ISSN 1063-7834, Physics of the Solid State, 2016, Vol. 58, No. 1, pp. 55–61. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © B.A. Belyaev, V.V. Tyurnev, A.V. Izotov, An.A. Leksikov, 2016, published in Fizika Tverdogo Tela, 2016, Vol. 58, No. 1, pp. 56–62.

MAGNETISM

Study of the Fields Scattered by a Periodic Strip Structure of Thin Magnetic Films

B. A. Belyaev[a, b, c, *], **V. V. Tyurnev**[a, b], **A. V. Izotov**[b], **and An. A. Leksikov**[a]

[a] *Kirensky Institute of Physics, Siberian Branch, Russian Academy of Sciences, Akademgorodok 50–38, Krasnoyarsk, 660036 Russia*

[b] *Siberian Federal University, pr. Svobodnyi 79/10, Krasnoyarsk, 660041 Russia*

[c] *Siberian State Aerospace University, pr. im. Gazety Krasnoyarskii Rabochii 31, Krasnoyarsk, 660037 Russia*

**e-mail: belyaev@iph.krasn.ru*

Received May 25, 2015

Abstract—Components of the fields scattered by a periodic planar strip structure of thin magnetic films possessing a uniaxial magnetic anisotropy in the plane have been calculated using the phenomenological model. Regularities in the dependence of these fields on the design parameters of the structure have been studied. The results obtained agree with the numerical analysis of the micromagnetic model of this structure. It has been shown that, near the edges of strips magnetized orthogonally to the major axis, the components of the scattered field can exceed the external magnetizing field by a few orders of magnitude. This fact makes it possible to design highly efficient magnetoresistive elements on the basis of a strip structure of magnetic films and thin semiconductor films.

DOI: 10.1134/S1063783416010054

1. INTRODUCTION

As is known, magnetoresistive materials are employed in magnetometers of weak fields and various sensors [1], including nanosized sensors based on thin magnetic films (TMF) for studies of biological objects [2]. TMF magnetoresistive elements serve as the reading heads in magnetic memory devices; they also serve as a basis for newly developed microelectronic devices, e.g., magnetic isolators [3]. The giant magnetoresistive effect is inherent in the widely used multilayered film structures consisting of magnetic and nonmagnetic layers [3, 4]; they can also find wide application in controlled microwave devices [5]. However, despite a relatively large variation in the resistance, which can reach ~100% [5], the control of such structures requires relatively strong control magnetic fields on the order of 10^4 Oe. This drawback is absent in a magnetoresistor consisting of a semiconductor film upon the surface of which a periodic structure of parallel strips of a magnetically soft material is applied [6]. The scattered fields in such a structure, which are the subject of this work, can exceed by a few orders of magnitude the external control magnetic field and, as a result, significantly increase the variation in the resistance of the semiconductor film.

As is known, near a ferromagnetic body magnetized orthogonally to the surface, the magnitude of the field in Gaussian units is on the order of $2\pi M$, where M is the saturation magnetization of the material. This property is employed in electromagnets with cores of magnetically soft materials, which can generate sufficiently strong fields at relatively lower expenditures of energy. Indeed, the coils of an electromagnet magnetize the core to saturation, which can result in a constant magnetic field of up to ~$4\pi M$ in the gap between the poles. A similar increase in the magnetic field is observed in small gaps of a periodic structure of strip elements formed by thin magnetic films. Moreover, such structures have important advantages over massive magnets. They are capable of remagnetizing for a few nanoseconds in a relatively weak magnetic field (slightly stronger than the field of uniaxial magnetic anisotropy of the film), as a rule, not exceeding a few oersteds. It is this effect that is employed in magnetoresistors on the basis of a semiconductor film with a lattice (raster) TMF structure [6]. Theoretically, such magnetoresistors can serve as an active element not only in magnetometers and sensors, but also in various microwave devices, including those with electrically controlled characteristics.

However, thin-film structures have an essential drawback: the fields on the order of $4\pi M$ are concentrated in them in a very small volume, determined by the film thickness. Nevertheless, since modern technologies make it possible to produce periodic structures with gaps of a few nanometers, the importance and urgency of studies of scattered fields in a periodic array of magnetic strip elements is beyond doubt. The results of such studies enable one to formulate basic requirements to materials and geometrical parameters of multilayered structures for the development of new magnetoresistive element on their basis.

2. CALCULATION

Consider a plane one-dimensional periodic structure consisting of parallel magnetic thin-film strips infinite in the direction of the y-axis (Fig. 1a). The strips have a thickness T, width W, and the gap of width S between them. The structure is produced of a magnetically soft material with a saturation magnetization M, and the magnetic film possesses an induces uniaxial magnetic anisotropy in the plane, which is characterized by the field $H_a \ll 4\pi M$. Suppose that the structure is found in a constant magnetic field H_0 oriented at an angle θ_H to the y-axis (Fig. 1b), the easy magnetization axis of the uniaxial magnetic anisotropy is oriented at an angle θ_a, and the equilibrium saturation magnetization of the film is oriented at an angle θ_M.

Assume that the strip elements have no domain structure; i.e., the magnetization vector $\mathbf{M}(x, y, z)$ is homogeneous in the bulk of them. Then, the planar

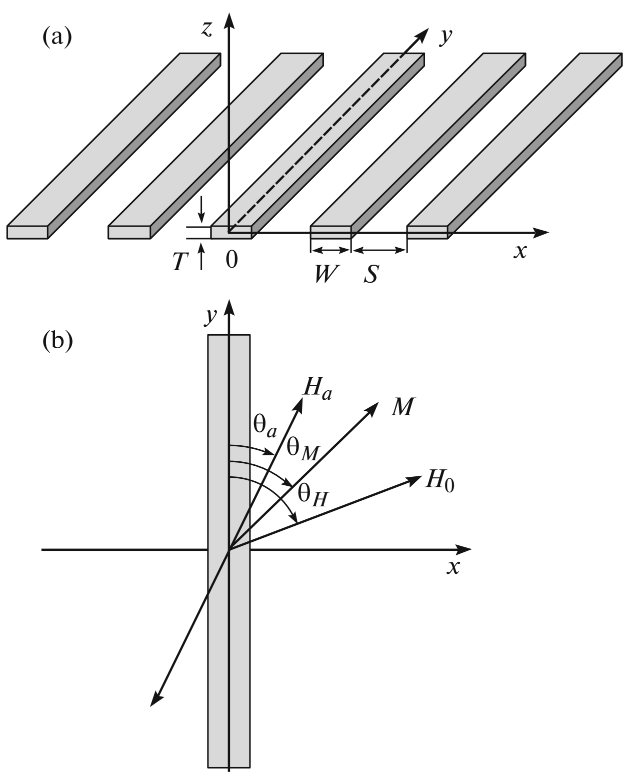

Fig. 1. (a) Array of magnetic strips and (b) the model of a strip with uniaxial magnetic anisotropy in the plane.

components of the magnetization vector are expressed by the formulas

$$M_x = M\sin\theta_M, \quad M_y = M\cos\theta_M. \tag{1}$$

In the case of an infinite film, the value of the angle θ_M is the solution of the equation [7]

$$H_0\sin(\theta_H - \theta_M) + \frac{1}{2}H_a\sin 2(\theta_a - \theta_M) = 0. \tag{2}$$

Introduce the scalar potential determining the scattered field:

$$\mathbf{H} = -\mathrm{grad}\varphi. \tag{3}$$

This potential satisfies Poisson's equation

$$\nabla^2\varphi = \mathrm{div}\mathbf{M}. \tag{4}$$

Introduce the quantity $\rho = -4\pi\mathrm{div}M$, which is often called the bulk density of magnetic charges. In the case of uniformly magnetized strip elements, the magnetic charges ρ are located only on that part of their surface on which the normal component M_n of the magnetization vector $\mathbf{M}$ suffers a jump. From Eq. (4) and the condition of continuity of the normal component of the magnetic field B_n on the interface of two media, it follows that the surface density of magnetic charges on the edges of the magnetic strips is expressed as

$$\rho_s = \pm 4\pi M_x, \tag{5}$$

where the plus sign corresponds to the right edge of the magnetic strip and the minus sign, to the left one.

It is worth listing the element of symmetry of the given problem. First, the array of magnetic strips is homogeneous along the y-axis. Therefore, the sought potential φ may be a function of only two coordinates, namely, x and z. Second, in the direction of the x-axis, the array of strips is periodic. Therefore, the potential $\varphi(x, z)$ has a period $W + S$ along the x-axis. Third, the structure has two planes of symmetry, periodically repeating along the x-axis and orthogonal to it. These planes divide each strip element and each gap between strips into two equal parts, left and right. Each plane of symmetry is located at the same distance from charges of opposite signs. Therefore, the potential φ vanishes on the planes of symmetry. Thus, for finding the scattered field $\mathbf{H}(x, z)$, it suffices to obtain the solution to Eq. (4) for the potential $\varphi(x, z)$ within one elementary cell, i.e., in the two-dimensional region

$$0 \le x \le (W + S)/2, \quad -\infty < z < \infty, \tag{6}$$

and then periodically continue this solution to the entire array of strip elements.

The potential must vanish on the left and right boundaries of the elementary cell, i.e.,

$$\varphi(x, z)|_{x=0} = 0, \quad \varphi(x, z)|_{x=(W+S)/2} = 0, \tag{7}$$

and, outside the surface, the potential $\varphi(x, z)$ is a harmonic function.

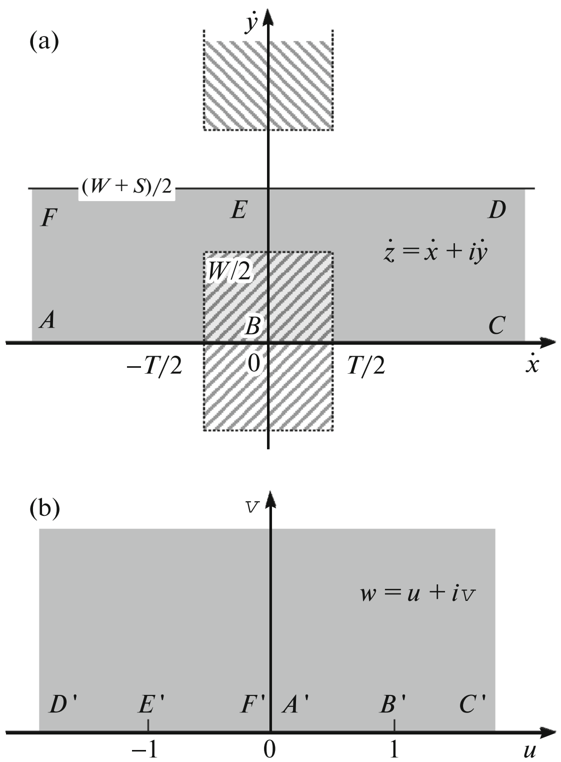

Fig. 2. Conformal mapping of domains. Dashed rectangles represent cross sections of magnets. Gray color emphasizes the region of an "elementary cell" of the periodic structure.

On the surface of the edge of a strip element ($-T/2 \leqslant z \leqslant T/2, x = W/2$), the volume charge density is determined by the surface charge density ρ_s and, in the general case, for the interval $-T/2 \leqslant z \leqslant T/2$, by the expression

$$\rho(x, z) = \rho_s \delta(x - W/2). \quad (8)$$

The solution to Poisson's equation (4) in elementary cell (6), which has the form of an infinite strip with two boundaries is significantly simplified if one conformingly maps the cell onto a semi-infinite region with one boundary. To this end, we denote the coordinates z and x by $\dot{x}$ and $\dot{y}$, respectively (Fig. 2a). As a result, the boundaries of the elementary cell will pass through the points A, B, C, D, E, and F so that the points A, B, and C will belong to the upper boundary of the cell and the points D, E, and F, to its upper boundary. Figure 2b shows, in the new coordinates u and v, a half-plane region onto which an elementary cell with the boundary passing through the points A', B', C', D', E', and F' is mapped.

To find the conformal map, it is necessary associate each point having real coordinates $\dot{x}$ and $\dot{y}$ in Fig. 2a with a complex number $\dot{z} = \dot{x} + i\dot{y}$ and associate each point in Fig. 2b having real coordinates u and v with a complex number $w = u + iv$. In Fig. 2, the conformal mapping of the regions is performed by the analytic functions [8]

$$\dot{z}(w) = \frac{W+S}{2\pi}\ln(w), \quad w(\dot{z}) = \exp\left(\frac{2\pi\dot{z}}{W+S}\right). \quad (9)$$

It is easy to check that, in conformal map (9), the points A, B, C, D, E, and F in Fig. 2a correspond to the points A', B', C', D', E', and F' in Fig. 2b.

Any analytic function of a complex variable, $\Phi(\dot{z})$, is a harmonic function of the coordinates $\dot{x} = \mathrm{Re}\dot{z}$ and $\dot{y} = \mathrm{Im}\dot{z}$. Therefore, the real function $\varphi(\dot{x}, \dot{y})$ describing the scalar potential of a magnetostatic field will be sought in the form

$$\varphi(\dot{x}, \dot{y}) = \mathrm{Re}\,\Phi(\dot{z}). \quad (10)$$

The potential of a single infinitely thin ribbon of width $d\dot{x}_1$ with a linear charge $dq = \rho_s d\dot{x}_1$, passing through the coordinate $w_1 = w(\dot{x} + iW/2)$ orthogonally to the w-plane, is defined as [8, 9]

$$d\varphi_1(w) = -2\rho_s d\dot{x}_1 \mathrm{Re}\ln(w - w_1). \quad (11)$$

The function $d\varphi_1(w)$ is the real part of an analytic function in the entire region w, except a unique isolated singular point $w = w_1$. Therefore, it is harmonic with respect to both the coordinates u, v and the conformingly mapped coordinates $\dot{x}$, $\dot{y}$. The magnetic field strength $\mathbf{H}$ corresponding the potential $d\varphi_1(w)$ vanishes only at $w \to \infty$. If there is another infinite ribbon passing through the coordinate w_1^* and carrying the charge $-dq$, the total potential of the ribbons is zero on the boundary of the w-domain; i.e., at $\mathrm{Im}w = 0$:

$$d\varphi(w) = 2\rho_s d\dot{x}_1 \mathrm{Re}\ln\left(\frac{w - w_1^*}{w - w_1}\right). \quad (12)$$

Since the boundary of the w-domain is a map of the boundary of the z-domain, potential (12) satisfies both boundary conditions in the $\dot{x}\dot{y}$-plane, i.e., vanishes on the boundaries ($-\infty < \dot{x} < \infty$, $\dot{y} = 0$) and ($-\infty < \dot{x} < \infty$, $\dot{y} = (W+S)/2$). As a result, calculating the gradient of the potential, we find the components of the magnetic field produced by the charges $\pm dq$ by formula (3):

$$\begin{aligned} dH_{\dot{x}} &= -2\rho_s d\dot{x}_1 \,\mathrm{Re}\frac{d}{d\dot{z}}[\ln(w - w_1^*) - \ln(w - w_1)], \\ dH_{\dot{y}} &= 2\rho_s d\dot{x}_1 \,\mathrm{Im}\frac{d}{d\dot{z}}[\ln(w - w_1^*) - \ln(w - w_1)]. \end{aligned} \quad (13)$$

Carrying out the differentiation with respect to $\dot{z}$ in (13), we obtain

$$\begin{aligned} dH_{\dot{x}} &= -2\rho_s \mathrm{Re}\left[\left(\frac{1}{w - w_1^*} - \frac{1}{w - w_1}\right)\frac{dw}{d\dot{z}}\right]d\dot{x}_1, \\ dH_{\dot{y}} &= 2\rho_s \mathrm{Im}\left[\left(\frac{1}{w - w_1^*} - \frac{1}{w - w_1}\right)\frac{dw}{d\dot{z}}\right]d\dot{x}_1. \end{aligned} \quad (14)$$

To find the magnetic field from all charges on the boundary of the strip element, it is necessary to integrate (14) with respect to $\dot{x}_1$ from $-T/2$ to $T/2$ at $\dot{y}_1 = W/2$. Using (9), after the integration and the return from the temporary coordinates $\dot{x}$, $\dot{y}$ to the original coordinates z, x, we obtain

$$H_z(x,z) = -2M_x \mathrm{Re}\Phi, \quad H_x(x,z) = 2M_x \mathrm{Im}\Phi, \quad (15)$$

$$\Phi = \ln \frac{\left[\exp\left(2\pi\frac{z+ix}{W+S}\right) - \exp\left(\pi\frac{-T-iW}{W+S}\right)\right]\left[\exp\left(2\pi\frac{z+ix}{W+S}\right) - \exp\left(\pi\frac{T+iW}{W+S}\right)\right]}{\left[\exp\left(2\pi\frac{z+ix}{W+S}\right) - \exp\left(\pi\frac{-T+iW}{W+S}\right)\right]\left[\exp\left(2\pi\frac{z+ix}{W+S}\right) - \exp\left(\pi\frac{T-iW}{W+S}\right)\right]}. \quad (16)$$

Formulas (15) and (16) enable one to calculate the components of the fields scattered by a periodic lattice of magnetic strip elements at any point of space, including the regions near the strip elements and gaps between them.

3. STUDYING THE DEPENDENCES OF THE SCATTERED FIELDS ON THE DESIGN PARAMETERS OF THE MAGNETIC STRIP STRUCTURE

The method for calculating the components of the fields scattered by a periodic array of magnetic strips was implemented in a computer program for the numerical analysis of the above-considered structure. Using this program, regularities of the behavior of the scattered fields on the parameters of the structure magnetized in the plane orthogonally to the major axis of the strips ($H > H_a$) were studied [10]. The lines in Fig. 3 represent the coordinate dependences of the longitudinal (parallel to the plane), H_x, and transverse (orthogonal to the plane), H_z, components of the scattered field, plotted for the considered structure for two values of the gaps between magnetic strips, $S = 1$ and 0.2 μm; the magnetic strips had the width of $W = 2$ μm and thickness of $T = 0.1$ μm. The TMF material was permalloy with a saturation magnetization of M = 800 G. The fields were calculated at a distance of z_0 = 0.05 μm from the upper surface of the magnetic films. We see that the maximum absolute values of the fields are approximately the same and the extrema of the transverse component are located across from the edges of the magnetic strips, where zeros of the longitudinal component of the field are observed. It should be noted that the component H_x preserves fairly large values almost in the entire gap between strips.

The reliability of the results obtained is confirmed by their good agreement with the results of numerical analysis of the micromagnetic model [11, 12] of the considered structure, which are represented in the same figure by dots. As is known, this method has proved its worth not only by its high efficiency and sufficiently high accuracy but its ability to calculate not only static by also dynamic characteristics of complex magnetic objects [13].

The dependence of the scattered field on the gap width between strips can be analyzed on the example of the longitudinal component $H_x(S)$ (Fig. 4). The dependences were constructed for a structure with $W = 10$ μm, $T = 0.1$ μm, and $M = 800$ G. The fields were calculated for the points across from the center of the strip (*1*), across from the center of the gap (*2*), and across from the edge of the strip (*3*) for two distances from the TMF surface: $z_0 = 0.001$ μm and $z_0 = 0.5$ μm. We see that, for $z_0 = 0.001$ μm, the magnitude of the longitudinal field across from the center of the gap (curve *2*) in the region of small S exceeds 2 kOe and

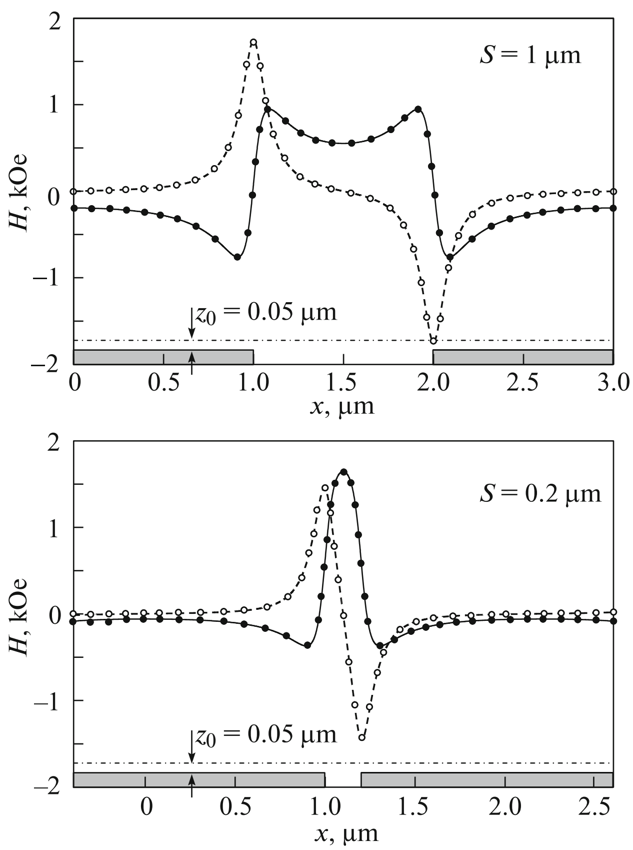

Fig. 3. Dependences of planar, H_x (solid lines), and normal, H_z (dashed lines), components of scattered fields calculated at a distance of z_0 = 0.05 μm from the surface of films with the magnetization of M = 800 G, plotted for two gap widths S between strips of thickness T = 0.1 μm and width W = 2 μm. Dots represent the results of micromagnetic modeling.

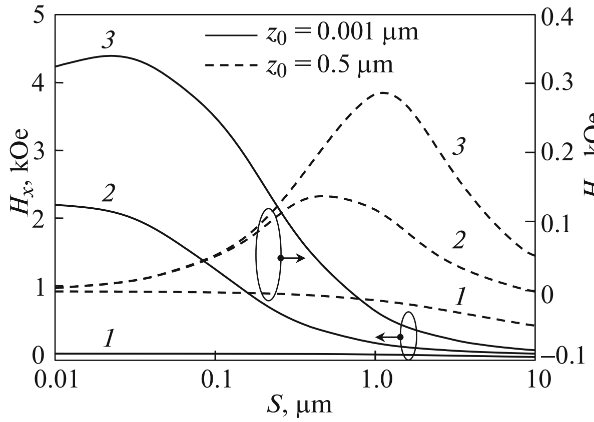

Fig. 4. Dependences of the longitudinal component of scattered field for a structure with $W = 10$ μm, $T = 0.1$ μm, and $M = 800$ G on the gap width between strips, calculated for two distances z_0 (*1*) across from the center of a strip, (*2*) across from the center of a gap, and (*3*) across from the edge of a strip.

decreases by about a half when the gap width equals the thickness of the magnetic film. However, with a further increase in S, the field H_x monotonically decreases to zero, as expected. In this case, the field in the point across from the edge of the strip (curve *3*) in the region of small gap widths has a weakly expressed maximum and exceeds in the gap 4 kOe, decreasing by half if the gap width equals approximately twice the thickness of the magnetic film. A further increase in S also leads to a monotonic decrease in H_x to zero. We also see that the magnitude of the longitudinal field in the point across from the center of the strip (curve *1*) is insignificant and depends weakly on the gap width between strips. The calculation of the fields at the distance of $z_0 = 0.5$ μm from the TMF surface has shown that the maxima of the longitudinal fields decrease by about an order of magnitude. In this case, the dependences $H_x(S)$ exhibit strongly pronounced maxima of the fields both across from the center of the gap (curve *2*) and across from the edge of the strip (curve *3*); the magnitude of the longitudinal field across from the center of the strip (curve *1*) is small and depends weakly on the gap width between strips.

Taking into account the fact that, being employed in a magnetoresistor [6], the magnetic strip structure under consideration is placed on the surface of a semiconductor film whose thickness may be both greater and smaller than the thickness of the TMF, it is of significant practical interest to consider the dependences of the scattered fields on the distance z_0 above the surface of the structure. Figure 5 shows the dependences of the longitudinal, H_x, and transverse, H_z, components of the scattered fields calculated at a distance of 0.1 μm from the edge of the strip across from the gap (Fig. 5a) and across from the strip (Fig. 5b). The dependences were plotted for two gap widths S

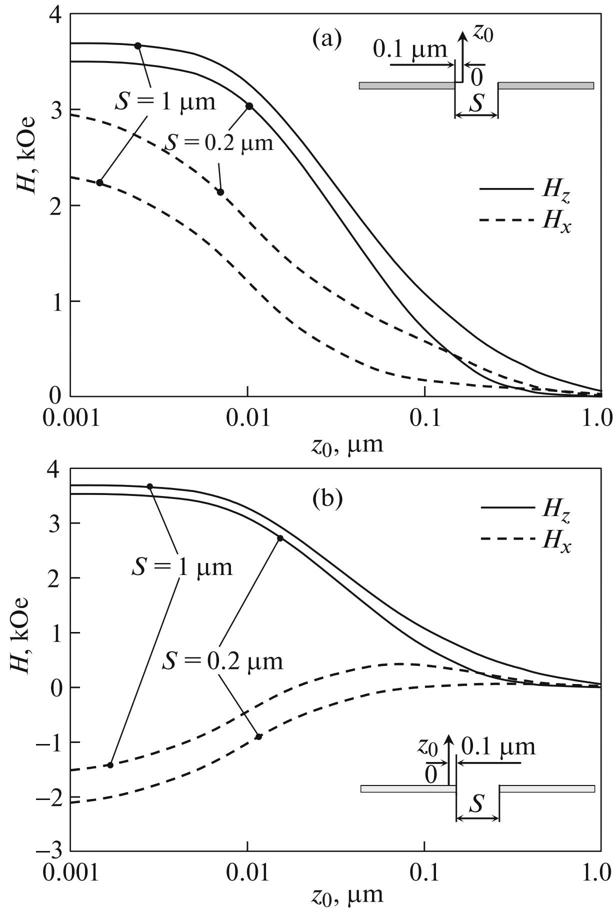

Fig. 5. Dependences of the scattered fields on the height z_0 above the surface of the structure, calculated near the edge of a magnetic strip (a) across from the gap and (b) across from the strip, plotted for two gap widths between strips with a width of 10 μm, thickness of 0.1 μm, and saturation magnetization of $M = 800$ G.

between the strips of the structure with the film thickness of 0.1 μm and strip widths of 10 μm. The saturation magnetization of the TMF is of $M = 800$ G. We see that the longitudinal component across from the strip and across from the gap has different signs and its absolute value is smaller than that of the transverse component. In this case, the behavior of both these components with an increase in z_0 has approximately the same character but, as the gap width between them increases from 0.2 to 1.0 μm, the transverse component of the scattered field slightly increases and the longitudinal component decreases.

The dependences presented in Fig. 6 can be used to estimate the efficiency of the strip structure of thin magnetic films designed to "amplify" the external magnetic field. They show the dependences of the longitudinal component of the scattered fields calculated near the edges of strips on the basis of micromagnetic modeling on the external DC magnetic field applied in the plane of the structure orthogonally to the strips.

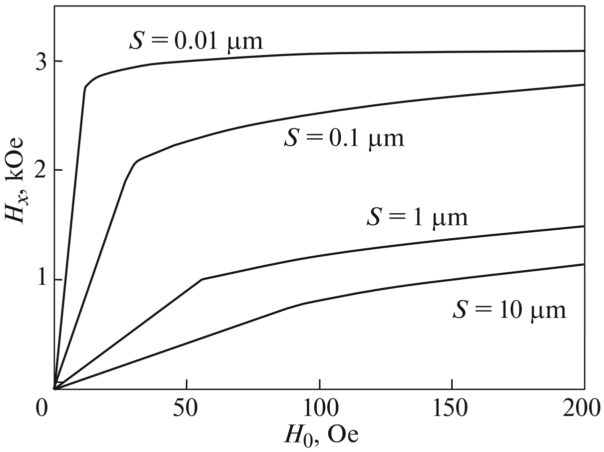

Fig. 6. Dependences of the longitudinal component of scattered fields near the edge of a magnetic strip on the external DC magnetic field applied in the plane of the structure orthogonally to strips, plotted for several gap widths between strips with a width of 10 μm, thickness of 0.1 μm, saturation magnetization of TMF of 800 G, and anisotropy field of 5 Oe.

The curves were obtained for several gap widths between strips, which have the width of $W = 10$ μm and thickness of $T = 0.1$ μm; the plane uniaxial magnetic anisotropy of the TMF, whose easy axis is directed along the strips, is characterized by the field strength of $H_a = 5$ Oe and the saturation magnetization of the film of $M = 800$ G.

We see that, for the gap widths between strips of $S = 0.01$ μm, the scattered field reaches the value of $H_x = 2.78$ kOe, the external magnetizing field being of 12 Oe. In other words, the external magnetizing field is "amplified" by the magnetic strip structure more than 230-fold. However, with a 10-fold increase in the gap widths between strips ($S = 0.1$ μm), the maximum gain factor drops to 82; with a gap wider by two orders of magnitude, when it becomes equal to the strip width ($W = S = 10.0$ μm), the gain factor is only of 8.3. Hence, the structure under consideration, undoubtedly, can provide a sufficiently high level of "amplification" of the external magnetizing field necessary for the efficient work of a magnetoresistor [6] but it can be done with sufficiently narrow gaps between the strips, comparable or smaller than the thickness of the magnetic films. This fact is easily explained by the presence of demagnetizing fields in the considered periodic strip structure, as a result of which its magnetization takes place in fields substantially exceeding the field of its uniaxial magnetic anisotropy ($H_a = 5$ Oe): the magnetizing field corresponds to it only in the case of a continuous film. As is known, demagnetizing fields rapidly decrease with a decrease in the gap width between strips; therefore, the fields in which the magnetization of such a structure takes place decrease.

4. CONCLUSIONS

In this work, the components of the fields scattered by a periodic planar strip structure of thin magnetic films possessing a uniaxial magnetic anisotropy in the plane have been calculated by means of a phenomenological model. The regularities of the behavior of these fields with variations in the design parameters of the periodic lattice of magnetic strips have been studied. The reliability of the results obtained is confirmed by their good agreement with numerical analysis of the micromagnetic model of the considered structure. It has been shown that, near the edges of strips magnetized orthogonally to the major axis, the components of the scattered fields can exceed the external magnetizing field by a few orders of magnitude. This fact makes it possible to create highly efficient magnetoresistive elements on the basis of a strip structure of magnetic films and thin semiconductor films.

As is known [14], the magnetoresistive effect in semiconductor materials is proportional to the square of the magnitude of the applied magnetic field; therefore, in weak external fields, this effect is negligibly small. However, as has been shown in this work, a relatively small control magnetic field can be "amplified" by a few orders of magnitude due to scattered fields existing in small gaps of a periodic structure of magnetic film strip elements. Moreover, the use of film elements with a plane uniaxial anisotropy oriented along the strips makes it possible to vary the scattered fields gradually from zero to the maximum by applying a control magnetic field in the plane orthogonally to the direction of strips. Placing such a structure above the surface of a thin semiconductor film, one can observe a substantial change in its resistance under the action of such significant variations in the scattered magnetic fields. Obviously, the magnetoresistive effect in such a magnetoresistor can be enhanced by the use of magnetic films with the maximum possible saturation magnetization.

It should be noted that the above-considered structures are capable of remagnetizing for a few nanoseconds and in a relatively weak magnetic field. As has been shown above, this field only slightly exceeds the field of uniaxial magnetic anisotropy, which, as a rule, is of a few oersteds. This fact opens the theoretical possibility of application of the film magnetoresistor as the active element of not only magnetometers and sensors but also various microwave devices using the ferromagnetic resonance in TMF. Obviously, the characteristics of such devices can be controlled by varying the external magnetic field.

ACKNOWLEDGMENTS

This study was performed at the Siberian Federal University and was supported by the Ministry of Education and Science of the Russian Federation (state task for 2014–2016, project no. 3.528.2014K).

REFERENCES

1. D. Wang, in *Handbook of Advanced Magnetic Materials*, Ed. by Y. Liu, D. J. Sellmyer, and D. Shindo (Springer-Verlag, New York, 2006), p. 1635. http://dx.doi.org/doi 10.1007/1-4020-7984-2_41
2. Shan X. Wang and G. Li, IEEE Trans. Magn. **44** (7), 1687 (2008).
3. C. Reig, M.-D. Cubells-Beltran, and D. R. Muñoz, Sensors **9**, 7917 (2009). doi 10.3390/s91007919
4. R. Weiss, R. Mattheis, and G. Reiss, Meas. Sci. Technol. **24**, 082001 (2013). doi 10.1088/0957–0233/24/8/082001
5. V. V. Ustinov, A. B. Rinkevich, L. N. Romashev, and D. V. Petrov, Tech. Phys. **49** (5), 613 (2004).
6. B. A. Belyaev and V. V. Tyurnev, Russian Inventor's Certificate No. 1085460, Byull. Izobret., No. 7 (2012).
7. B. A. Belyaev and A. V. Izotov, Phys. Solid State **49** (9), 1731 (2007).
8. G. Korn and T. Korn, *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review* (McGraw-Hill, New York, 1961; Nauka, Moscow, 1968).
9. V. V. Tyurnev, *Equations of Mathematical Physics: A Textbook* (Krasnoyarsk State Technical University, Krasnoyarsk, 2001) [in Russian].
10. B. A. Belyaev, V. V. Tyurnev, and An. A. Leksikov, Izv. Vyssh. Uchebn. Zaved., Fiz. **55** (8/2), 152 (2012).
11. B. A. Belyaev, A. V. Izotov, and An. A. Leksikov, Phys. Solid State **52** (8), 1664 (2010).
12. A. V. Izotov, B. A. Belyaev, M. M. Valikhanov, S. V. Polenga, and A. V. Stefanyuk, Vychisl. Metody Program. **13**, 551 (2012).
13. B. A. Belyaev and A. V. Izotov, Phys. Solid State **55** (12), 2491 (2013).
14. A. I. Ansel'm, *Introduction to the Theory of Semiconductors* (Nauka, Moscow, 1978; Mir, Moscow, 1981).

Translated by E. Chernokozhin