EXACT SOLUTION FOR THE THERMODYNAMICS OF THE ONE-DIMENSIONAL HUBBARD MODEL WITH INFINITE REPULSION IN A MAGNETIC FIELD

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We demonstrate that in the limit of infinite repulsion in a magnetic field, the canonical partition function of the one-dimensional Hubbard model in the nearest-neighbor approximation factors into the product of partition functions of holons and spinons, which can be calculated exactly. We thus obtain exact numerical dependences of the free energy, entropy, internal energy, specific heat, chemical potential, and magnetic susceptibility on the hole concentration, temperature, and magnetic field induction.

1. Introduction

The Hubbard model in a magnetic field has been studied in numerous papers. For example, numerous exact quantitative results obtained using the Bethe ansatz formalism can be found in [1]. The thermodynamics were calculated in [2] using the quantum transfer-matrix method. As mentioned in [3], the results in [1] and [2] agree well with each other.

But it is easy to see that the above calculations are related to using rather complicated mathematical equations. It is therefore still interesting to find the thermodynamics in a simpler way. Here, we use only linear algebra to show that the Hubbard model partition function in the limit as $U \to \infty$ in a magnetic field factors into the product of partition functions of holons and spinons. It turns out that we can write the spinon partition function in the form of an explicit analytic expression, which in the case of half filling coincides with the partition function of the one-dimensional Ising model with a zero exchange integral. We can calculate the exact holon partition function numerically using the exact diagonalization of the Hamiltonian of interatom hopping for spinless fermions by the method described in detail in [4]. As a result, we obtain simple relations between the thermodynamic characteristics of electrons and holons.

This paper is organized as follows. In Sec. 2, we obtain the exact analytic solution in the absence of hopping. In Sec. 3, we present the method for obtaining a numerically exact solution in the presence of hopping. In Sec. 4, we analyze the thermodynamic characteristics presented as numerically exact graphs. We prove the factorization of the partition function into the product of spinon and holon partition functions in the appendix.

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2. The exact solution at $t = 0$

The Hamiltonian we study in the Hubbard operator representation is

$$H = H_0 + H_t + H_m, \tag{1}$$

where

$$H_0 = \varepsilon \sum_{f,\sigma} X_f^{\sigma\sigma},$$

$$H_t = -\sum_{f,m,\sigma} t_{fm} X_f^{\sigma 0} X_m^{0\sigma}, \tag{2}$$

$$H_m = -h \sum_f (X_f^{\uparrow\uparrow} - X_f^{\downarrow\downarrow}),$$

ε is the on-site electron energy, t_{fm} is the hopping integral from site f to site m, $h = g_s \mu_B B/2$, g_s is the Landé g-factor, μ_B is the Bohr magneton, B is the magnetic field induction in the medium, and we define the Hubbard operator X_f^{pq} as $X_f^{pq} = |p\rangle\langle q|$.

We begin our analysis with the simplest case $H_t = 0$ (the so-called atomic limit). In the absence of hopping, exact relation (50)–(52) between the holon and electron partition functions obtained in the appendix allows obtaining exact expressions for all thermodynamic characteristics of electrons. Indeed, the matrix $H^{(h)}$ in this case is a diagonal matrix of size $C_N^{N_e} \times C_N^{N_e}$ with the numbers $N_e \varepsilon$ on the diagonal. The exponential of such a matrix is again a diagonal matrix of the same size with the numbers $e^{-N_e \varepsilon / k_B T}$ on the diagonal. The holon partition function is

$$Z^{(h)} = C_N^{N_e} e^{-N_e \varepsilon / k_B T}, \qquad C_N^{N_e} = \frac{N!}{N_e! \, (N - N_e)!}. \tag{3}$$

Representing the factorials in (3) using the Stirling formula $n! \approx (n/e)^n$ and introducing the notation $x = 1 - N_e/N$ for the hole concentration, we obtain

$$Z^{(h)} = \left(\left(x^{-x} (1-x)^{-(1-x)} \right) e^{-(1-x)\varepsilon / k_B T} \right)^N. \tag{4}$$

Using (4) and (50)–(52), we consequently obtain all the thermodynamic characteristics of electrons: the free energy

$$f = \varepsilon(1-x) + k_B T \left(x \log x + (1-x) \log(1-x) - (1-x) \log\left(2 \cosh\left(\frac{h}{k_B T} \right) \right) \right), \tag{5}$$

the entropy

$$s = -k_B \left(x \log x + (1-x) \log(1-x) - (1-x) \log\left(2 \cosh\left(\frac{h}{k_B T} \right) \right) \right) - (1-x)\frac{h}{T} \tanh\left(\frac{h}{k_B T} \right), \tag{6}$$

the internal energy

$$u = (1-x)\left(\varepsilon - h \tanh\left(\frac{h}{k_B T} \right) \right), \tag{7}$$

the specific heat

$$c = (1 - x) \frac{h^2}{k_{\mathrm{B}} T^2} \frac{1}{\cosh^2(h/k_{\mathrm{B}} T)}, \qquad (8)$$

the chemical potential

$$\mu = \varepsilon - k_{\mathrm{B}} T \log \frac{2 \cosh(h/k_{\mathrm{B}} T) x}{1 - x}, \qquad (9)$$

and the magnetic susceptibility (the Curie law)

$$\chi = \frac{1}{V} \frac{g_{\mathrm{s}}^2 \mu_{\mathrm{B}}^2 N_e}{4 k_{\mathrm{B}} T}. \qquad (10)$$

Formulas (5)–(9) generalize formulas (28), (30)–(32), and (34) in [4] to the case of a nonzero magnetic field. We do not present a detailed analysis of the obtained results here. All of them are particular cases of the results obtained in Sec. 4 with hopping taken into account.

We note that the magnetic susceptibility is completely determined by the factor $Z^{(\mathrm{s})}$ in (50) because only this factor determines the dependence of the partition function on the magnetic field induction. The susceptibility is therefore insensitive to the value of the hopping parameter t between the nearest neighbors. We therefore omit the expression for the magnetic susceptibility when writing formulas for the thermodynamic characteristics in what follows because this expression always has form (10).

We also note that the studied model obviously coincides at $t = 0$ and $x = 0$ with the one-dimensional Ising model at $J = 0$ in a magnetic field [5]. Correspondingly, the partition functions of these models also coincide,

$$Z_{J=0}^{(\mathrm{Ising})} = Z_{t=0}^{(\mathrm{t})} = \left(2 \cosh \left(\frac{h}{k_{\mathrm{B}} T} \right) \right)^N,$$

together with all the thermodynamic characteristics.

3. The exact solution in the case of a nonzero hopping parameter

Finding the thermodynamic characteristics of holons is possible because of the Fermi commutation relation

$$\{X_f^{10}, X_m^{01}\} = X_f^{10} X_m^{01} + X_m^{01} X_f^{10} = \delta_{fm}. \qquad (11)$$

Using relation (11), we can diagonalize the holon Hamiltonian,

$$H = H_0 = \varepsilon \sum_f X_f^{11} - \sum_{f,m} t_{fm} X_f^{10} X_m^{01}$$

using the Fourier transform, which results in the spectral function

$$\langle X_k^{10} X_k^{01} \rangle = \frac{1}{e^{(t(k) - \mu^{(\mathrm{h})})/k_{\mathrm{B}} T} + 1} \qquad (12)$$

coinciding with the Fermi–Dirac free fermion distribution. The function $t(k)$ in expression (12) is the zonal spectrum, which becomes

$$t(k) = -2t \cos ka \qquad (13)$$

in the nearest-neighbor approximation. We find the holon thermodynamic characteristics by first solving the equation for the chemical potential

$$\sum_k \frac{1}{e^{(t(k)-\mu^{(\mathrm{h})})/k_{\mathrm{B}}T}+1} = N_e \tag{14}$$

and then calculating the free energy

$$u^{(\mathrm{h})} = \sum_k \frac{t(k)}{e^{(t(k)-\mu^{(\mathrm{h})})/k_{\mathrm{B}}T}+1}, \tag{15}$$

the specific heat

$$c^{(\mathrm{h})} = \frac{\partial u^{(\mathrm{h})}}{\partial T}, \tag{16}$$

the entropy (using the Nernst theorem)

$$s^{(\mathrm{h})} = \int_0^T \frac{c^{(\mathrm{h})}}{T} dT, \tag{17}$$

and finally the free energy

$$f^{(\mathrm{h})} = u^{(\mathrm{h})} + T s^{(\mathrm{h})}. \tag{18}$$

The superscript (h) in formulas (12) and (14)–(18) indicates that these quantities pertain to holons. Formulas (14)–(18) result in the well-known dependences describing the behavior of the ideal Fermi gas [6].

Exact relation (50) between the holon and electron partition functions allows obtaining exact numerical values of the electron thermodynamic characteristics. Using standard formulas from textbooks on statistical mechanics and thermodynamics, we obtain the following formulas for the corresponding thermodynamic characteristics: the free energy is

$$f = f^{(\mathrm{h})} - k_{\mathrm{B}}T(1-x)\log\left(2\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right)\right) = f^{(\mathrm{e})} - k_{\mathrm{B}}T(1-x)\log\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right), \tag{19}$$

the entropy is

$$s = s^{(\mathrm{h})} + k_{\mathrm{B}}(1-x)\log\left(2\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right)\right) - (1-x)\frac{h}{T}\tanh\left(\frac{h}{k_{\mathrm{B}}T}\right) =$$

$$= s^{(\mathrm{e})} + k_{\mathrm{B}}(1-x)\log\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right) - (1-x)\frac{h}{T}\tanh\left(\frac{h}{k_{\mathrm{B}}T}\right), \tag{20}$$

the internal energy is

$$u = u^{(\mathrm{h})} - (1-x)h\tanh\left(\frac{h}{k_{\mathrm{B}}T}\right) = u^{(\mathrm{e})} - (1-x)h\tanh\left(\frac{h}{k_{\mathrm{B}}T}\right), \tag{21}$$

the specific heat is

$$c = c^{(\mathrm{h})} + (1-x)\frac{h^2}{k_{\mathrm{B}}T^2}\frac{1}{\cosh^2(h/k_{\mathrm{B}}T)} = c^{(\mathrm{e})} + (1-x)\frac{h^2}{k_{\mathrm{B}}T^2}\frac{1}{\cosh^2(h/k_{\mathrm{B}}T)}, \tag{22}$$

and the chemical potential is

$$\mu = \mu^{(\mathrm{h})} - k_{\mathrm{B}}T\log\left(2\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right)\right) = \mu^{(\mathrm{e})} - k_{\mathrm{B}}T\log\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right). \tag{23}$$

In formulas (19)–(23), quantities with the superscript (h) pertain to holons, and quantities with the superscript (e) pertain to electrons in a zero magnetic field ($B = 0$). Each of formulas (19)–(23) has the general form

$$a = a^{(\mathrm{h})} + a^{(\mathrm{s})} = a^{(\mathrm{e})} + a^{(B)},$$

where $a^{(\mathrm{h})}$ is the contribution of factor (51) to (50) (the holon contribution), $a^{(\mathrm{s})}$ is the contribution of factor (52) to (50) (the spinon contribution), $a^{(\mathrm{e})}$ is the contribution of the factor $Z^{(\mathrm{e})} = 2^{N_e} Z^{(\mathrm{h})}$ to (50) (the electron contribution), and $a^{(B)}$ is the contribution of the factor $Z^{(B)} = \cosh^{N_e}(h/k_{\mathrm{B}}T)$ to (50) (the magnetic field contribution). We also note that formulas (8)–(12) corresponding to the case $t = 0$ agree perfectly with general formulas (19)–(23) (also see formulas (15)–(19) and (27)–(34) in [4]).

4. Discussing the obtained results

We now discuss the results obtained in formulas (19)–(23). In Fig. 1, we present the results obtained for the free energy. In Fig. 1a, we present the free energy dependence on the concentration of holes, which demonstrates that as temperature increases, the curve drifts from its initial position at $T = 0$,

$$\frac{f}{t} = \frac{f^{(\mathrm{h})}}{t}\bigg|_{T=0} - (1-x)\frac{h}{t} = \frac{u^{(\mathrm{h})}}{t}\bigg|_{T=0} - (1-x)\frac{h}{t} = \left(\frac{1}{Nt}\sum_{k=-k_{\mathrm{F}}}^{k_{\mathrm{F}}} \varepsilon(k)\right) - (1-x)\frac{h}{t} =$$

$$= -\frac{1}{\pi}\int_{-(1-x)\pi}^{(1-x)\pi} \cos y \, dy - (1-x)\frac{h}{t} = -\frac{2}{\pi}\sin \pi x - (1-x)\frac{h}{t}, \tag{24}$$

to minus infinity as $T \to \infty$. We also see that as the field increases at a constant temperature, the curve stretches from its initial shape at $h = 0$,

$$\frac{f}{t} = \frac{f^{(\mathrm{h})}}{t}\left(\frac{k_{\mathrm{B}}T}{t}\right) - (1-x)\frac{k_{\mathrm{B}}T}{t}\log 2, \tag{25}$$

to minus infinity as $h \to \infty$.

The curves of dependences of the free energy on $k_{\mathrm{B}}T/t$ (Fig. 1b) start at the point $(0, -(2/\pi)\sin \pi x) - (1-x)h/t$ and as $k_{\mathrm{B}}T/t \to \infty$ manifest the asymptotic behavior

$$\frac{f}{t} = \frac{2}{\pi}\sin \pi x + \frac{k_{\mathrm{B}}T}{t}(x\log x + (1-x)\log(1-x) - (1-x)\log 2). \tag{26}$$

The curve of the free energy dependence on h/t (Fig. 1c) demonstrates that the free energy decreases as the magnetic field induction increases. As $h \to \infty$, they asymptotically approach the lines described by the equation

$$\frac{f}{t} = \frac{f^{(\mathrm{h})}}{t}\left(x, \frac{k_{\mathrm{B}}T}{t}\right) - (1-x)\frac{h}{t}. \tag{27}$$

In Fig. 2, we present the dependences of the entropy on the concentration of holes, on the temperature, and on the magnetic field induction. In Fig. 2a representing the entropy dependence on the hole concentration, we see that at $h = 0$, the curve develops a hump starting with the straight line

$$\frac{s}{k_{\mathrm{B}}} = (1-x)\log 2 \tag{28}$$

at the point $T = 0$, where the entropy is determined only by localized spins $\sigma = 1/2$, and approaching the curve

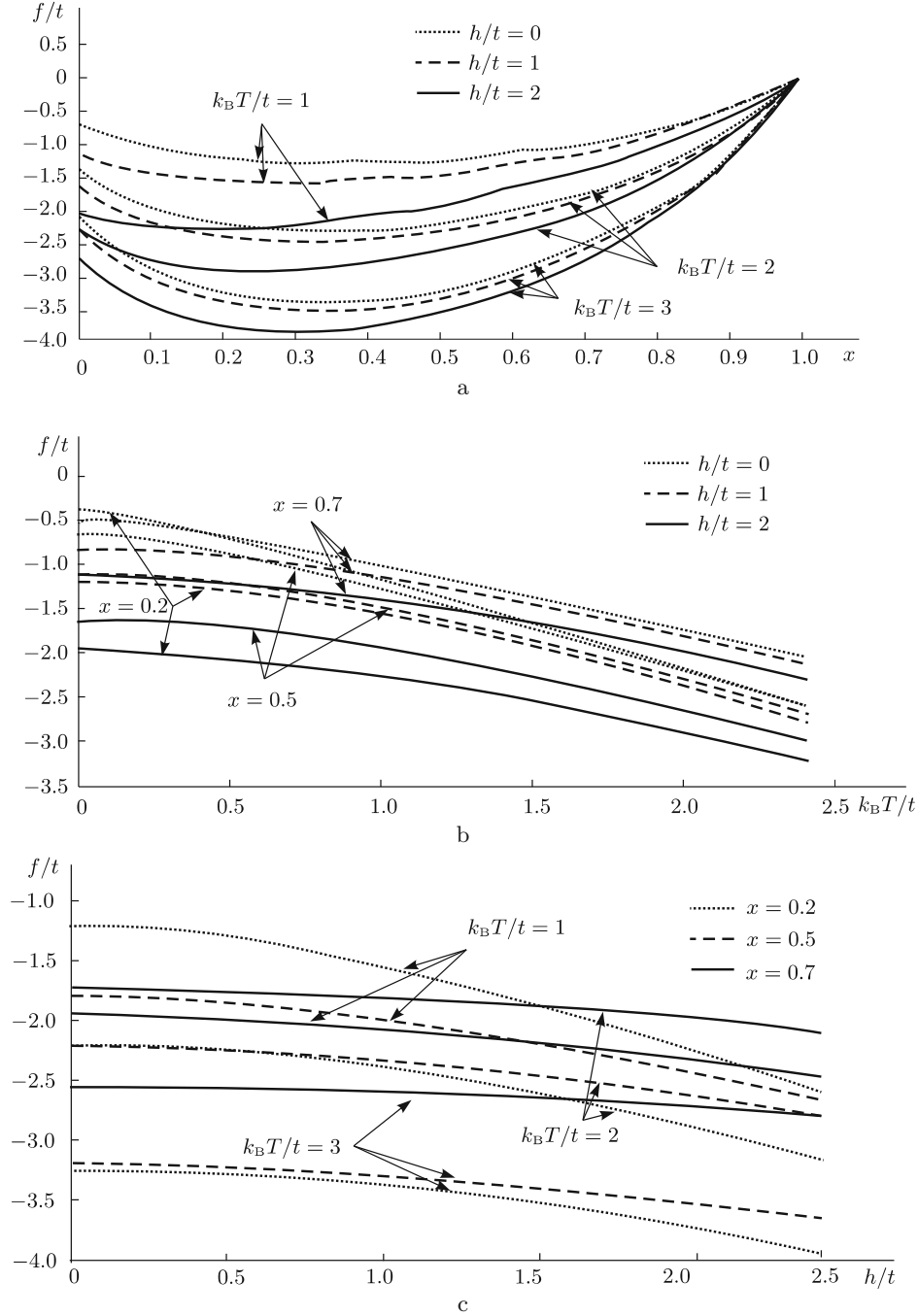$$\frac{s}{k_{\mathrm{B}}} = (1-x)\log 2 - x\log x - (1-x)\log(1-x) \tag{29}$$

Fig. 1. Dependences of the free energy (a) on the concentration of holes, (b) on the temperature, and (c) on the magnetic field induction.

as $T \to \infty$ (indicating that the entropy is determined by both spin and charge contributions in this limit). Switching on an arbitrarily small magnetic field changes the situation drastically, resulting in the curve of the dependence of s/k_{B} on x at $T = 0$ merely coinciding with the abscissa axis and asymptotically approaching the curve described by the same Eq. (29) as $T \to \infty$. This does not follow directly from Fig. 2a, but it is clear from the temperature dependence of the entropy in Fig. 2b. The physical reason for this jump can be clearly described at $t = 0$. In this case, even at an arbitrarily small magnetic field, we must
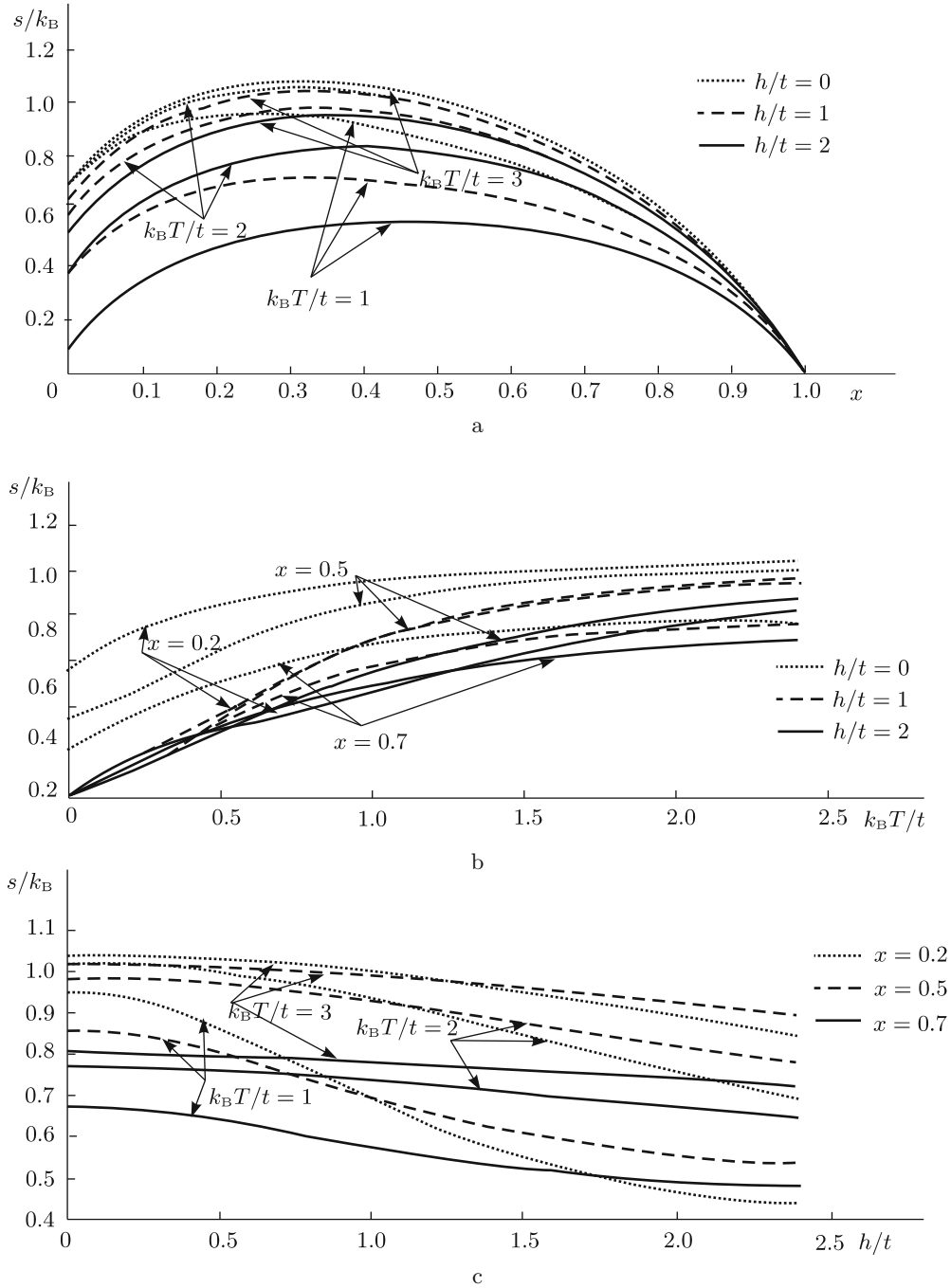
Fig. 2 . Dependences of entropy (a) on the concentration of holes, (b) on the temperature, and (c) on the magnetic field induction.

exclude states with the electron spin directed upward (at $T = 0$) from the total basis of states. As a result, the configuration contribution to the entropy changes discontinuously. As it must be, the dependences of s/k_B on $k_\mathrm{B}T/t$ are increasing functions. When discussing the entropy dependence on the hole concentration, we already mentioned that the curves of these dependences at $h = 0$ start at the point $(0, (1 - x)\log 2)$ and approach the asymptote described by (29) as $k_\mathrm{B}T/t \to \infty$. In the case $h \neq 0$, the entropy at $T = 0$ decreases discontinuously by $k_\mathrm{B}(1 - x)\log 2$, but the expression for the asymptote at $k_\mathrm{B}T/t \to \infty$ still has

form (29).

The entropy dependences on the magnetic field induction in Fig. 2c are decreasing curves in accordance with (20). A physical reason for this decrease can be explained in the simplest way again in the case $t = 0$, where entropy decreases because of the inequality

$$\frac{\partial}{\partial h}(-w_\uparrow \log w_\uparrow - w_\downarrow \log w_\downarrow) < 0, \tag{30}$$

where $w_\uparrow$ and $w_\downarrow$ are the occupation probabilities for the respective upper and lower atomic energy levels arising because of the Zeeman splitting. Writing the grand canonical distribution with Hamiltonian (1) in which we set $H_t = 0$, we obtain the expressions for these probabilities

$$
\begin{aligned}
w_\uparrow &= \frac{1-x}{2\cosh(h/k_{\mathrm{B}}T)} e^{-h/k_{\mathrm{B}}T}, \\
w_\downarrow &= \frac{1-x}{2\cosh(h/k_{\mathrm{B}}T)} e^{h/k_{\mathrm{B}}T}.
\end{aligned}
\tag{31}
$$

Inequality (30) is trivially satisfied. We note that formula (6) for the entropy at $t = 0$ totally agrees with the Boltzmann formula according to which

$$s = -w_0 \log w_0 - w_\uparrow \log w_\uparrow - w_\downarrow \log w_\downarrow,$$

where $w_0 = x$ is the occupation probability for an empty state on a site.

We next discuss the internal energy behavior. Figure 3a exhibiting the internal energy dependence on the hole concentration shows that as the temperature increases at $h = 0$, the curve slowly drifts up from its initial shape at $k_{\mathrm{B}}T/t = 0$

$$\frac{u}{t} = -\frac{2}{\pi}\sin \pi x, \tag{32}$$

approaching its limit shape

$$\frac{u}{t} = \frac{2}{\pi}\sin \pi x \tag{33}$$

as $k_{\mathrm{B}}T/t \to \infty$. For $h \neq 0$, the equation of the curve $\frac{u}{t}(x)$ for $k_{\mathrm{B}}T/t = 0$ is

$$\frac{u}{t} = -\frac{2}{\pi}\sin \pi x - (1-x)\frac{h}{t}, \tag{34}$$

and it retains form (33) as $k_{\mathrm{B}}T/t \to \infty$.

The curves in Fig. 3b describing the temperature dependence of the internal energy start at the point $(0, -(2/\pi)\sin \pi x - (1-x)h/t)$, increase monotonically, and approach shape (33) as $k_{\mathrm{B}}T/t \to \infty$. As expected, the internal energy decreases as the magnetic field induction increases (see Fig. 3c). This is because of the prevailing filling of lower energy levels under the Zeeman splitting.

We now discuss the behavior of the specific heat. We begin by discussing its temperature dependence (Fig. 4b). Both terms in (22) considered as functions of temperature produce curves with a single maximum point. We might then expect the appearance of two maximums on the temperature-dependence curve for
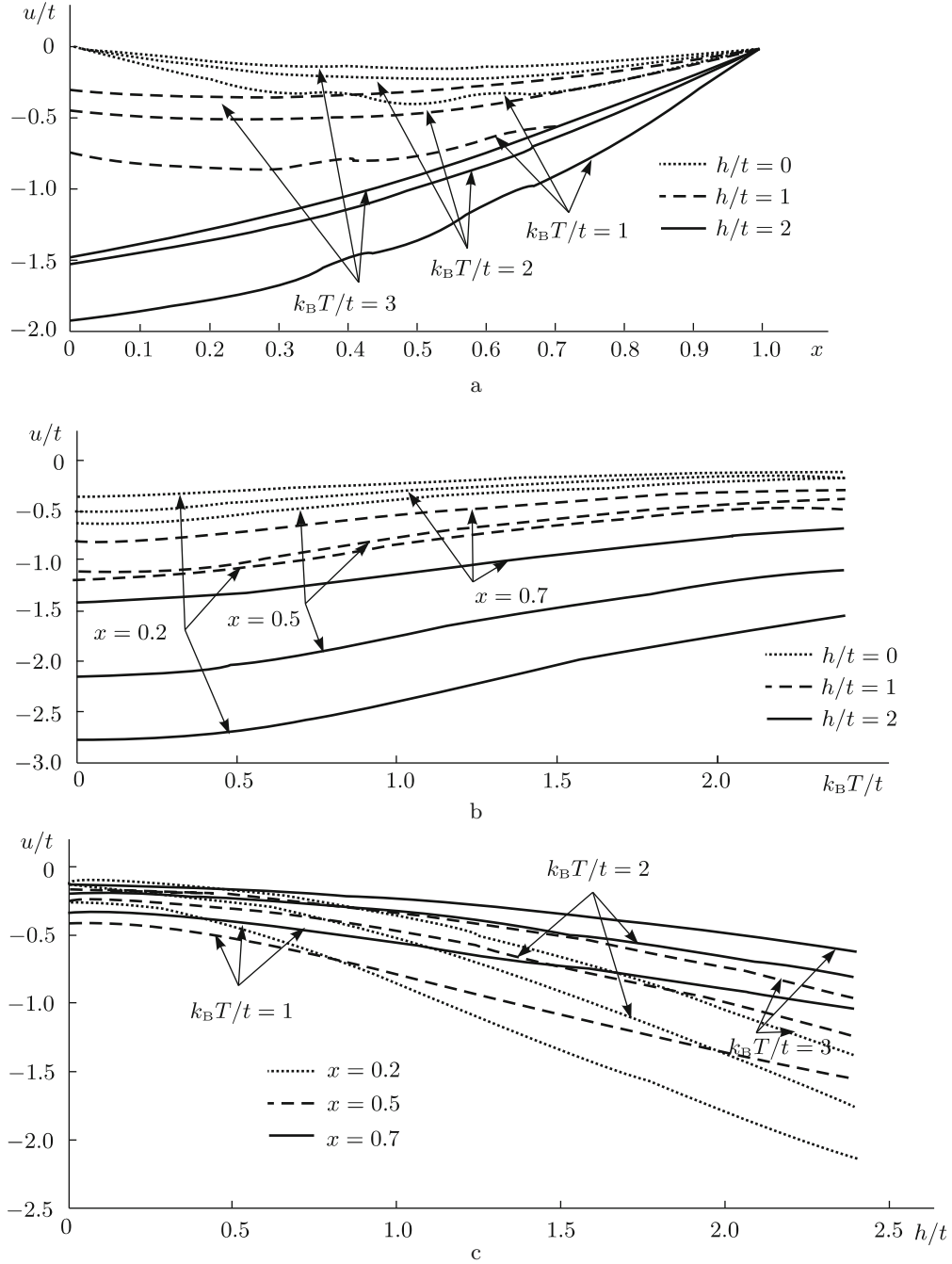
Fig. 3 . Dependences of the internal energy (a) on the concentration of holes, (b) on the temperature, and (c) on the magnetic field induction.

specific heat. But for fields $h \approx (0.1 \text{ to } 4)t$ at temperatures close to those at which the curves $c^{(\mathrm{h})}(T)$ and

$$c^{(\mathrm{s})}(T) = \frac{h^2}{k_{\mathrm{B}} T^2}(1-x)\frac{1}{\cosh^2(h/k_{\mathrm{B}}T)}$$

reach their maximums, the quantities $c^{(\mathrm{h})}$ and $c^{(\mathrm{s})}$ are of the same order. The curve $c(T)$ hence has only one maximum. But for fields outside this interval, the curve $c^{(\mathrm{h})}(T)$ attains its maximum in the domain where
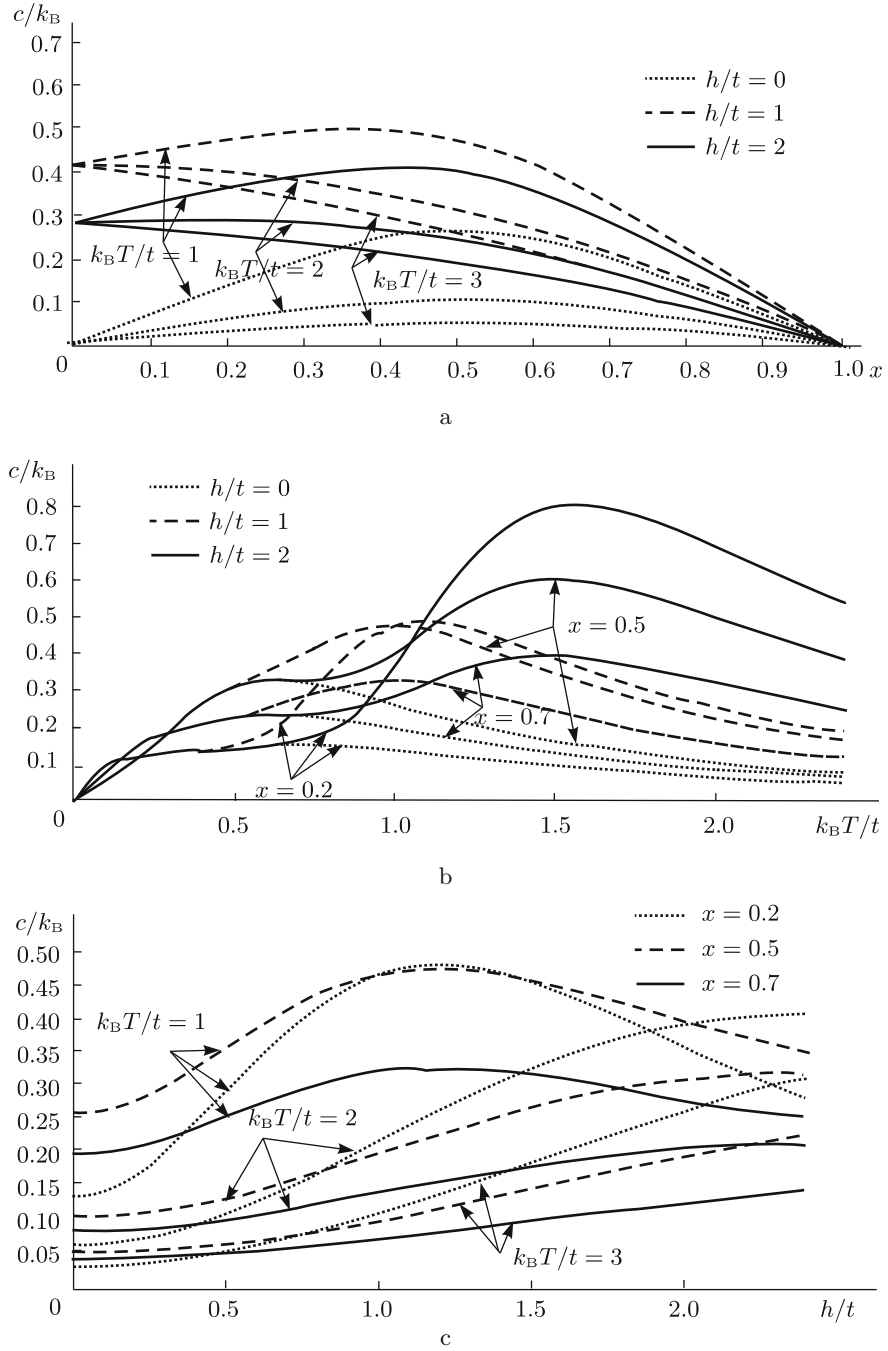
Fig. 4 . Dependences of the specific heat (a) on the concentration of holes, (b) on the temperature, and (c) on the magnetic field induction.

$c^{(\mathrm{h})} \gg c^{(\mathrm{s})}$, and we have two distinct peaks on the curve $c(T)$. It can be clearly seen in Fig. 5 that the curve $c(T)$ has either one or two maximums at different values of h. As mentioned in [7], [8], in the Hubbard model with a finite energy U of the on-site repulsion of two electrons in the absence of a magnetic field, the temperature dependence of the specific heat also develops two maximum points. The low-temperature peak is due to spin excitations and is attained at a temperature $T \approx J/k_{\mathrm{B}}$. The high-temperature peak is due to charge excitations and is attained at a temperature $T \approx t/k_{\mathrm{B}}$. In this model with $u \to \infty$, the exchange
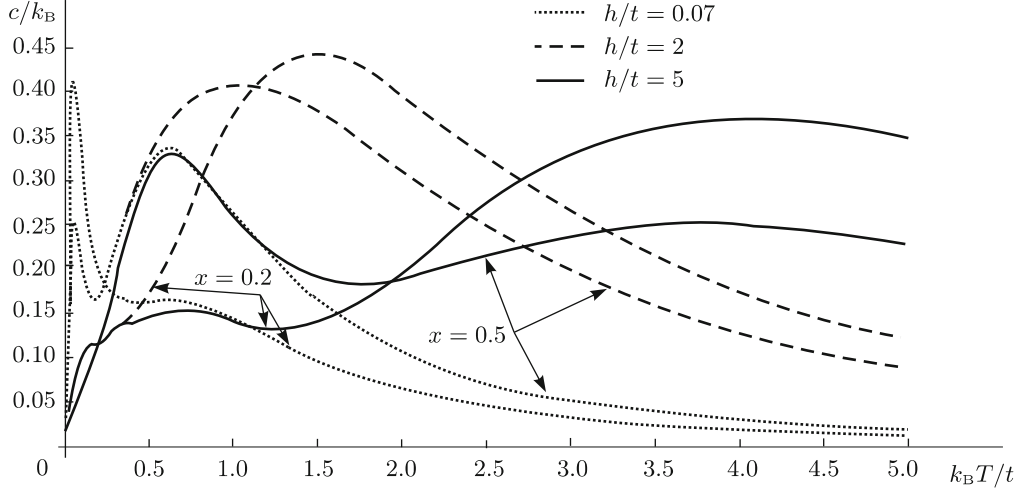
Fig. 5 . Curves of the temperature dependence of the specific heat: these curves have either one or two maximum points depending on the value of the magnetic field induction.

parameter $J = t^2/U = 0$, and the spin peak must therefore be absent. But switching on the magnetic field results in a nonzero spin excitation energy, and this peak therefore again appears.

From the specific heat dependence on the temperature discussed above, it is now easy to understand the specific heat dependence on the hole concentration. As can be seen in Fig. 4a, for a constant $0.1t < h < 4t$, the curves drift up from their initial shape $\frac{c^{(\mathrm{h})}}{k_{\mathrm{B}}}(x)$ at $k_{\mathrm{B}}T/t = 0$, attain the extremum (the highest) curve, and subsequently flatten down to the curve $\frac{c^{(\mathrm{h})}}{k_{\mathrm{B}}}(x)$ as $k_{\mathrm{B}}T/t \to \infty$. For $h < 0.1t$, as the temperature increases, we initially reach the first extremum curve, which is related to spin excitations. After reaching this curve, the dependence curve $\frac{c}{k_{\mathrm{B}}}(x)$ first moves down and then begins to rise, reaching the second extremum curve related to charge excitations. For $h > 4t$, in contrast, we first reach the holon extremum curve and then reach the spinon extremum curve. In both cases, after reaching the second extremum curve, the curve $\frac{c}{k_{\mathrm{B}}}(x)$ then flattens to the curve $\frac{c^{(\mathrm{h})}}{k_{\mathrm{B}}}(x)$ as $k_{\mathrm{B}}T/t \to \infty$.

Finally, we discuss the behavior of the chemical potential. We first see that switching on the magnetic field lowers the Fermi level (numerically equal to the chemical potential at $T = 0$) by h. As follows from Fig. 6a, where we show the chemical potential dependence on the hole concentration, as x increases, the chemical potential monotonically decreases from plus infinity at $x = 0$ to minus infinity at $x = 1$.

The curves indicating the dependence of μ/t on $k_{\mathrm{B}}T/t$ (see Fig. 6b) start at the point $(0,\, 2\cos\pi x - h/t)$ and asymptotically approach the lines

$$\frac{\mu}{t} = -\frac{k_{\mathrm{B}}T}{t}\log\frac{2x}{1-x} \tag{35}$$

as $k_{\mathrm{B}}T/t \to \infty$.

The dependences of μ/t on h/t monotonically decrease (Fig. 6c), as might be expected. This is again because of the prevailing occupation of lower energy levels obtained under the Zeeman splitting. The curves of the dependence of μ/t on h/t start at the point $\left(0,\, \frac{\mu^{(\mathrm{h})}}{t}(x, k_{\mathrm{B}}T/t) - (k_{\mathrm{B}}T/t)\log 2\right)$ and asymptotically approach the lines

$$\frac{\mu}{t} = \frac{\mu^{(\mathrm{h})}}{t}\left(x, \frac{k_{\mathrm{B}}T}{t}\right) - \frac{h}{t} \tag{36}$$
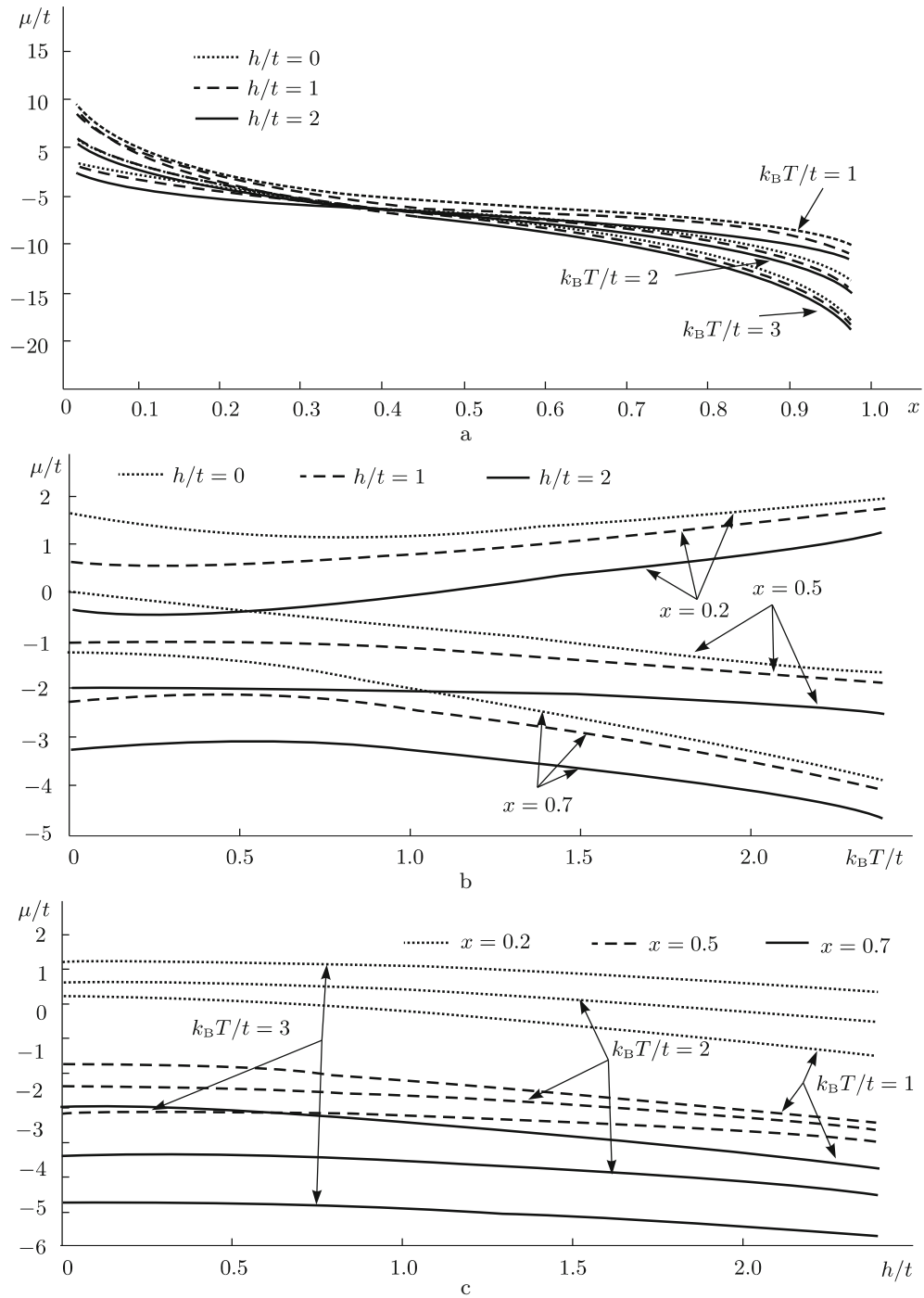
as $h/t \to \infty$.

Fig. 6 . Dependences of the chemical potential (a) on the concentration of holes, (b) on the temperature, and (c) on the magnetic field induction.

Appendix

A way to calculate the partition function of the one-dimensional t-model in the absence of a magnetic field exactly was presented in [4]. As we show below, the matrix method used there can also be successfully applied in the case of a nonzero magnetic field.

The partition function for the system described by Hamiltonian (1) by definition is

$$Z = \mathrm{Sp}(e^{(-H_0+H_t+H_m)/k_\mathrm{B}T}). \tag{37}$$

We note that the pairwise commutation relations of all terms in Hamiltonian (1) vanish,

$$[H_0, H_t] = [H_t, H_m] = [H_m, H_0] = 0. \tag{38}$$

In addition, the matrix of the operator H_0 is obviously diagonal in every basis and, moreover, has the numbers $N_e\varepsilon$ on the diagonal. These two circumstances allow rewriting expression (37) for the partition function as

$$Z = e^{-N_e\varepsilon/k_\mathrm{B}T}\,\mathrm{Sp}(e^{-H_t/k_\mathrm{B}T}e^{-H_m/k_\mathrm{B}T}). \tag{39}$$

The construction below requires an explicit rule for constructing the basis of wave functions in which we express the matrices $e^{-H_t/k_\mathrm{B}T}$ and $e^{-H_m/k_\mathrm{B}T}$. As the basis functions, we take functions of the form

$$\psi = \psi_1\psi_2\cdots\psi_N = \prod_{i=1}^{N}\psi_i, \tag{40}$$

where

$$|\psi\rangle_i = \begin{cases} |0\rangle_i, & \text{site } i \text{ is not occupied by an electron,} \\ |\uparrow\rangle_i, & \text{site } i \text{ is occupied by an electron with spin } 1/2, \\ |\downarrow\rangle_i, & \text{site } i \text{ is occupied by an electron with spin } -1/2. \end{cases} \tag{41}$$

We now discuss characteristic properties of the matrices $e^{-H_t/k_\mathrm{B}T}$ and $e^{-H_m/k_\mathrm{B}T}$ in the basis of functions (40) that we need in what follows. Both $e^{-H_t/k_\mathrm{B}T}$ and $e^{-H_m/k_\mathrm{B}T}$ are quadratic matrices of size $2^{N_e}C_N^{N_e} \times 2^{N_e}C_N^{N_e}$. We present each of these matrices in the $C_N^{N_e} \times C_N^{N_e}$ block form with each block of the size $2^{N_e} \times 2^{N_e}$. Such a block decomposition for a given ordering of all $2^{N_e}C_N^{N_e}$ states of type (40) is unique. By "block" in what follows, we always assume one of the $2^{N_e} \times 2^{N_e}$ blocks of this matrix uniquely decomposed into blocks. One very important rule remains in force. If $\{\langle\psi|_i\}$ and $\{|\psi\rangle_j\}$ denote a set of bra and ket vectors on which the ijth block of this matrix is realized, then we choose all the bra states $\{\langle\psi|_i\}$ (ket states $\{|\psi\rangle_j\}$) such that they do not differ in the site numbers where electrons are located and differ only in the spin directions of some of those electrons.

As indicated in [4], each separately taken block of the matrix $e^{-H_t/k_\mathrm{B}T}$ expressed in the basis of functions (40) and written in the nearest-neighbor approximation is diagonal with all the numbers on the diagonal of a given block equal to each other. We let $\sigma_1, \sigma_2, \ldots, \sigma_{C_N^{N_e}}$ denote the numbers in the diagonal blocks of $e^{-H_t/k_\mathrm{B}T}$.

We comment as follows concerning the matrix $e^{-H_m/k_\mathrm{B}T}$. First, analyzing the action of H_m on states of type (40) in which each of the functions $|\psi\rangle_i$ is determined by (41), we conclude that the operator matrix is diagonal. Correspondingly, the matrix of the operator $e^{-H_m/k_\mathrm{B}T}$ is also diagonal in this basis. Second, $X_f^{\uparrow\uparrow}$ and $X_f^{\downarrow\downarrow}$ are merely operators of the numbers of particles with the respective up and down spins. Their action on states of type (40) is independent of the location of the sites on which electrons with up and down spins are situated and on which holes are situated, depending only on the total number of electrons with up and down spins. Within one group comprising 2^{N_e} states among $C_N^{N_e}$ groups of states, the mutual distribution of sites on which electrons with up and down spins and holes are situated remains stable, as mentioned above. This means that if we order all states within each such group by the same principle, then

all $C_N^{N_e}$ diagonal blocks of the matrix H_m (and hence of the matrix $e^{-H_m/k_{\mathrm{B}}T}$) become equal to each other in the matrix sense.

Let Σ denote the sum of all diagonal elements of the single diagonal block of the matrix $e^{-H_m/k_{\mathrm{B}}T}$. The partition function then becomes

$$Z = e^{-N_e \varepsilon/k_{\mathrm{B}}T} \Sigma \sum_{i=1}^{C_N^{N_e}} \sigma_i. \tag{42}$$

In [4], a rule was formulated following which we can obtain the matrix of the electron hopping operator $H_t = -\sum_{f,m,\sigma} t_{fm} X_f^{\sigma 0} X_m^{0\sigma}$ from the holon hopping matrix $H_t^{(\mathrm{h})} = -\sum_{f,m} t_{fm} X_f^{10} X_m^{01}$ in the nearest-neighbor approximation. This rule states that each element of $H_t^{(\mathrm{h})}$ transforms into a diagonal block whose diagonal comprises numbers equal to the transformed matrix element. Taking this rule and the above definition of the numbers σ_i into account, we conclude that the product $e^{-N_e \varepsilon/k_{\mathrm{B}}T} \sum_{i=1}^{C_N^{N_e}} \sigma_i$ is just the holon partition function,

$$Z^{(\mathrm{h})} = \mathrm{Sp}(e^{-H^{(\mathrm{h})}/k_{\mathrm{B}}T}) = e^{-N_e \varepsilon/k_{\mathrm{B}}T} \sum_{i=1}^{C_N^{N_e}} \sigma_i, \tag{43}$$

where $H^{(\mathrm{h})} = \sum_f \varepsilon X_f^{11} + H_t^{(\mathrm{h})}$. Hence, the electron partition function in a magnetic field is

$$Z = \Sigma Z^{(\mathrm{h})}. \tag{44}$$

Expression (44) obviously implies that to obtain the final result for Z, we need only the quantity Σ.

We begin calculating Σ by introducing a special ordering of 2^{N_e} states with the same distribution of electrons to sites and differing only in the directions of electron spins at some of these sites. We illustrate this ordering with the example of a system of three electrons:

$$(1) \uparrow\uparrow\uparrow, \quad (2) \downarrow\uparrow\uparrow, \quad (3) \uparrow\downarrow\uparrow, \quad (4) \downarrow\downarrow\uparrow, \quad (5) \uparrow\uparrow\downarrow, \quad (6) \downarrow\uparrow\downarrow, \quad (7) \uparrow\downarrow\downarrow, \quad (8) \downarrow\downarrow\downarrow. \tag{45}$$

This ordering of the three-spin system implies the ordering of the general system of N_e spins. We obtain this ordering by induction: after ordering the system of k spins, we obtain an ordered system of $k+1$ spins as follows: to the already obtained 2^k states, we add 2^k new states differing from the already written states only by a change in the direction of the last, $(k+1)$th spin. If one of the first 2^k states has the number n ($1 \le n \le 2^k$), then the state obtained from this one by flipping the $(k+1)$th spin has the number $2^k + n$. As a result of this procedure for adding states, we obtain the ordered system of $2^k + 2^k = 2^{k+1}$ states.

We let Σ_k denote the sum of the first 2^k diagonal elements of a block of the matrix $e^{-H_m/k_{\mathrm{B}}T}$. Acting with the operator H_m on states of type (45), we obtain

$$\Sigma_1 = e^{-hN_e/k_{\mathrm{B}}T}(1 + e^{2h/k_{\mathrm{B}}T}),$$

$$\Sigma_2 = e^{-hN_e/k_{\mathrm{B}}T}(1 + e^{2h/k_{\mathrm{B}}T})^2.$$

We can prove the general formula

$$\Sigma_k = e^{-hN_e/k_{\mathrm{B}}T}(1 + e^{2h/k_{\mathrm{B}}T})^k \tag{46}$$

by induction. Indeed, from the above rule for state ordering, we find that the last 2^k of the first 2^{k+1} diagonal elements of the block are obtained by multiplying the first 2^k elements by the factor $e^{2h/k_{\mathrm{B}}T}$. This together with the induction assumption (46) implies the equality

$$\Sigma_{k+1} = \Sigma_k + e^{2h/k_{\mathrm{B}}T}\Sigma_k = e^{-hN_e/k_{\mathrm{B}}T}(1 + e^{2h/k_{\mathrm{B}}T})^{k+1},$$

which proves general formula (46).

Further, it is obvious that

$$\Sigma = \Sigma_{N_e}. \tag{47}$$

Elementary transformations of formula (46) at $k = N_e$ yield the final answer

$$\Sigma = \left(2\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right)\right)^{N_e}. \tag{48}$$

We note that the quantity Σ is a factor appearing because electrons have spins, i.e., it is the spinon partition function $Z^{(\mathrm{s})}$:

$$Z^{(\mathrm{s})} = \Sigma. \tag{49}$$

We present the final answer for the partition function:

$$Z = Z^{(\mathrm{h})}Z^{(\mathrm{s})}, \tag{50}$$

$$Z^{(\mathrm{h})} = \mathrm{Sp}(e^{-H^{(\mathrm{h})}/k_{\mathrm{B}}T}), \tag{51}$$

$$Z^{(\mathrm{s})} = \left(2\cosh\left(\frac{h}{k_{\mathrm{B}}T}\right)\right)^{N_e}. \tag{52}$$

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