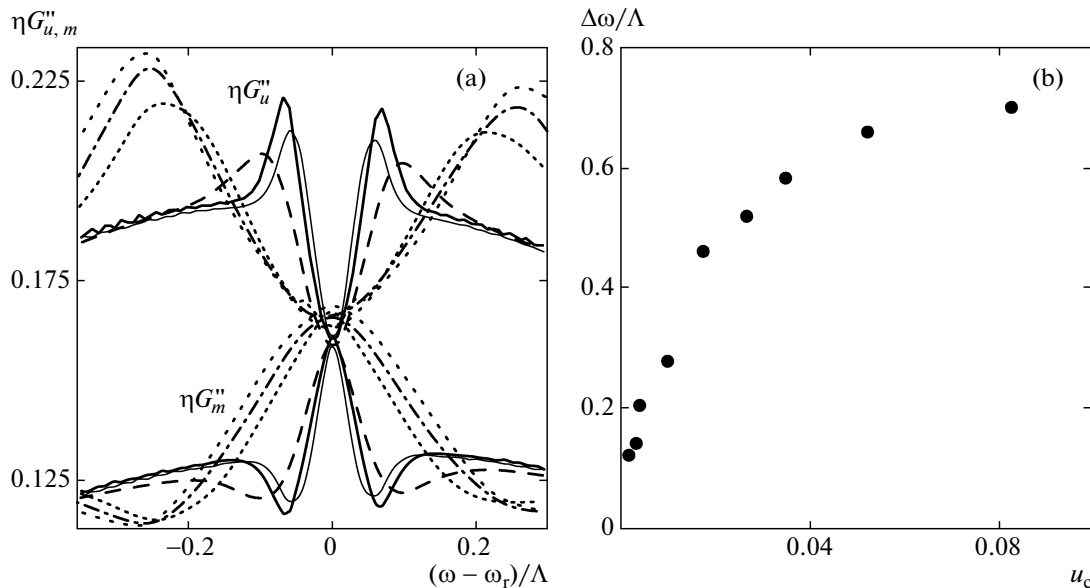
same u_c value. Instead of degeneracy removal and the appearance of two peaks at the distance Λ, a wide single-mode peak with the width Λ is observed for each of the functions $G_m''(\omega)$ and $G_u''(\omega)$. However, the vertices of these peaks are not smooth: a specific fine structure appears on the vertex of each peak. This structure for $G_m''(\omega)$ and $G_u''(\omega)$ has the form of narrow (compared to Λ) resonance and narrow antiresonance, respectively. The fine structure corresponding to these formations is also manifested in the $G_m'(\omega)$ and $G_u'(\omega)$ curves, which are also shown in Fig. 8.

We now examine the properties of the functions $G_m''(\omega)$ and $G_u''(\omega)$ in more detail. All next figures show the Green's functions corresponding to the n value ensuring the convergence of successive approximations. This n value depends on the chosen parameters of the system, primarily on the k_c value. For large k_c values, several n values can be sufficient, whereas tens and even hundreds successive numerical integrations of recurrent relations (93) and (94) are required for small k_c values.

Figure 9 shows the functions (thin lines) $G_m(\omega)$ and (thick lines) $G_u(\omega)$ in the same scale for three u_c values. It can be seen that these characteristics of the fine structure are more sensitive to a change in the correlation wavenumber k_c of inhomogeneities than the characteristics of wide peaks. An increase in small k_c values by an order of magnitude almost does not change the width of

Fig. 10. (a) Resonance and antiresonance peaks of the fine structure at the vertices of wide peaks of the functions $G_m''(\omega)$ and $G_u''(\omega)$ for the correlation wavenumber $u_c =$ (thin solid lines) 0.8×10^{-3}, (thick solid lines) 1.6×10^{-3}, (dashed lines) 2.5×10^{-3}, (dotted lines) 1.6×10^{-2}, (dash–dotted lines) 2.5×10^{-2}, and (rare dotted line) 3.3×10^{-2}. (b) Width $\Delta\omega/\Lambda$ of the peaks of the fine structure versus u_c.
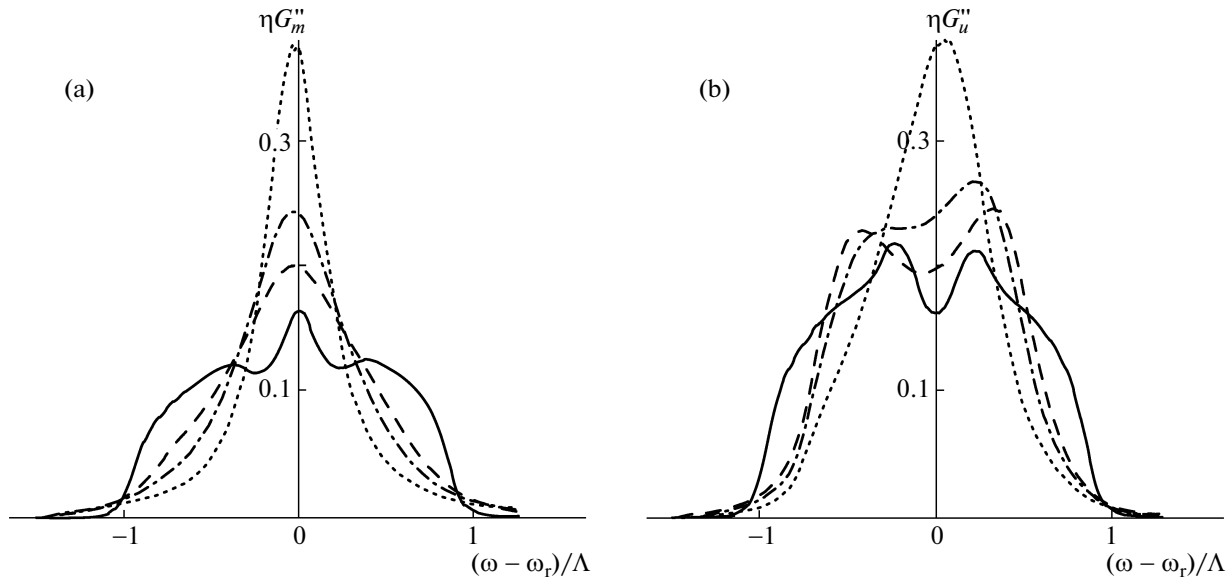
Fig. 11. Imaginary parts of the Green's functions of (a) spin, $G_m''(\omega)$ and (b) elastic, $G_u''(\omega)$, waves at $u_c =$ (solid lines) 0.016, (dashed lines) 0.16, (dash–dotted lines) 0.25, and (dotted lines) 0.5

the wide peak. At the same time, the width of a narrow resonance in the $G_m''(\omega)$ curve, as well as the antiresonance in the $G_u''(\omega)$ curve, increases by several times.

The dependence of this fine structure on the correlation wavenumber is shown in more detail in Fig. 10. According to Fig. 10a, the resonance and antiresonance peaks for small k_c values ($u_c \sim 0.001–0.01$) exhibit approximate mirror symmetry (the amplitude of resonance is somewhat smaller than the amplitude of antiresonance). The widths of resonance and antiresonance are approximately the same and increase with k_c (Fig. 10b). A further increase in k_c (Fig. 11) is accompanied by a significant change in the shape and the narrowing of the main wide peak. This is due to the exchange narrowing of the magnetic resonance lines, which was studied using the noncrossing correlation approximation for ferromagnetic and spin-wave resonances in [21, 32]. As can be seen in Fig. 11, an increase in k_c leads to the opposite effects for wide peaks and fine-structure peaks. The former are narrowed and the latter are broadened and, then, disappear. The fine structure first disappears at the peak of the function $G_m''(\omega)$ and, then, at the peak of the function $G_u''(\omega)$.

Above, we considered the effects occurring at the first crossing point of the dispersion curves of spin and elastic waves, $k = k_r$. At this point, $v_m < v_u$ and, according to Eq. (116), $G_m''(\omega_r) < G_u''(\omega_r)$ at $k_c = 0$. Near the second crossing point of the dispersion curves, $k = k_{r'}$, $v_m > v_u$ and, correspondingly, $G_m''(\omega_{r'}) > G_u''(\omega_{r'})$ at $k_c = 0$. In this case, the fine struc-

ture at the vertices of the wide peaks of the functions $G_m''(\omega)$ and $G_u''(\omega)$ is opposite: narrow resonance appears in the $G_u''(\omega)$ curve, whereas narrow antiresonance occurs in the $G_m''(\omega)$ curve. All Figs. 8–11 change similarly. In the cases of both inequalities $v_m < v_u$ and $v_m > v_u$, there is a common property: at $k_c \neq 0$ and $\omega = \omega_r$, the point of the maximum of the resonance of the fine structure of one Green's function coincides with the point of the minimum of the antiresonance of the fine structure of the other Green's function and

$$G_m''(\omega_r) = G_u''(\omega_r). \qquad (117)$$

As can be seen in Fig. 11, this property holds even when the fine structure disappears. It is reasonable to assume that this property is a consequence of the general law of the uniform distribution of the energy of oscillations over the degrees of freedom, in this case, over spin and elastic oscillations, at the crossing point of dispersion curves $k = k_r$, $\omega = \omega_r$. In the case $k_c = 0$ corresponding to the model of independent grains in a polycrystal with a random (both in magnitude and in sign) coupling parameter in each grain, the fine structure is absent and the functions $G_m''(\omega)$ and $G_u''(\omega)$ at $k = k_r$ have single-mode peaks with the width Λ. The inequality $G_m''(\omega) < G_u''(\omega)$ is satisfied for all frequencies in the region of existence of these peaks. At arbitrarily small value $k_c \neq 0$, the interaction appears between grains and the law of the equiprobable energy distribution from which the equality $G_m''(\omega_r) = G_u''(\omega_r)$ follows should be satisfied at the crossing point of the

dispersion curves. How are the Green's functions transformed to satisfy this law? The function $G_m''(\omega)$ "raises" a thin "feeler" (fine-structure resonance) at the point $\omega = \omega_r$, whereas the function $G_u''(\omega)$ "drops" the analogous feeler (fine-structure antiresonance) at the same point. These feelers touch each other approximately at the middle of the distance between the maxima of wide peaks of both functions, and the required equality is satisfied at the point $\omega = \omega_r$. The inequality $G_m''(\omega_r) < G_u''(\omega_r)$ remains valid for all other frequencies except for $\omega = \omega_r$.

5. CONCLUSIONS

The dynamic susceptibilities (Green's functions) of the system of two interacting wave fields of different physical natures with the stochastically inhomogeneous coupling parameter with zero mean value have been studied. The study has been performed for spin and elastic waves in a ferromagnet with the inhomogeneous magnetostriction parameter. The model problem has been considered including only transverse elastic vibrations.

The system of Dyson equations and series for averaged Green's functions of spin, $\bar{G}_m$, and elastic, $\bar{G}_u$, waves have been obtained. The main feature of the system of Dyson equations is that the elastic mass operator Q_u appears in the equation for the spin Green's function $\bar{G}_m$, whereas the spin mass operator Q_m appears in the equation for the elastic Green's function $\bar{G}_u$. The main feature of the Green's functions $\bar{G}_m$ and $\bar{G}_u$ is that the initial Green's functions G_m^0 alternate with the derivatives of the initial Green's functions G_u^0 in each term of the series. The series for $\bar{G}_m$ contains the products of alternating initial spin Green's functions and second derivatives of the initial elastic Green's functions. The series for $\bar{G}_u$ contains the products of alternating first derivatives of the initial spin elastic Green's functions and the initial spin Green's functions.

The self-consistent approximation taking into account all diagrams with noncrossing correlation/interaction lines in the expansion of a Green's function (called the noncrossing correlation approximation in this work), which is known as the Migdal approximation and the self-consistent Born approximation in the theory of condensed matter and the Kraichnan approximation in stochastic hydrodynamics and radiophysics, has been generalized to the case of two interacting wave fields of different physical natures with the stochastically inhomogeneous coupling parameter between these fields; the mean value of this parameter is zero.

Disorder-induced crossing resonance occurring at the crossing of the dispersion curves of spin and elastic waves has been examined within the developed method. The system of integral equations for the spin and elastic mass operators has been solved numerically for the case of one-dimensional inhomogeneities of the coupling parameter. The results obtained taking into account the processes of multiple scattering of waves from inhomogeneities are significantly different from those obtained for this situation in the Bourret approximation [10, 12–14]. Instead of frequency degeneracy removal in the wave spectrum and the splitting of resonance peaks of dynamic susceptibilities, a wide single-mode peak with the fine structure in the form of narrow resonance (antiresonance) peaks should appear in the Green's function G_m'' (G_u'') at the crossing point of the unperturbed dispersion curves. It has been shown that, at small correlation wavenumbers k_c of inhomogeneities, the width of the wide peaks is determined by the rms fluctuation $\Delta\varepsilon$ of the coupling parameter, whereas the width of the narrow resonance and antiresonance peaks is determined by k_c. This will allow the independent measurements of both these main characteristics of inhomogeneities. As k_c increases, the sharp exchange narrowing of the wide peak occurs and the narrow resonance and antiresonance peaks are broadened and gradually disappear.

The properties of the fine-structure peaks can be explained assuming that this structure is due to the law of the uniform energy distribution between spin and elastic oscillations at the crossing point of the dispersion curves $k = k_r$, $\omega = \omega_r$.

Investigation in this work has been performed for disorder-induced crossing resonance of spin and elastic waves. Particular relations for crossing resonance of waves or quasiparticles of another nature (e.g., optical and acoustic phonons) will have another form. However, the general form of the dynamic susceptibilities should remain unchanged: the fine resonance structure against the background of the wide peak of the susceptibility of one wave field and the fine antiresonance structure against the background of the wide peak of the susceptibility of the other wave field; the sharp dependence of the fine-structure peaks on k_c for small k_c values; and the equality of the imaginary parts of the susceptibilities of both wave fields at the crossing point of the dispersion curves $k = k_r$, $\omega = \omega_r$.

The experimental detection of the predicted effects would be of most interest for media with small k_c values and with sufficient excess of the velocity of waves of one field over the velocity of waves of the other field at $k = k_r$. The former property leads to small widths of lines, whereas the latter property is responsible for large amplitudes of the resonance and antiresonance peaks of the fine structure.

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